PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCXXXI.

PART I.

LONDON:

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MDCCCXXXI.

ADVERTISEMENT.

The Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions*, take this opportunity to acquaint the Public, that it fully appears, as well from the council-books and journals of the Society, as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries, till the Forty-seventh Volume: the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion,

as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

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APPENDIX.

The President and Council of the Royal Society adjudged the Royal Medals for the year 1830 as follows:

A Royal Medal to David Brewster, LL.D., Fellow of the Royal Society, for his Communications to the Royal Society on the Polarization, and other Properties, of Light.

A Royal Medal to M. Balard, of Montpellier, for his Discovery of Brome.

PHILOSOPHICAL TRANSACTIONS.

I. *Observations of the Second Comet of* 1822, *made at Rio de Janeiro. By Lieutenant (now Captain)* William Robertson, *R.N. Communicated in a Letter to Captain* Basil Hall, *F.R.S.*

Read June 17, 1830.

I SHALL feel obliged to you to lay before the Royal Society the following observations, which, with the assistance of Lieut. Charles Drinkwater, R.N., I made upon the "second comet of 1822," as it is called. They were made at Rio de Janeiro, when I was Lieutenant of His Majesty's ship Creole, under the orders of Commodore Sir Thomas Hardy; but as the means I had in my possession for making such nice observations were not great, I did not imagine the results could be of much value, till I accidentally gained information of the following remarks in Baron Zach's Journal, vol. vi. page 596.

"Cette comète, comme nous l'avons dit dans notre V^me^ cahier, page 481, n'a été que très peu observée, à cause de son mouvement fort-rapide en déclinaison australe; elle s'est par conséquent bientôt soustraite à nos regards, pour se montrer peut-être avec plus d'éclat aux antipodes. Nous n'avons reçu d'autres observations que celles que nous avons déjà publiées dans notre cahier précédent. Probablement l'orbite de cet astre passager nous restera inconnue pour toujours, à moins que MM. Fallow, Rumker, ou un autre Basil Hall, ne parviennent à l'observer dans l'hémisphere austral. Mais la correspondence astronomique des deux hémispheres n'est pas encore bien établie *."

The above allusion to your observations on the comet which we observed together at Valparaiso in 1821, and which are published in the Philosophical

* Correspondence Astronomique du Baron Zach, vol. vi. p. 595.

Transactions, induces me to address this communication to you, to be laid before the Society, if you consider it worthy of that honour.

M. Pons, it appears, (Zach, vol. vi. p. 385) first discovered this comet on the 31st of May, about two o'clock in the morning. It was then at the distance of $2\frac{1}{2}°$ from the star β Piscium, 5° from η Aquarii, nearly in $340\frac{1}{4}°$ of Æ, and in $2\frac{1}{2}°$ north declination. As M. Pons had no instruments ready to observe the comet, it does not appear to have been further noticed till the 8th of June, by Professor Caturegli at Bologna, and two days afterwards by M. Gambart at Marseilles. Owing to its extreme feebleness, however, Baron Zach does not appear to think the observations at Bologna very exact.

The following two Tables of the observations just mentioned are from Zach's Correspondence, vol. vi. p. 482.

1822.	Tems vrai à Bologne.	Ascen. droite de la comète.	Declin. de la comète australe.
	h m	° ′	° ′
Juin 8	15 10	347 39	8 49
10	14 45	351 43	13 28
11	14 44	354 32	16 46
12	14 59	358 25	21 5

The following are the only two observations made by M. Gambart at Marseilles.

1822.	Tems moyen de minuit.	Differ. d'ascen. droite.	N° d'obs.	Differ. de declin.	N° d'obs.	Etoiles companées.
	h m s	° ′ ″		′ ″		
Juin 10	3 3 49	+2 11 9.2	1	−21 38.1	1	ψ^3 du Verseau.
11	2 48 56	−0 20 27.0	4	+30 31.3	4	133 Hor. XXIII. Piazzi.

From these few observations, M. Hulingenstein has deduced the orbit in Schumacher's Astronomische Nachrichten, vol. iv. pp. 533, 534, and which I have copied at the end of this letter. But as the above observations comprehend only a very small portion of the orbit, these elements are susceptible of improvement from observations made during a longer interval.

The following are the whole of the observations which were made by Mr. Drinkwater and myself on this comet.

On the evening of the 18th of June it was first observed, with the naked eye, near the star Canopus, and though it had been in conjunction with the sun on that day, its great southern latitude permitted it to be seen after sunset. Of course we did not know that it had been observed in Europe, and immediately proceeded to determine its position by the only means of which we had command. By means of a reflecting sextant, we took its angular distances from known fixed stars. The faintness of the comet's light, and the uncertainty in estimating its apparent centre, presented considerable difficulties in the employment of this method of observation. It was attempted to diminish the errors which, no doubt, arose from this cause, by taking the angular distances from four stars, and these observations were repeated, as often as the weather permitted, on the succeeding nights. In this manner angular distances were obtained on the evenings of the 18th, 19th, 22nd, 23rd, and 24th of June, 1822; after which the increasing brightness of the moonlight, and the faintness, prevented its being accurately observed with the sextant, and eventually obliterated it altogether.

During the whole of the above period, the comet presented the same appearance as it seems to have done in Europe,—namely, that of a nebulous mass, without either tail or nucleus. I was of course inclined to believe, at first, that these observations would be of little value, as I did not doubt that the comet must have been observed from other places in the southern hemisphere, with more efficient instruments than mine. I hope it may still prove so; but, after repeated inquiries, I have not been able to learn that the comet was seen in any other quarter of that portion of the globe. Nor, indeed, was it until its orbit had been computed from my observations by Mr. Thomas Henderson of Edinburgh, that the comet was suspected to have been one previously observed in Europe. This discovery, it will perhaps be thought, gives a new value to the observations which we made; for though the means used were deficient in that precision which is desirable, the observations, taken in conjunction with those made in Europe, embrace a far greater extent of the comet's orbit than either series do alone. And in the present state of cometary astronomy it is impossible to foretell the value which may one day be assigned to observations which at present appear to have little interest.

The following is a faithful transcript of the original observations as they

were actually written down at the time; without the alteration of a letter or a figure.

Rio de Janeiro, June 18th, 1822, at 6^h 30^m P.M. Observed a bright orbicular nebula near Canopus. On directing the telescope to it, we find it to have the appearance of a comet. At 6^h 40^m mean time, the following distances were taken with sextants:

	°	′	″
From Canopus . . .	3	6	20
—— Sirius	34	27	10
—— α Hydræ . . .	58	9	20
—— α Crucis . . .	47	58	50

June 19th. The comet appeared fainter than last night. There was a thin haze in the sky. The following observations were taken at 6^h 40^m P.M.:

	°	′	″
From Canopus . . .	11	33	30
—— Sirius	30	3	37
—— α Hydræ . . .	46	2	47
—— α Crucis . . .	44	15	30

June 20th. Thick, rainy weather; comet not seen.

June 21st. Thick, cloudy weather.

June 22nd. Fine, clear moonlight. Observed the comet without a telescope. It is still of a round shape, no tail or nucleus observed when looked at with a telescope. The following angular distances were taken at 7^h 0^m P.M.:

	°	′	″
From Canopus . . .	33	35	00
—— Sirius	33	12	00
—— α Hydræ . . .	25	9	45
—— α Crucis . . .	44	36	25

June 23rd. Clear weather. The following angular distances were taken at 6^h 34^m P.M.:

	°	′	″
From Canopus . . .	37	29	20
—— Sirius	35	15	45
—— α Hydræ . . .	21	38	50
—— α Crucis . . .	45	13	10

June 24th. Clear weather; moonlight. The following distances were taken at 6^h 30^m P.M.:

From α Hydræ	. . . .	18° 57′ 25″
—— α Crucis	. . . .	46 37 30

June 25th. Saw the comet; but owing to the clear moonlight, it was too faint to be observed with the sextant.

June 26th. Dark cloudy weather, with rain and thunder.

June 27th. Rainy weather. In the evening, fine weather; comet not seen.

June 28th. Cloudy evening.

June 29th. Fine clear moonlight; could not discover the comet.

On my attention being called to this comet during the last year, more than seven years after I had observed it, I placed the above observations in the hands of Mr. Thomas Henderson, of Edinburgh, who has furnished me with the following remarks, which I transcribe verbatim, and request you will communicate to the Royal Society in the same manner.

"From the observations," says Mr. Henderson, "made at Rio de Janeiro, by Captain Robertson and Lieutenant Drinkwater, on the second comet of 1822, I have obtained the following position of that comet referred to the ecliptic, and cleared of the effect of refraction, but not of parallax, aberration, nutation, or precession. Those positions have been adopted which represent the observed angular distances with the minimum of error, as found by the method of least squares.

Mean solar time at Rio de Janeiro. June 1822.	Apparent longitude.	Apparent latitude.
d h m	° ′ ″	° ′ ″
18 6 40	93 39 26	73 51 6 South
19 6 40	125 15 42	66 42 19
22 7 0	147 5 5	47 30 47
23 6 34	149 31 36	43 49 20
24 6 30	150 48 47	40 39 58

"The errors of observation, on the differences between the observed and computed angular distances, do not exceed five minutes of space, except on the

23rd, when, in one observation, the error amounts to eleven minutes; for which reason the position of that day is not employed in computing the orbit.

"The following elements of the comet's parabolic orbit have been obtained by OLBERS' method of computation, founded upon the observations of June 19th, 22nd, and 24th.

Time of perihelion passage, mean solar time at Greenwich, 1822, July 15.651.

	°	′	″
Longitude of the perihelion	220	19	49
Inclination of the orbit	35	36	0
Longitude of the ascending node . . .	98	14	47
Logarithm of perihelion distance . . .	9.92879		

Motion retrograde.

"The following are the errors of the places computed from these elements, or the corrections to be applied to the computed places, in order to obtain those which were observed.

	Longitude. ′	Latitude. ′
June 18	− 7	+ 1
19	+ 5	+ 1
22	0	− 1
23	+ 15	+ 5
24	0	+ 1

"The greatest error is on the 23rd; the observations of which day, for the reasons already stated, are supposed not to be so exact as those of the other days. The other errors, it may be remarked, are not greater than what might have been expected from the uncertainty of the observations, and great latitude of the comet, when the errors in longitude are apparently much increased, from being reckoned upon a small circle.

"On comparing the foregoing elements, computed from Captain ROBERTSON and Lieutenant DRINKWATER's observations, with those deduced by M. HULINGENSTEIN from the observations made in Europe, referred to at page 2, it will be seen that the differences between them are wonderfully small, considering the different instruments used by the observers in the two hemispheres.

"The elements, placed side by side, stand thus:

	By M. Hulingenstein's computation from observations in Europe.		By the computations from Captain Robertson's observations at Rio de Janeiro.
Time of perihelion passage,	July 16.03925	. . . . . .	July 15.651
	Mean Solar Time at Marseilles.		Mean Solar Time at Greenwich.
	° ′ ″		° ′ ″
Longitude of the perihelion . .	219 53 48	. . . . .	220 19 49
Inclination of the orbit . . .	37 43 4	. . . . . .	35 36 0
Longitude of the ascending node	97 51 23	. . . . . .	98 14 47
Logarithm of perihelion distance	9.92743	. . . . .	9.92879

"Perhaps more correct elements might be obtained from a comparison of all the observations, European as well as South American, were it deemed of sufficient importance to undergo the requisite labour. But without entering into such an investigation, enough has been already stated to show that the instruments and other means in the possession of every naval officer, are sufficient to enable him to determine, with considerable accuracy, the orbit of any comet which is not too faint for being observed with the usual reflecting instruments used at sea."

Before concluding this communication, it may not be improper to mention that about the same time that we were making the observations above detailed, on the "second comet of 1822," we were fortunate enough to see the celebrated comet of Encke, but it had not sufficient light to enable us to observe it in the same manner that we did the other. We were therefore obliged to content ourselves with observing it through an ordinary telescope. But, as it does not appear that on this return of Encke's comet to the neighbourhood of the earth, it was seen in any other part of the world, except at Paramatta, the following notes of what we saw of it at Rio de Janeiro, may not be altogether uninteresting, though probably of little or no value to astronomers.

Memorandum of Encke's comet seen at Rio de Janeiro in 1822.

June 7th. At 6^h 30^m P.M. Observed the comet calculated by Professor Encke, in the constellation Gemini. It was only seen through a telescope, and appeared like a faint nebula of a round form. There were two stars of the 5th

or 6th magnitude near it, with which it formed a right-angled triangle; the right angle at the northernmost of the two stars, and the comet to the westward.

June 10th. Observed Encke's comet after sunset. It has increased its Æ. The stars seen along with it on the 7th are not now in the field of view of the telescope at the same time with the comet.

June 12th. Observed Encke's comet after sunset. It was very faint. No stars that we have in our catalogues (which are very limited) in the field of the telescope.

June 13th. Observed Encke's comet forming an angle of about 100° at β Canis Minoris, with Procyon; at about once and a quarter the distance from β Canis Minoris, that β is from Procyon. It is not brighter than when it was first seen.

June 17th. Encke's comet again seen. A line drawn from Sirius to β Canis Minoris cuts a star of the 3rd or 4th magnitude: about $\frac{1}{5}$th of the distance from that star to Procyon, was the comet, in a triangle formed by three stars of the 5th or 6th magnitude, seen by the telescope thus,  , the Æ being about 103°, and declination 5° north, and it has still the same nebulous, orbicular appearance as when first seen.

June 18th. Saw Encke's comet after sunset—very faint. It had increased its Æ considerably since last night, from the small stars seen last night in the field of the telescope.

June 19th. Hazy, and the direction of the comet not seen.

June 20th and 21st. Thick weather; comet not seen.

June 22nd. Fine clear moonlight; Encke's comet could not be made out, nor was it again seen.

If you think any of these observations likely to interest the Royal Society, I request you will do me the honour to present them.

II. *On the performance of Fluid Refracting Telescopes, and on the applicability of this principle of construction to very large instruments. By* Peter Barlow, *Esq. F.R.S. Cor. Mem. Inst. of France, of the Imperial Academy of St. Petersburgh, &c. &c.*

Read December 9, 1830.

In the Philosophical Transactions for 1827, a paper of mine was published containing an account of a series of experiments which I had carried on with Messrs. W. and T. Gilbert on the curvature of object-glasses for telescopes. In the course of these experiments, I saw so much the difficulty which opticians experience in obtaining large pieces of good flint-glass, that I turned my attention to supplying this material by a fluid. Having, after several attempts, at length found an admirable substitute in sulphuret of carbon, I wrote a short account of my intended construction, addressed to His present Majesty, at that time Lord High Admiral, and, as such, President of the Board of Longitude, soliciting from that Board assistance in carrying forward my experiments. Having obtained this aid, the result of my first trial was the construction of an eight-inch fluid telescope, at that time the largest refractor in this country. A description of this instrument is given in the Philosophical Transactions for 1829, and some objects are pointed out which had been selected as tests of its performance.

I have however since had more time and better means of testing the instrument; first, through the kindness of Mr. Herschel, who pointed out to me several objects that he had observed with his new twenty-inch speculum; and secondly, by direct observations on the same objects in Sir James South's new twenty-feet refractor, and in my own telescope. A few of these, which serve to mark distinctly the progress I have made, are given below; but I will first state two or three of my own observations, which, I conceive, tend also to the same object.

In the paper last referred to, I have stated my observation on η Persei,

marked as double in South and Herschel's catalogue, with a small star at a greater distance; this star is seen distinctly sextuple in my telescope. These stars I had the satisfaction of showing to M. Struve in his recent visit to England, and I have since seen them in Sir James South's telescope. Another good test of the light of my telescope is found in σ Orionis, marked in the above catalogue as two distinct sets of stars, each triple; whereas, in my telescope, both sets are quadruple, with a double star, or rather two very fine stars between them; the fourth star in the bright set, is a remarkably fine brilliant point, very near to the principal star, and in the same line as the nearest of the original small stars, on the opposite side, so that the three are in one line; or more accurately, the line joining the two small stars touches the margin of the bright star. I might mention several other cases of fine double stars which I have discovered, but I select the above because it is evident that both objects have been well examined with fine instruments, and that the stars I have mentioned had, notwithstanding, escaped detection.

Of the tests furnished me by Mr. Herschel I shall only select two, one of which in particular serves to point out in a very precise manner the limit of power of my telescope. This is the star β Capricorni, which, in the finder, is a coarse double star of about $3'$; but between these two stars, nearly in the middle, but a little below the line of junction, is a very fine double star, discovered by Mr. Herschel, and which he considers a very severe test; he says indeed that he requires no other, of the light of a telescope. This star I can see, and, under favourable circumstances, distinctly; but still I have not sufficient command of it to see it double. We have thus the exact limit defined at which the light of this splendid instrument surpasses that of my telescope. The other object to which I have alluded is ϕ Virginis: this he considers a very easy double star, although it had before escaped detection; it is however rather close. This star I could see very distinctly one evening (June 4th), the moon being very bright and full, on the meridian, and within an hour of the star. I mention this object because it requires a certain degree of defining power; in point of light it involves no difficulty. Mr. Herschel could see it when his aperture was reduced to six inches.

Amongst the objects which I have seen in Sir James South's twenty-feet, there is also one in particular which forms a good test of the relative power of

his instrument and mine; this is MESSIER's twenty-second nebula. This object, which in a good $3\frac{1}{2}$-inch refractor has only the appearance of a white cloud, I saw in the above instrument resolved into an immense number of brilliant small stars. In my telescope also it is resolved into apparently as great a number of stars, but the full power of the instrument is exerted, and still the resolution seems scarcely complete. In fact, my instrument appears to labour to effect what seems to be quite within the power of the other.

I wish particularly to direct attention to this object and that of β Capricorni, because, where the same object can be seen without any very apparent difference in two instruments, or where it can be only seen in one, the great test of comparison is lost; but in those I have mentioned, the exact limit of power is defined.

Amongst the objects I have examined in Sir JAMES SOUTH's telescope, and repeated in mine, for the purpose of comparing the defining powers of the two instruments, were the planets Jupiter and Mars. These were both more sharply defined in Sir JAMES SOUTH's than in my telescope, but the superiority was by no means so great as I had expected. I fortunately saw the shadow of Jupiter's third satellite pass over the disc, on the 8th of August, at Kensington, and it exhibited a fine black round spot, extremely well defined; and on the 13th of the same month, I witnessed precisely the same phenomenon in mine, and, as far as the definition of the shadow was concerned, with an effect in which I could not distinguish an inferiority; it had the appearance of a small black wafer on a sheet of white paper; still, however, the edge of the planet was certainly sharper in the twenty-feet. Both evenings were amongst the finest this climate affords, and the powers employed as nearly as possible equal; viz. about 260 and 450 in both instruments. We also, at Kensington, observed Mars with various powers from 260 to 1400, and it carried 1200 well; with this the white spot near its south pole was seen beautifully distinct, as also a long dark spot on its apparent eastern limb. The bright spot at its south pole I saw also remarkably well defined in my own telescope on the 13th, and a dark spot on its disc very distinct; but it occupied the centre of its disc. The highest power however I used was 500, and it was probably best seen with 260.

There can be no question of the superior defining power of the twenty-feet,

and the light is of course also greater; still, however, when I consider that I have been comparing with two telescopes, one of twenty inches aperture, and the other of twelve inches, and each of them twenty feet in focal length, or nearly, while my telescope is barely eight inches aperture, and only twelve feet in length, I cannot but consider the comparison as highly satisfactory.

In addition to the above observations, which have been certainly highly gratifying to myself, I have also had the honour of showing the instrument to many persons, both Englishmen and foreigners well acquainted with astronomy, and in every instance the practicability of the principle of construction has been admitted; a point by no means generally granted when the suggestion was first advanced.

Other obstacles also, independent of the arrangement of the lenses, were foreseen, which time is gradually dissipating; such as the difficulty of permanently securing the fluid, and then, admitting this to be effected, the probability of a decomposition of the glass by the fluid, &c. &c. I have, however, now the satisfaction to state, that the lens of my 3-inch telescope, filled August 5th 1827, continues in precisely its original state, no perceptible change having yet taken place in either the quantity or quality of the fluid, or in the transparency of the glass.

As far as the above observations and remarks extend, therefore, it appears that the essential properties of the flint lens are supplied by the fluid. I beg now to state a few particulars in which the sulphuret of carbon has advantages which the glass has not:—these are, first, that in consequence of the very high dispersive power of this fluid, the correcting lens is placed so far behind the principal plate or crown lens, as to require to be only one half as much in diameter; a highly important consideration in the construction of a very large telescope.

Secondly, the combination is such as to give a focal power one and a half times the length of the tube, or, which is the same, the telescope may be reduced to two thirds the length of a glass telescope of the usual kind, without incurring a greater amount of spherical aberration in the front lens.

Of the latter advantage, however, I have not ventured fully to avail myself in my 8-inch, because, as I knew the general opinion was against the success of the experiment, I was fearful of failing in the beginning by attempting too much.

I have therefore made the length twelve feet, to an aperture of eight inches, which, although shorter than opticians would choose to work in the usual achromatic, is not so short as this principle of construction would admit, and which in any new case I should not hesitate to adopt. Indeed, according to the form of construction I am now about to propose, a telescope of two feet aperture and twenty-four feet in length would not have more spherical aberration to contend with, than a telescope of the usual construction of six inches aperture and twelve feet length, which is fully within the range of the usual practice; at the same time I will not undertake to say that I could on so large a scale confine the length to twelve times the aperture, although I should certainly attempt it in the first instance. But if the length extended to even fifteen or eighteen times the aperture, I have little doubt of making the instrument manageable by one person, by adequate mechanical arrangements, and of producing a telescope which would as much exceed the most powerful telescopes of the present day, as these exceed the refractors of highest repute at the close of the last century.

Whether such an instrument will be undertaken at present, depends upon circumstances which I cannot command. I can only say, that if such a construction were entrusted to my direction, no exertion should be wanted on my part, to render it complete and worthy of the present state of English science. At all events I cannot doubt that the spirit of scientific enterprise will lead ultimately to the attempt; and in order to facilitate the accomplishment of it, as far as lies in my power, I have in the following pages described the nature of the arrangements which in my opinion would most contribute to success.

In my former paper I have given a formula expressing the relations between the length, foci, and distances of the lenses, and have remarked upon the almost infinite variety of forms to which it leads; some of them, I have stated, would probably be found in practice preferable to others, although they are all equally correct in theory. Of these cases, some have since suggested themselves to me; and others will also probably be detected, by a due examination of the formula and tables, which Professor Littrow, of Berlin, has recently presented to the Astronomical Society, relative to this form of telescope; with tables of curvatures, both direct from the formulæ of Euler (reduced to the

case of open lenses), as also indirectly from principles of his own. I have not as yet had an opportunity of examining these cases, but, from the well-known ingenuity of their author, I cannot doubt of finding in this memoir many useful suggestions.

The great change, however, which I propose to make in the construction of this giant telescope, is to have two front lenses, which will be attended with advantages not involved in the above considerations. At present, in consequence of the diameter of the fluid lens being only half that of the front lens, it is difficult to get a sufficient quantity of spherical aberration in the former, to correct that of the latter;—for although we give to the plate lens the curvature requisite for reducing its aberration to a minimum, yet the fluid lens is obliged to be made considerably concavo-convex (a form not to be used when it can be avoided), in order to produce a sufficient aberration in the fluid to correct it. Moreover I have hitherto employed parallel meniscus cheeks to contain the fluid, which present a practical difficulty, if not a positive impediment, to good centering. This will be seen immediately when we consider that when a lens is double-concave or convex, and also when it is concavo-convex, if the radii of curvature are very unequal, the centering may always be effected: for the line joining the centres of the two spheres, or this line produced in the latter case, must pass through the lens, and indicate its true centre: but when the lens has parallel surfaces, or the radii equal, if the two spherical centres be not coincident from the tool itself (a very improbable case) the line which joins them can never cut the lens, and consequently it can have no true centre. All these evils will, however, be avoided in the proposed application of two front lenses, which, by being placed each in what opticians call their best position, will at once reduce the spherical aberration of the front lens to about one third of its present amount, and thereby enable us to correct it by the fluid lens without adopting the distorted form rendered necessary under present circumstances.

Another important consideration is also involved in this form, relative to the facility it affords of obtaining the plate-glass. If the front lens were single, the thickness would be such as would require the glass to be made specifically for the purpose, and of course all the delay and expense of previous experiments would be incurred; whereas, by dividing the whole amount of curvature

between the two, the usual thickness of plate-glass, as at present manufactured, would be sufficient, and we might have the selection from large stores of the best glass at a trifling expense; and as to that of the correcting fluid, or substitute for the flint-glass, it is so very inconsiderable as not to deserve being mentioned; although, if it were possible to obtain a piece of flint-glass large enough for such a purpose, scarcely any price, however great, would be thought exorbitant. In the instrument proposed, nearly the whole expense would be the workmanship, and I must think it very inconsiderable in comparison with the magnitude and importance of the undertaking.

I had intended to have concluded this paper by giving the curvature, foci, &c. which I have computed; but as they are merely supposititious, as far as they are dependent on the index and dispersion of the front glass, it is perhaps better to withhold them. The only object I had in making them was to form some idea of the requisite curvature, thickness of glass, &c. They can only of course be permanently made after the plate-glass has been selected.

III. *Researches in Physical Astronomy.* By JOHN WILLIAM LUBBOCK, *Esq., V.P. and Treas. R.S.*

Read December 9, 1830.

IN last April I had the honour of presenting to the Society a paper containing expressions for the variations of the elliptic constants in the theory of the motions of the planets. The stability of the solar system is established by means of these expressions, if the planets move in a space absolutely devoid of any resistance *, for it results from their form that however far the approximation be carried, the eccentricity, the major axis, and the tangent of the inclination of the orbit to a fixed plane, contain only periodic inequalities, each of the three other constants, namely, the longitude of the node, the longi-

* When the body moves in a medium which resists according to any power of the velocity, the contrary obtains, the major axis and eccentricity acquiring a term which varies with the time, while the longitude of the perihelion and longitude of the epoch have only periodic inequalities. This results from the equations given in the former part of this paper, Phil. Trans. Part II. 1830, page 340.

$$\mathrm{d}\,a = -2\,c\,a\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}} \left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n+1}{2}} \frac{(1-e\cos v)}{n}\,\mathrm{d}\,v$$

$$\mathrm{d}\,e = -2\,c\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}} \left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}} \left(\frac{1-e^2}{2}\right)\cos\mathrm{d}\,v$$

$$c\,\mathrm{d}\,\varpi = -2\,c\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}} \left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}} \frac{\sqrt{1-e^2}}{n}\sin v\,\mathrm{d}\,v$$

$$\mathrm{d}\,\varepsilon - \mathrm{d}\,\varpi = 2\,c\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}} \left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}} \left\{\frac{1-e^3\cos v}{e}\right\}\mathrm{d}\,v$$

$$\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n+1}{2}} = 1 + (n+1)\,e\cos v + (n+1)^2\,e^2\cos^2 v$$

$$\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}} = 1 + (n-1)\,e\cos v + (n-1)^2\,e^2\cos^2 v$$

tude of the perihelion, and the longitude of the epoch, contains a term which varies with the time, and hence the line of apsides and the line of nodes revolve continually in space. The stability of the system may therefore be inferred, which would not be the case if the eccentricity, the major axis, or the tangent of the inclination of the orbit to a fixed plane contained a term varying with the time, however slowly.

The problem of the precession of the equinoxes admits of a similar solution; of the six constants which determine the position of the revolving body, and the axis of instantaneous rotation at any moment, three have only periodic inequalities, while each of the other three has a term which varies with the time. From the manner in which these constants enter into the results, the equilibrium of the system may be inferred to be stable, as in the former case. Of the constants in the latter problem, the mean angular velocity of rotation

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$$\mathrm{d}\,a = -\frac{2\,c\,a}{n}\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\{1 + n\,e\cos v + n\,(n+1)\,e^2\cos^2 v + \&c.\}\,\mathrm{d}\,v$$

$$= -\frac{2\,c\,a}{n}\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{1 + \frac{n\,(n+1)}{2}e^2 + n\,e\cos v + \frac{n\,(n+1)}{2}e^2\cos^2 v + \&c.\right\}\mathrm{d}\,v$$

$$\mathrm{d}\,e = -\frac{2\,c}{n}\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\{\cos v + (n+1)\,e\cos^2 v + (n+1)^2 e^2\cos^3 v + \&c.\}\,(1-e^2)\,\mathrm{d}\,v$$

$$= -\frac{2\,c}{n}\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{n+1}{2}\,e + \left(1 + 3\,\frac{(n+1)^2}{4}\right)\cos v + \frac{(n+1)}{2}\,e\cos 2\,v + \frac{(n+1)^2}{4}\cos 3\,v + \&c.\right\}(1-e^2)\,\mathrm{d}\,v$$

neglecting the terms which are periodic

$$\mathrm{d}\,a = -\frac{2\,c\,a}{n}\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{1 + \frac{n\,(n+1)}{2}\,e^2 + \&c.\right\}\mathrm{d}\,v$$

$$\mathrm{d}\,e = -\frac{2\,c}{n}\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{(n+1)}{2}\,e + \&c.\right\}(1-e^2)\,\mathrm{d}\,v$$

The major axis decreases perpetually, the eccentricity diminishes perpetually until it reaches zero, while the perihelion retains the same *mean position*, and the longitude of the epoch the same *mean value*. I stated inadvertently in the former part of this paper, p. 340, that the variations of the eccentricity are all periodical.

may be considered analogous to the mean motion of a planet, or its major axis; the geographical longitude, and the cosine of the geographical latitude of the pole of the axis of instantaneous rotation, to the longitude of the perihelion and the eccentricity; the longitude of the first point of Aries and the obliquity of the ecliptic, to the longitude of the node and the inclination of the orbit to a fixed plane; and the longitude of a given line in the body revolving, passing through its centre of gravity, to the longitude of the epoch. By the stability of the system I mean that the pole of the axis of rotation has always nearly the same geographical latitude, and that the angular velocity of rotation, and the obliquity of the ecliptic vary within small limits, and periodically. These questions are considered in the paper I now have the honour of submitting to the Society. It remains to investigate the effect which is produced by the action of a resisting medium; in this case the latitude of the pole of the axis of rotation, the obliquity of the ecliptic, and the angular velocity of rotation might vary considerably, although slowly, and the climates undergo a considerable change.

The co-efficients of the terms in the development of R, multiplied by the squares and products of the eccentricities, are susceptible of very great simplification, in consequence of the equations of condition which obtain between the quantities of which the general symbol is b. I have now given the development of R, as far as the terms depending upon the squares and products of the eccentricities, in its simplest form. See p. 30.

I have also given methods of obtaining the inequalities of the radius vector, of longitude, and of latitude in the planetary theory. The expressions in this paper differ in form from those of LAPLACE, but their identity may be shown by means of equations of condition which obtain between some of the quantities involved.

I have taken as a numerical example, the calculation of the co-efficients of some of the inequalities in the theory of Jupiter, disturbed by Saturn.

On the Precession of the Equinoxes.

Let O be the origin of the co-ordinate axes, coinciding with some point in the interior of the mass M.

Let x, y, z be the co-ordinates of any element $d\,m$ parallel to three rectangular

axes $O\,x$, $O\,y$, $O\,z$, fixed in space, $x_{\prime}, y_{\prime}, z_{\prime}$, the co-ordinates of the same element parallel to three other rectangular axes $O\,x_{\prime}$, $O\,y_{\prime}$, $O\,z_{\prime}$, fixed in the mass M' and revolving with it. Let the line N O N′ be the intersection of the plane $x_{\prime}y_{\prime}$ with the plane $x\,y$,

Let the angle $N\,O\,x = \psi$, $N\,O\,x_{\prime} = \varphi$, and the inclination of the plane $x_{\prime}\,y_{\prime}$ upon $x\,y = \theta$.

$$x_{\prime} = x(\cos\theta\sin\psi\sin\varphi + \cos\psi\cos\varphi) + y(\cos\theta\cos\psi\sin\varphi - \sin\psi\cos\varphi) - z\sin\theta\sin\varphi$$
$$y_{\prime} = x(\cos\theta\sin\psi\cos\varphi - \cos\psi\sin\varphi) + y(\cos\theta\cos\psi\cos\varphi + \sin\psi\sin\varphi) - z\sin\theta\cos\varphi$$
$$z_{\prime} = x\sin\theta\sin\psi + y\sin\theta\cos\psi + z\cos\theta$$

Let $X_{\prime}, Y_{\prime}, Z_{\prime}$ be the accelerating forces which act upon the element $d\,m$ in the direction of the axes $O\,x_{\prime}$, $O\,y_{\prime}$, $O\,z_{\prime}$,

$$\int(y_{\prime}^2 + z_{\prime}^2)\,d\,m = A, \qquad \int(x_{\prime}^2 + z_{\prime}^2)\,d\,m = B, \qquad \int(x_{\prime}^2 + y_{\prime}^2)\,d\,m = C$$

$$p\,\mathrm{d}\,t = \sin\varphi\sin\theta\,\mathrm{d}\,\psi - \cos\varphi\,\mathrm{d}\,\theta$$
$$q\,\mathrm{d}\,t = \cos\varphi\sin\theta\,\mathrm{d}\,\psi + \sin\varphi\,\mathrm{d}\,\theta$$
$$r\,\mathrm{d}\,t = \mathrm{d}\,\varphi - \cos\theta\,\mathrm{d}\,\psi$$

$$C\,\mathrm{d}r + (B - A)\,p\,q\,\mathrm{d}\,t = \mathrm{d}\,t\int(x_{\prime}\,Y_{\prime} - y_{\prime}\,X_{\prime})\,\mathrm{d}\,m$$

$$B\,\mathrm{d}q + (A - C)\,r\,p\,\mathrm{d}\,t = \mathrm{d}\,t\int(z_{\prime}\,X_{\prime} - x_{\prime}\,Z_{\prime})\,\mathrm{d}\,m$$

$$A\,\mathrm{d}p + (C - B)\,q\,r\,\mathrm{d}\,t = \mathrm{d}\,t\int(y_{\prime}\,Z_{\prime} - x_{\prime}\,Y_{\prime})\,\mathrm{d}\,m$$

If the axis of instantaneous rotation coincides with the line O I at any instant

$$\cos I\,O\,x_{\prime} = \frac{p}{\sqrt{p^2 + q^2 + r^2}}$$

$$\cos I\,O\,y_{\prime} = \frac{q}{\sqrt{p^2 + q^2 + r^2}}$$

$$\cos I\,O\,z_{\prime} = \frac{r}{\sqrt{p^2 + q^2 + r^2}}$$

$$\sin I\,O\,z_{\prime} = \frac{\sqrt{p^2 + q^2}}{\sqrt{p^2 + q^2 + r^2}}$$

If $z_{\prime}$ I L be a great circle cutting the plane $x_{\prime}y_{\prime}$ in L,

$$\cos x_{\prime}\,O\,L = \frac{\cos I\,O\,x_{\prime}}{\sin I\,O\,z_{\prime}} = \frac{p}{\sqrt{p^2 + q^2}}$$

If the accelerating forces $X, Y, Z = 0$ and $B = A$, the integrals of the preceding equations are

$$r = n, \qquad p = c\cos\frac{C - A}{A}(n\,t + \gamma), \qquad q = c\sin\frac{C - A}{A}(n\,t + \gamma)$$

and neglecting c^2,

$$\theta = \omega + \frac{c}{n}\frac{A}{C}\sin\left(\frac{C}{A}n\,t + \frac{C-A}{A}\gamma\right)$$

$$\psi = \psi_0 - \frac{c}{n\sin\omega}\frac{A}{C}\cos\left(\frac{C}{A}n\,t + \frac{C-A}{A}\gamma\right)$$

$$\varphi = \varphi_0 + n\,t - \frac{c\cos\omega}{n\sin\omega}\frac{A}{C}\cos\left(\frac{C}{A}n\,t + \frac{C-A}{A}\gamma\right)$$

n, c, γ, ω, ψ_0, and φ_0 being constants. In the problem of the Precession of the Equinoxes ω is the mean *obliquity of the ecliptic*, ψ_0 is the *longitude of the first point of Aries* when $t = 0$ reckoned from some fixed line.

Sin I O $z_{,} = \frac{c}{\sqrt{n^2 + c^2}}$, hence it appears that if a body whose form is that of a figure of revolution be made to revolve, and be acted upon by no extraneous force, the axis of instantaneous rotation revolves about the axis of the figure, the *latitude* of the former axis remaining constant.

$$\text{The angle } x_{,}\,\mathrm{O\,I}_{,} = \frac{C-A}{A}(n\,t + \gamma)$$

If the forces $X_{,}$, $Y_{,}$, $Z_{,}$ arise from the attractions of a distant luminary M', of which the coordinates referred to the axes O $x_{,}$, O $y_{,}$, O $z_{,}$, are $x_{,}'$, $y_{,}'$, $z_{,}'$, the force varying inversely as the square of the distance,

$$X_{,} = \frac{M'(x_{,}' - x_{,})}{\{(x_{,} - x_{,}')^2 + (y_{,} - y_{,}')^2 + (z_{,} - z_{,}')^2\}^{\frac{3}{2}}}$$

$$Y_{,} = \frac{M'(y_{,}' - y_{,})}{\{(x_{,} - x_{,}')^2 + (y_{,} - y_{,}')^2 + (z_{,} - z_{,}')^2\}^{\frac{3}{2}}}$$

$$Z_{,} = \frac{M'(z_{,}' - z_{,})}{\{(x_{,} - x_{,}')^2 + (y_{,} - y_{,}')^2 + (z_{,} - z_{,}')^2\}^{\frac{3}{2}}}$$

$$\int(x_{,}Y_{,} - y_{,}X_{,})\,\mathrm{d}m = M'\int\frac{(x_{,}y_{,}' - y_{,}x_{,}')}{r'^3}\left\{1 + \frac{3(x_{,}x_{,}' + y_{,}y_{,}' + z_{,}z_{,}')}{r^2} + \&\text{c.}\right\}\mathrm{d}m$$

By the properties of the principal axes $\int x_{,}y_{,}\,\mathrm{d}m = 0$, $\int x_{,}z_{,}\,\mathrm{d}m = 0$, $\int y_{,}z_{,}\,\mathrm{d}m = 0$, and by the properties of the centre of gravity $\int x_{,}\,\mathrm{d}m = 0$, $\int y_{,}\,\mathrm{d}m = 0$, $\int z_{,}\,\mathrm{d}m = 0$, whence

$$\int(x_{,}Y_{,} - y_{,}X_{,})\,\mathrm{d}m = \frac{3M'x_{,}'y_{,}'}{r'^5}\int(x_{,}^2 - y_{,}^2)\,\mathrm{d}m = \frac{3M'(B-A)}{r'^5}x_{,}'y_{,}'\,\mathrm{d}t.$$

$$C\,dr + (B - A)pq\,dt = \frac{3\,M'(B-A)}{r'^5} x_{\prime} y_{\prime}'\,dt$$

$$B\,dq + (A - C)rp\,dt = \frac{3\,M'(A-C)}{r'^5} z_{\prime}' x_{\prime}'\,dt$$

$$A\,dp + (C - B)qr\,dt = \frac{3\,M'(C-B)}{r'^5} y_{\prime}' z_{\prime}'\,dt$$

Substituting for $x_{\prime}'$, $y_{\prime}'$, $z_{\prime}'$ their values from equations, p. 20, upon the supposition of $\psi = 0$,

$$C\,dr + (B - A)pq\,dt = \frac{3\,M'(B-A)}{2\,r'^5} \left\{ \{ (y'\cos\theta - z'\sin\theta)^2 - x'^2 \} \sin 2\varphi \right.$$
$$\left. + 2x'(y'\cos\theta - z'\sin\theta)\cos^2\varphi \right\}$$

$$B\,dq + (A - C)rp\,dt = \frac{3\,M'(A-C)}{r_{\prime}^5} \left\{ x'(y'\sin\theta + z'\cos\theta)\cos\varphi \right.$$
$$\left. + (y'\cos\theta - z'\sin\theta)(y'\sin\theta + z'\cos\theta)\sin\varphi \right\}$$

$$A\,dp + (C - B)qr\,dt = \frac{3\,M'(C-B)}{r'^5} \left\{ (y'\cos\theta - z'\sin\theta)(y'\sin\theta + z'\cos\theta)\cos\varphi \right.$$
$$\left. - x'(y'\sin\theta + z'\cos\theta')\sin\varphi \right\}$$

If

$$\frac{3\,M'}{r'^5} \left\{ (y'\cos\theta - z'\sin\theta)(y'\sin\theta + z'\cos\theta) \right\} = P$$

$$\frac{3\,M'}{r'^5} \left\{ x'(y'\sin\theta + z'\cos\theta) \right\} = P'$$

$$B\,dq + (A - C)rp\,dt = (A - C)\,dt\,\{P'\cos\varphi + P\sin\varphi\}$$

$$A\,dp + (C - B)qr\,dt = (C - B)\,dt\,\{P\cos\varphi - P'\sin\varphi\}$$

P and P' may be developed according to sines and cosines of angles increasing proportionally to the time. Let $k\cos(it + \varepsilon)$ be any term of P, $k'\sin(it + \varepsilon)$ the corresponding term of P',

$$B\,dq + (A - C)rp\,dt = \frac{A - C}{2}\,dt \left\{ (k + k')\sin(\varphi + it + \varepsilon) + (k - k')\sin(\varphi - it - \varepsilon) \right\}$$

$$A\,dp + (C - B)qr\,dt = \frac{C - B}{2}\,dt \left\{ (k + k')\cos(\varphi + it + \varepsilon) + (k - k')\cos(\varphi - it - \varepsilon) \right\}$$

The equations which were given p. 21, may still be considered as afford-

ing a solution of the problem by making the constants n, c, γ, ω, ψ_0 and φ_0 vary,

$$\mathrm{d}n = 0$$

$$\sin\frac{C-A}{A}(nt+\gamma)\,\mathrm{d}c + c\frac{C-A}{A}\cos\frac{C-A}{A}(nt+\gamma)\,\mathrm{d}\gamma + \frac{(A-C)}{A}nc\cos\frac{C-A}{A}(nt+\gamma)\,\mathrm{d}t$$
$$= \frac{A-C}{2A}\mathrm{d}t\left\{(k+k')\sin(\phi+it+\varepsilon) + (k-k')\sin(\varphi-it-\varepsilon)\right\}$$

$$\cos\frac{C-A}{A}(nt+\gamma)\,\mathrm{d}c - c\frac{C-A}{A}\sin\frac{C-A}{A}(nt+\gamma)\,\mathrm{d}\gamma - \frac{A-C}{A}nc\sin\frac{C-A}{A}(nt+\gamma)\,\mathrm{d}t$$
$$= -\frac{A-C}{2A}\mathrm{d}t\left\{(k+k')\cos(\varphi+it+\varepsilon) + (k-k')\cos(\varphi-it-\varepsilon)\right\}$$

since $\varphi = \varphi_0 + nt$ nearly

$$\mathrm{d}c = \frac{A-C}{2A}\mathrm{d}t\left\{-(k+k')\cos\left(\varphi_0+nt+\frac{C-A}{A}(nt+\gamma)+it+\varepsilon\right)\right.$$
$$\left.-(k-k')\cos\left(\varphi_0+nt+\frac{C-A}{A}(nt+\gamma)-it-\varepsilon\right)\right\}$$

$$c\frac{C-A}{A}\mathrm{d}\gamma + \frac{A-C}{A}nc\,\mathrm{d}t = \frac{A-C}{2A}\mathrm{d}t\left\{(k+k')\sin\left(\varphi_0+nt+\frac{C-A}{A}(nt+\gamma)+it+\varepsilon\right)\right.$$
$$\left.+(k-k')\sin\left(\varphi_0+nt+\frac{C-A}{A}(nt+\gamma)-it-\varepsilon\right)\right\}$$

$$\mathrm{d}\omega + \frac{\mathrm{d}c}{n}\frac{A}{C}\sin\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right) + \frac{c}{n}\frac{C-A}{C}\cos\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right)\mathrm{d}\gamma = 0$$

$$\mathrm{d}\psi_0 - \frac{\mathrm{d}c}{n\sin\omega}\frac{A}{C}\cos\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right) + \frac{c\cos\omega}{n\sin\omega^2}\frac{A}{C}\cos\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right)\mathrm{d}\omega$$
$$+\frac{c}{n\sin\omega}\frac{C-A}{A}\sin\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right)\mathrm{d}\gamma = 0$$

$$\mathrm{d}\varphi_0 - \frac{\mathrm{d}c}{n\sin\omega}\frac{A}{C}\cos\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right) + \frac{c}{n\sin\omega^2}\frac{A}{C}\cos\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right)\mathrm{d}\omega$$
$$+\frac{c}{n\sin\omega}\frac{C-A}{A}\sin\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right)\mathrm{d}\gamma = 0$$

From the preceding expressions it may be inferred that

$n =$ constant.

$\frac{\mathrm{d}c}{\mathrm{d}t} =$ series of cosines without any constant quantity, unless the mean mo-

tion of rotation is commensurate to the mean motion of revolution of the luminary M'.

$\frac{d\gamma}{dt}$ = series of sines without any constant quantity, except in a similar case.

c being equal to a constant + a series of sines.

$\frac{d\omega}{dt}$ = a series of sines without any constant quantity.

$\frac{d\psi_0}{dt}$ = a series of cosines + a constant quantity.

$\frac{d\varphi_0}{dt}$ = a series of cosines + a constant quantity.

In the general case where A is not equal to B, n = constant + series of cosines.

The form of the preceding expressions is not affected however, for the approximation may be carried so that except in the case of commensurability above mentioned, the mean motion of rotation being also nearly twice the mean motion of the planet M' in its orbit or greater,

n = constant + series of cosines without any constant quantity multiplied by the time.

c = constant + series of sines or cosines without any constant quantity multiplied by the time.

ω = constant + series of cosines without any constant quantity multiplied by the time.

γ = constant + series of cosines or sines + a constant quantity multiplied by the time.

ψ_0 = constant + series of sines + a constant quantity multiplied by the time.

φ_0 = constant + series of sines + a constant quantity multiplied by the time.

The constant quantity multiplied by the time in the value of ψ_0 is the precession of the equinox.

If $z' = 0$, (which amounts to taking for the fixed plane the orbit of the planet M',) and n' be its mean motion, then neglecting the eccentricity, $x' = a' \cos n' t$, $y' = a' \sin n' t$, $r' = a'$,

$$P = \frac{3\,M'}{2\,r'^3} \sin\omega \cos\omega\,(1 - \cos 2\,n' t), \qquad P' = \frac{3\,M'}{2\,r'^3} \sin\omega \sin 2\,n' t$$

Supposing $\varphi_0 = 0$, $\gamma = 0$, $c_0 = 0$, c_0 being in fact imperceptible to observation, and neglecting $\cos 2n't$, $\sin 2n't$, in order to find the constant part of $\frac{d\psi_0}{dt}$,

$$\frac{dc_0}{dt} = -\frac{3(A-C)}{2nAa'^3} M' \sin\omega \cos\omega \cos\frac{C}{A} n't$$

$$\frac{c\,d\gamma}{dt} = -\frac{3(A-C)}{2nAa'^3} M' \sin\omega \cos\omega \sin\frac{C}{A} n't$$

$$\frac{d\psi_0}{dt} = \frac{3(C-A)}{2nCa'^3} M' \cos\omega = \frac{3(C-A)}{2nC} n'^2 \cos\omega$$

This result agrees with that given in the Méc. Cél. vol. ii. p. 318, and with that given by M. Poisson, Mémoires de l'Academie, vol. vii. p. 247. In Laplace's notation $\omega = h$, $m = n'$. In M. Poisson's notation $\omega = \theta$, $m = n'$.

On the Theory of the Motion of the Planets, continued from Part II. 1830, *p.* 357.

From the general equations given in the Méc. Cél. vol. i. p. 268, the following may be inferred.

$$\frac{3}{2}\frac{a}{a_{\prime}} b_{1,2} = \frac{(a^2+a_{\prime}^2)}{a_{\prime}^2} b_{1,1} - \frac{a}{a_{\prime}} b_{1,0} \qquad \frac{5}{2}\frac{a}{a_{\prime}} b_{1,3} = \frac{2(a^2+a_{\prime}^2)}{a_{\prime}^2} b_{1,2} - \frac{3}{2}\frac{a}{a_{\prime}} b_{1,1}$$

$$\frac{7}{2}\frac{a}{a_{\prime}} b_{1,4} = \frac{3(a^2+a_{\prime}^2)}{a_{\prime}^2} b_{1,3} - \frac{5}{2}\frac{a}{a_{\prime}} b_{1,2} \qquad \frac{9}{2}\frac{a}{a_{\prime}} b_{1,5} = \frac{4(a^2+a_{\prime}^2)}{a_{\prime}^2} b_{1,4} - \frac{7}{2}\frac{a}{a_{\prime}} b_{1,3}$$

$$\frac{a}{2a_{\prime}} b_{3,2} = \frac{(a^2+a_{\prime}^2)}{a_{\prime}^2} b_{3,1} - \frac{3a}{a_{\prime}} b_{3,0} \qquad \frac{3}{2}\frac{a}{a_{\prime}} b_{3,3} = \frac{2(a^2+a_{\prime}^2)}{a_{\prime}^2} b_{3,2} - \frac{5}{2}\frac{a}{a_{\prime}} b_{3,1}$$

$$\frac{5}{2}\frac{a}{a_{\prime}} b_{3,4} = \frac{3(a^2+a_{\prime}^2)}{a_{\prime}^2} b_{3,3} - \frac{7}{2}\frac{a}{a_{\prime}} b_{3,2} \qquad \frac{7}{2}\frac{a}{a_{\prime}} b_{3,5} = \frac{4(a^2+a_{\prime}^2)}{a_{\prime}^2} b_{3,4} - \frac{9}{2}\frac{a}{a_{\prime}} b_{3,3}$$

$$-\frac{a}{2a_{\prime}} b_{5,2} = \frac{(a^2+a_{\prime}^2)}{a_{\prime}^2} b_{5,1} - \frac{5a}{a_{\prime}} b_{5,0} \qquad \frac{a}{2a_{\prime}} b_{5,3} = \frac{2(a^2+a_{\prime}^2)}{a_{\prime}^2} b_{5,2} - \frac{7}{2}\frac{a}{a_{\prime}} b_{5,1}$$

$$\frac{3}{2}\frac{a}{a_{\prime}} b_{5,4} = \frac{3(a^2+a_{\prime}^2)}{a_{\prime}^2} b_{5,3} - \frac{9}{2}\frac{a}{a_{\prime}} b'_{5\,2} \qquad \frac{5}{2a_{\prime}} b_{5,5} = \frac{4(a^2+a_{\prime}^2)}{a_{\prime}^2} b_{5,4} - \frac{11}{2}\frac{a}{a_{\prime}} b_{5,3}$$

$$b_{1,0} = \frac{a^2+a_{\prime}^2}{a_{\prime}^2} b_{3,0} - \frac{a}{a_{\prime}} b_{3,1} \qquad b_{1,1} = \frac{a^2+a_{\prime}^2}{a_{\prime}^2} b_{3,1} - \frac{2a}{a_{\prime}} b_{3,0} - \frac{a}{a_{\prime}} b_3$$

$$b_{1,2} = \frac{a^2 + a_i^2}{a_i^2} b_{3,2} - \frac{a}{a_i} b_{3,1} - \frac{a}{a_i} b_{3,3} \qquad b_{1,3} = \frac{a^2 + a_i^2}{a_i^2} b_{3,3} - \frac{a}{a_i} b_{3,2} - \frac{a}{a_i} b_{3,4}$$

$$b_{3,0} = \frac{a^2 + a_i^2}{a_i^2} b_{5,0} - \frac{a}{a_i} b_{5,1} \qquad b_{3,1} = \frac{a^2 + a_i^2}{a_i^2} b_{5,1} - \frac{2a}{a_i} b_{5,0} - \frac{a}{a_i} b_{5,2}$$

$$b_{3,2} = \frac{a^2 + a_i^2}{a_i^2} b_{5,2} - \frac{a}{a_i} b_{5,1} - \frac{a}{a_i} b_{5,3} \qquad b_{3,3} = \frac{a^2 + a_i^2}{a_i^2} b_{5,3} - \frac{a}{a_i} b_{5,2} - \frac{a}{a_i} b_{5,4}$$

$$b_{1,1} = \frac{a}{a_i} \{b_{3,0} - \tfrac{1}{2} b_{3,2}\} \qquad 2\, b_{1,2} = \frac{a}{2\,a_i} \{b_{3,1} - b_{3,3}\}$$

$$3\, b_{1,3} = \frac{a}{2\,a_i} \{b_{3,2} - b_{3,4}\} \qquad 4\, b_{1,4} = \frac{a}{2\,a_i} \{b_{3,2} - b_{3,4}\}$$

$$b_{3,1} = 3 \frac{a}{a_i} \{b_{5,0} - \tfrac{1}{2} b_{5,2}\} \qquad 2\, b_{3,2} = \frac{3}{2} \frac{a}{a_i} \{b_{5,1} - b_{5,3}\}$$

$$3\, b_{3,3} = \frac{3}{2} \frac{a}{a_i} \{b_{5,2} - b_{5,4}\} \qquad 4\, b_{3,4} = \frac{3}{2} \frac{a}{a_i} \{b_{5,3} - b_{5,5}\}$$

$$2\, b_{3,0} = \frac{2\left(1 + \frac{a^2}{a_i^2}\right) b_{1,0} - 2 \frac{a}{a_i} b_{1,1}}{\left(1 - \frac{a^2}{a_i^2}\right)^2}, \qquad b_{3,1} = \frac{3\left(1 + \frac{a^2}{a_i^2}\right) b_{1,1} - 2\,.\,3 \frac{a}{a_i} b_{1,2}}{\left(1 - \frac{a^2}{a_i^2}\right)^2}$$

$$b_{3,2} = \frac{5\left(1 + \frac{a^2}{a_i^2}\right) b_{1,2} - 2\,.\,5 \frac{a}{a_i} b_{1,3}}{\left(1 - \frac{a^2}{a_i^2}\right)^2}, \qquad b_{3,3} = \frac{7\left(1 + \frac{a^2}{a_i^2}\right) b_{1,3} - 2\,.\,7 \frac{a}{a_i} b_{1,4}}{\left(1 - \frac{a^2}{a_i^2}\right)^2}$$

$$2\, b_{5,0} = \frac{2\left(1 + \frac{a^2}{a_i^2}\right) b_{3,0} + \frac{2}{3} \frac{a}{a_i} b_{3,1}}{\left(1 - \frac{a^2}{a_i}\right)^2}, \qquad b_{5,1} = \frac{\frac{5}{3}\left(1 + \frac{a^2}{a_i^2}\right) b_{3,1} - \frac{2}{3} \frac{a}{a_i} b_{3,2}}{\left(1 - \frac{a^2}{a^2_i}\right)^2}$$

$$\tfrac{1}{2}\, b_{5,2} = \frac{\frac{7}{3}\left(1 + \frac{a^2}{a_i^2}\right) b_{3,1} - 2 \frac{a}{a_i} b_{3,3}}{\left(1 - \frac{a^2}{a_i^2}\right)^2}, \qquad b_{5,3} = \frac{\frac{9}{3}\left(1 + \frac{a^2}{a_i^2}\right) b_{3,3} - \frac{2\,.\,5}{3} \frac{a}{a_i} b_{3,4}}{\left(1 - \frac{a^2}{a_i^2}\right)^2}$$

$$2\, b_{7,0} = \frac{2\left(1 + \frac{a^2}{a_i^2}\right) b_{5,0} + \frac{2\,.\,3}{5} \frac{a}{a_i} b_{5,1}}{\left(1 - \frac{a^2}{a_i^2}\right)^2}, \qquad b_{7,1} = \frac{\frac{7}{5}\left(1 + \frac{a^2}{a_i^2}\right) b_{5,1} + \frac{2}{5} \frac{a}{a_i} b_{5,2}}{\left(1 - \frac{a^2}{a_i^2}\right)^2}$$

$$b_{7,2} = \frac{\frac{9}{5}\left(1 + \frac{a^2}{a_i^2}\right) b_{5,2} - \frac{2}{5}\frac{a}{a_i} b_{5,3}}{\left(1 - \frac{a^2}{a_i^2}\right)^2}, \qquad b_{7,3} = \frac{\frac{11}{5}\left(1 + \frac{a^2}{a_i^2}\right) b_{5,3} - \frac{2 \cdot 3}{5}\frac{a}{a_i} b_{5,4}}{\left(1 - \frac{a^2}{a_i^2}\right)^2}$$

$$r = a F(n t), \qquad \left(\frac{\mathrm{d} R}{\mathrm{d} r}\right) = \frac{\mathrm{d} R \,\mathrm{d} a}{\mathrm{d} a \,\mathrm{d} a F(n t)}, \qquad \frac{\mathrm{d} a F n t}{\mathrm{d} a} = F n t, \qquad \frac{r \,\mathrm{d} R}{\mathrm{d} r} = a \left(\frac{\mathrm{d} R}{\mathrm{d} a}\right),$$

$$\{a^2 - 2 a a_i \cos\theta + a_i^2\}^{-\frac{1}{2}} = \frac{1}{a_i}\{b_{1,0} + b_{1,1}\cos\theta + b_{1,2}\cos 2\theta + \&c.\}$$

$$\frac{-(a - a_i\cos\theta)}{\{a^2 - 2 a a_i\cos\theta + a_i^2\}^{\frac{3}{2}}} = \frac{1}{a_i}\left\{\frac{\mathrm{d}\,.\,b_{1,0}}{\mathrm{d} a} + \frac{\mathrm{d} b_{1,1}}{\mathrm{d} a}\cos\theta + \frac{\mathrm{d}\,.\,b_{1,2}}{\mathrm{d} a}\cos 2\theta + \&c.\right\}$$

$$\frac{-\{a - a_i\cos\theta\}}{a_i^3}\{b_{3,0} + b_{3,1}\cos\theta + b_{3,2}\cos 2\theta + \&c.\} = \frac{1}{a_i}\left\{\frac{\mathrm{d}\,.\,b_{1,0}}{\mathrm{d} a}\right.$$

$$\left. + \frac{\mathrm{d}\,.\,b_{1,1}}{\mathrm{d} a}\cos\theta + \frac{\mathrm{d}\,.\,b_{1,2}}{\mathrm{d} a}\cos 2\theta + \&c.\right\}$$

whence

$$\frac{a\,\mathrm{d}\,.\,b_{1,0}}{\mathrm{d} a} = -\frac{a}{a_i}\left\{\frac{a}{a_i} b_{3,0} - \frac{1}{2} b_{3,1}\right\}$$

$$\frac{a\,\mathrm{d}\,.\,b_{1,1}}{\mathrm{d} a} = -\frac{a}{a_i}\left\{\frac{a}{a_i} b_{3,1} - b_{3,0} - \frac{1}{2} b_{3,2}\right\}$$

$$\frac{a\,\mathrm{d}\,.\,b_{1,2}}{\mathrm{d} a} = -\frac{a}{a_i}\left\{\frac{a}{a_i} b_{3,2} - \frac{1}{2} b_{3,1} - \frac{1}{2} b_{3,3}\right\}$$

$$\frac{a\,\mathrm{d}\,.\,b_{1,3}}{\mathrm{d} a} = -\frac{a}{a_i}\left\{\frac{a}{a_i} b_{3,3} - \frac{1}{2} b_{3,2} - \frac{1}{2} b_{3,4}\right\}$$

similarly

$$\frac{a\,\mathrm{d}\,.\,b_{3,0}}{\mathrm{d} a} = -3\frac{a}{a_i}\left\{\frac{a}{a} b_{5,0} - \frac{1}{2} b_{5,1}\right\}$$

$$\frac{a\,\mathrm{d}\,.\,b_{3,1}}{\mathrm{d} a} = -3\frac{a}{a_i}\left\{\frac{a}{a_i} b_{5,1} - b_{5,0} - \frac{1}{2} b_{5,1}\right\}$$

$$\frac{a\,\mathrm{d}\,.\,b_{5,0}}{\mathrm{d} a} = -5\frac{a}{a_i}\left\{\frac{a}{a_i} b_{7,0} - \frac{1}{2} b_{7,1}\right\}$$

$$\frac{a\,\mathrm{d}\,.\,b_{5,1}}{\mathrm{d} a} = -5\frac{a}{a_i}\left\{\frac{a}{a_i} b_{7,1} - b_{7,0} - \frac{1}{2} b_{7,2}\right\}$$

$$\frac{a^2\,\mathrm{d}^2\,.\,b_{1,0}}{\mathrm{d}\,a^2} = \frac{a}{a_{\prime}}\left\{\frac{2\,a}{a_{\prime}}b_{3,0} - \frac{1}{2}\,b_{3,1}\right\}$$

$$\frac{a^2\,\mathrm{d}^2\,.\,b_{1,1}}{\mathrm{d}\,a^2} = \frac{a}{a_{\prime}}\left\{\frac{2\,a}{a_{\prime}}b_{3,1} - b_{3,2}\right\}$$

$$\frac{a^2\,\mathrm{d}^2\,.\,b_{1,2}}{\mathrm{d}\,a^2} = \frac{a}{a_{\prime}}\left\{\frac{2\,a}{a_{\prime}}b_{3,2} + \frac{1}{2}b_{3,1} - \frac{3}{2}b_{3,3}\right\}$$

$$\frac{a^2\,\mathrm{d}^2\,.\,b_{1,3}}{\mathrm{d}\,a^2} = \frac{a}{a_{\prime}}\left\{\frac{2\,a}{a_{\prime}}b_{3,3} + b_{3,2} - 2b_{3,4}\right\}$$

$$\frac{a^2\,\mathrm{d}^2\,.\,b_{1,4}}{\mathrm{d}\,a^2} = \frac{a}{a_{\prime}}\left\{\frac{2\,a}{a_{\prime}}b_{3,4} + \frac{3}{2}\,b_{3,3} - \frac{5}{2}b_{3,5}\right\}$$

The value of R given p. 349 of the former part of this paper is susceptible of much simplification. The first term of R for instance

$$= m_{\prime}\left\{-\frac{b_{1,0}}{a_{\prime}} + \frac{3\,(a^2\,e^2 + a_{\prime}^2\,e_{\prime}^2)}{2\,.\,2\,a_{\prime}^3}\,b_{3,0} + \frac{a\,a_{\prime}}{2\,a_{\prime}^3}\left(\sin^2\frac{\iota_{\prime}}{2} + \frac{(e^2 + e_{\prime}^2)}{2}\right)b_{3,1}\right.$$
$$- \frac{3}{2\,.\,4\,a_{\prime}^5}\left(2\,a^4\,e^2 + 5\,a^2\,a_{\prime}^2\,(e^2 + e_{\prime}^2) + 2\,a_{\prime}^4\,e_{\prime}^2\right)b_{5,0} + \frac{3\,.\,2}{2\,.\,4\,a_{\prime}^5}\,(a^2\,e^2 + a_{\prime}^2\,e_{\prime}^2)\,a\,a_{\prime}\,b_{5,1}$$
$$\left. + \frac{1\,.\,3\,.\,3}{2\,.\,4\,.\,2}\,\frac{a^2\,a_{\prime}^2}{a_{\prime}^5}\,(e^2 + e_{\prime}^2)\,b_{5,2}\right\}$$

$$= m_{\prime}\left\{-\frac{b_{1,0}}{a_{\prime}} + \frac{3\,(a^2\,e^2 + a_{\prime}^2\,e_{\prime}^2)}{2\,.\,2\,a_{\prime}^3}\,b_{3,0} + \frac{a\,a_{\prime}}{2\,a_{\prime}^3}\left(\sin^2\frac{\iota_{\prime}}{2} + \frac{e^2 + e_{\prime}^2}{2}\right)b_{3,1}\right.$$
$$- \frac{3\,.\,2}{2\,.\,4}\,\frac{(a^2 + a_{\prime}^2)}{a_{\prime}^2}\,\frac{(a^2\,e^2 + a_{\prime}^2\,e_{\prime}^2)}{a_{\prime}^3}\,b_{5,0} - \frac{3\,.\,3}{2\,.\,4}\,\frac{a^2\,a_{\prime}^2}{a_{\prime}^5}\,(e^2 + e_{\prime}^2)\,b_{5,0}$$
$$\left. + \frac{3\,.\,2}{2\,.\,4}\,\frac{(a^2\,e^2 + a_{\prime}^2\,e_{\prime}^2)}{a_{\prime}^3}\,\frac{a}{a_{\prime}}\,b_{5,1} + \frac{3\,.\,3}{2\,.\,4\,.\,2}\,\frac{a^2\,a_{\prime}^2}{a_{\prime}^5}\,(e^2 + e_{\prime}^2)\,b_{5,2}\right\}$$

and since

$$b_{3,0} = \frac{(a^2 + a_{\prime}^2)}{a_{\prime}^2}\,b_{5,0} - \frac{a}{a_{\prime}}\,b_{5,1} \quad \text{See p. 26}$$

$$b_{3,1} = \frac{3\,a}{a_{\prime}}\left\{b_{5,0} - \tfrac{1}{2}\,b_{5,2}\right\}$$

this term reduces itself to

$$m_{\prime}\left\{-\frac{b_{1,0}}{a_{\prime}} + \frac{a\,a_{\prime}}{2\,a_{\prime}^3}\left(\sin^2\frac{\iota_{\prime}}{2} + \frac{e^2 + e_{\prime}^2}{2}\right)b_{3,1} - \frac{3}{2\,.\,4}\,\frac{a\,a_{\prime}}{a_{\prime}^3}\,(e^2 + e_{\prime}^2)\,b_{3,1}\right\}$$

$$= m_{\prime}\left\{-\frac{b_{1,0}}{a_{\prime}} + \frac{a}{2\,a_{\prime}^2}\left(\sin^2\frac{\iota_{\prime}}{2} - \frac{e^2 + e_{\prime}^2}{4}\right)b_{3,1}\right\}$$

The succeeding terms admit of similar simplifications, so that

$$R = m_{\prime}\left\{-\frac{b_{1,0}}{a_{\prime}}+\frac{a}{2a_{\prime}^2}\left(\sin^2\frac{\iota_{\prime}}{2}+\frac{e^2+e_{\prime}^2}{2}\right)b_{3,1}-\frac{3}{2\,.\,4}\frac{a}{a_{\prime}^2}(e^2+e_{\prime}^2)\,b_{3,1}\right\}$$

$$+m_{\prime}\left\{\frac{a}{a_{\prime}^2}\left(\cos^2\frac{\iota_{\prime}}{2}-\frac{e^2+e_{\prime}^2}{2}\right)-\frac{b_{1,1}}{a_{\prime}}+\frac{a}{a_{\prime}^2}\left(\sin^2\frac{\iota_{\prime}}{2}+\frac{e^2+e_{\prime}^2}{2}\right)\left(b_{3,0}+\tfrac{1}{2}b_{3,2}\right)\right.$$

$$\left.-\frac{3\,.\,2}{2\,.\,4}\frac{a}{a_{\prime}^2}(e^2+e_{\prime}^2)b_{3,2}\right\}\cos(nt-n_{\prime}t+\varepsilon-\varepsilon_{\prime}) \qquad [1]$$

$$+m_{\prime}\left\{-\frac{b_{1,2}}{a_{\prime}}+\frac{a}{2a_{\prime}^2}\left(\sin^2\frac{\iota_{\prime}}{2}+\frac{e^2+e_{\prime}^2}{2}\right)(b_{3,1}+b_{3,3})+\frac{3}{2\,.\,4}\frac{a}{a_{\prime}^2}(e^2+e_{\prime}^2)b_{3,1}\right.$$

$$\left.-\frac{3\,.\,3}{2\,.\,4}\frac{a}{a_{\prime}^2}(e^2+e_{\prime}^2)\,b_{3,3}\right\}\cos(2nt-2n_{\prime}t+2\varepsilon-2\varepsilon_{\prime}) \qquad [2]$$

$$+m_{\prime}\left\{-\frac{b_{1,3}}{a_{\prime}}+\frac{a}{2a_{\prime}^2}\left(\sin^2\frac{\iota_{\prime}}{2}+\frac{e^2+e_{\prime}^2}{2}\right)(b_{3,2}+b_{3,4})+\frac{3\,.\,2}{3\,.\,4}\frac{a}{a_{\prime}^2}(e^2+e_{\prime}^2)\,b_{3,2}\right.$$

$$\left.-\frac{3\,.\,4}{2\,.\,4}\frac{a}{a_{\prime}^2}(e^2+e_{\prime}^2)\,b_{3,4}\right\}\cos(3nt-3n_{\prime}t-3\varepsilon-3\varepsilon_{\prime}) \qquad [3]$$

$$+m_{\prime}\left\{-\frac{b_{1,4}}{a_{\prime}}+\frac{a}{2a_{\prime}^2}\left(\sin^2\frac{\iota_{\prime}}{2}+\frac{e^2+e_{\prime}^2}{2}\right)(b_{3,3}+b_{3,5})+\frac{3\,.\,3}{2\,.\,4}\frac{a}{a_{\prime}^2}(e^2+e_{\prime}^2)b_{3,3}\right.$$

$$\left.-\frac{3\,.\,5}{2\,.\,4}\frac{a}{a_{\prime}^2}(e^2+e_{\prime}^2)\,b_{3,5}\right\}\cos(4nt-4n_{\prime}t+4\varepsilon-4\varepsilon_{\prime}) \qquad [4]$$

The coefficient of $\cos(\varpi-\varpi_{\prime})$, Argument 41,

$$=m_{\prime}\left\{-\frac{9}{4}\frac{a}{a_{\prime}^3}b_{3,0}-\frac{a}{8\,a_{\prime}^2}b_{3,2}+\frac{3\,.\,6}{2\,.\,4}\frac{(a^2+a_{\prime}^2)}{a_{\prime}^5}a\,a_{\prime}\,b_{5,0}-\frac{3\,.\,7}{2\,.\,4\,.\,2}\frac{a^2a_{\prime}^2}{a_{\prime}^5}b_{5,1}\right.$$

$$\left.-\frac{3}{2\,.\,4}\frac{(a^2+a_{\prime}^2)}{a_{\prime}^5}a\,a_{\prime}\,b_{5,2}-\frac{3}{2\,.\,4\,.\,2}\frac{a^2a_{\prime}^2}{a_{\prime}^5}b_{5,3}\right\}e\,e_{\prime}$$

$$=m_{\prime}\left\{-\frac{9}{4}\frac{a}{a_{\prime}^2}b_{3,0}-\frac{a}{8\,a_{\prime}^2}b_{3,2}+\frac{3\,.\,6}{2\,.\,4}\frac{a}{a_{\prime}^2}\left\{\frac{(a^2+a_{\prime}^2)}{a_{\prime}^2}b_{5,0}-\frac{a}{a_{\prime}}b_{5,1}\right\}\right.$$

$$\left.-\frac{3}{8}\frac{a}{a_{\prime}^2}\left\{\frac{(a^2+a_{\prime}^2)}{a_{\prime}^2}b_{5,2}-\frac{a}{a_{\prime}}b_{5,1}-\frac{a}{a_{\prime}}b_{5,3}\right\}+\frac{9}{16}\frac{a}{a_{\prime}^3}\left\{b_{5,1}-b_{5,3}\right\}\right\}$$

$$=m_{\prime}\left\{-\frac{9}{4}\frac{a}{a_{\prime}^2}b_{3,0}-\frac{a}{8\,a_{\prime}^2}b_{3,2}+\frac{9}{4}\frac{a}{a_{\prime}^2}b_{3,0}-\frac{3}{8}\frac{a}{a_{\prime}^2}b_{3,2}+\frac{2\,.\,3}{8}\frac{a}{a_{\prime}^2}b_{3,2}\right\}e\,e_{\prime}$$

$$=m_{\prime}\frac{a}{4\,a_{\prime}^2}b_{3,2}$$

So that the part of R which is independent of $n\,t$, $n_{\prime}\,t$

$$=m_{\prime}\left\{-\frac{b_{1,0}}{a_{\prime}^2}+\frac{a}{2\,a_{\prime}^2}\left(\sin^2\frac{\iota_{\prime}}{2}-\frac{e^2+e_{\prime}^2}{4}\right)b_{3,1}+\frac{a}{4\,a_{\prime}^2}b_{3,2}\,e\,e_{\prime}\cos(\varpi-\varpi_{\prime})\right\}$$

$$=m_{\prime}\left\{-\frac{b_{1,0}}{a_{\prime}}-\frac{a}{2\,a_{\prime}^2}\left(\sin^2\frac{\iota_{\prime}}{2}-\frac{e^2+e_{\prime}^2}{4}\right)b_{3,1}-\left\{\frac{3\,a}{2\,a_{\prime}^2}b_{3,0}-\frac{a^2+a_{\prime}^2}{2\,a_{\prime}^3}b_{3,1}\right\}e\,e_{\prime}\cos(\varpi-\varpi_{\prime})\right\}$$

In the general case, when ι_1, ι_2 are the inclinations of the orbits of the planets P and $P_{,}$ to any plane, the direction of which is arbitrary,

$$\cos \iota_{,} = \cos \iota_1 \cos \iota_2 + \sin \iota_1 \sin \iota_2 \cos (\nu_1 - \nu_2)$$

$$\sin^2 \frac{\iota_{,}}{2} = \frac{1 - \cos \iota_1 \cos \iota_2 - \sin \iota_1 \sin \iota_2 \cos (\nu_1 - \nu_2)}{2}$$

The part of R which is independent of $n\,t$, $n_{,}t$

$$= m_{,}\left\{ -\frac{b_{1,0}}{a_{,}} + \frac{a}{4\,a_{,}^2}\left\{1 - \cos \iota_1 \cos \iota_2 - \sin \iota_1 \sin \iota_2 \cos (\nu_1 - \nu_2) - \frac{e^2 + e_{,}^2}{2}\right\} b_{3,1} \right.$$
$$\left. - \left\{\frac{3\,a}{2\,a_{,}^2} b_{3,0} - \frac{(a^2 + a_{,}^2)}{2\,a_{,}^3} b_{3,1}\right\} e\,e_{,} \cos (\varpi - \varpi_{,}) \right\}$$

$$\cos \iota = \frac{1}{\sqrt{1 + \tan^2 \iota}} = 1 - \tfrac{1}{2}\tan^2 \iota, \qquad \sin \iota = \frac{\tan \iota}{\sqrt{1 + \tan^2 \iota}} = \tan \iota \text{ nearly.}$$

$$= m_{,}\left\{ -\frac{b_{1,0}}{a_{,}} + \frac{a}{8\,a_{,}^2}\left\{(\tan^2 \iota_1 + \tan^2 \iota_2 - 2 \tan \iota_1 \tan \iota_2 \cos (\nu_1 - \nu_2) - e^2 - e_{,}^2\right\} b_{3,1} \right.$$
$$\left. - \left\{\frac{3\,a}{2\,a_{,}^2} b_{3,0} - \frac{(a^2 + a_{,}^2)}{2\,a_{,}^3} b_{3,1}\right\} e\,e_{,} \cos (\varpi - \varpi_{,}) \right\}$$

$$= m_{,}\left\{ -\frac{b_{1,0}}{a_{,}} + \frac{a}{8\,a_{,}^2}\left\{(\tan \iota_1 \cos \nu_1 - \tan \iota_2 \cos \nu_2)^2 + (\tan \iota_1 \sin \nu_1 - \tan \iota_2 \sin \iota_2)^2 - e^2 - e_{,}^2\right\} b_{3,1} \right.$$
$$\left. - \left\{\frac{3\,a}{2\,a_{,}^2} b_{3,0} - \frac{(a^2 + a_{,}^2)}{2\,a_{,}^3} b_{3,1}\right\} e\,e_{,} \cos (\varpi - \varpi_{,}) \right\}$$

and if $\tan \iota \sin \nu = p$, $\tan \iota \cos \nu = q$, this quantity

$$= m_{,}\left\{ -\frac{b_{1,0}}{a_{,}} + \frac{a}{8\,a_{,}^2}\left\{(p_1 - p_2)^2 + (q_1 - q_2)^2 - e^2 - e_{,}^2\right\} b_{3,1} \right.$$
$$\left. - \left\{\frac{3\,a}{2\,a_{,}^2} b_{3,0} - \frac{a^2 + a_{,}^2}{2\,a_{,}^3} b_{3,1}\right\} e\,e_{,} \cos (\varpi - \varpi_{,}) \right\}.$$

which evidently agrees with the result given by M. de Pontécoulant, Théor. Anal. vol. i. p. 363. All the other coefficients of terms multiplied by the squares and products of the eccentricities are susceptible of reductions similar to those in the two preceding pages, and finally;

$$R = m_{,}\left\{ -\frac{b_{1,0}}{a_{,}} + \frac{a}{2\,a_{,}^2}\left(\sin^2 \frac{\iota_{,}}{2} - \frac{e^2 + e_{,}^2}{4}\right) b_{3,1} \right\} \qquad [0]$$

$$+ m_{,}\left\{ -\frac{a}{a_{,}^2}\left(\cos^2 \frac{\iota_{,}}{2} - \frac{e^2 + e_{,}^2}{2}\right) - \frac{b_{1,1}}{a_{,}} + \frac{a}{a_{,}^2} \sin^2 \frac{\iota_{,}}{2} (b_{3,0} + \tfrac{1}{2} b_{3,2}) \right.$$
$$\left. + \frac{a}{a_{,}^2} \frac{(e^2 + e_{,}^2)}{8} (4\,b_{3,0} - 4\,b_{3,2}) \right\} \cos (n\,t - n_{,}t + \varepsilon - \varepsilon_{,}) \qquad [1]$$

Development of R.

$$+ m_{\prime} \left\{ - \frac{b_{1,2}}{a_{\prime}} + \frac{a}{2 a_{\prime}^2} \sin^2 \frac{\iota_{\prime}}{2} (b_{3,1} + b_{3,3}) + \frac{a}{a_{\prime}^2} \frac{(e^2 + e_{\prime}^2)}{8} (5 b_{3,1} - 7 b_{3,3}) \right\} \cos (2 n t - 2 n_{\prime} t + 2 \varepsilon - 2 \varepsilon_{\prime}) \qquad [2]$$

$$+ m_{\prime} \left\{ - \frac{b_{1,3}}{a_{\prime}} + \frac{a}{2 a_{\prime}^2} \sin^2 \frac{\iota_{\prime}}{2} (b_{3,2} + b_{3,4}) + \frac{a}{a_{\prime}^2} \frac{(e^2 + e_{\prime}^2)}{8} (8 b_{3,2} - 10 b_{3,4}) \right\} \cos (3 n t - 3 n_{\prime} t + 3 \varepsilon - 3 \varepsilon_{\prime}) \qquad [3]$$

$$+ m_{\prime} \left\{ - \frac{b_{1,4}}{a_{\prime}} + \frac{a}{2 a_{\prime}^2} \sin^2 \frac{\iota_{\prime}}{2} (b_{3,3} + b_{3,5}) + \frac{a}{a_{\prime}^2} \frac{(e^2 + e_{\prime}^2)}{8} (11 b_{3,3} - 13 b_{3,5}) \right\} \cos (4 n t - 4 n_{\prime} t + 4 \varepsilon - 4 \varepsilon_{\prime}) \qquad [4]$$

$$+ m_{\prime} \left\{ - \frac{b_{1,5}}{a_{\prime}} + \frac{a}{2 a_{\prime}^2} \sin^2 \frac{\iota_{\prime}}{2} (b_{3,4} + b_{3,6}) + \frac{a}{a_{\prime}^2} \frac{(e^2 + e_{\prime}^2)}{8} (14 b_{3,4} - 16 b_{3,6}) \right\} \cos (5 n t_{\prime} - 5 n_{\prime} t + 5 \varepsilon - 5 \varepsilon_{\prime}) \qquad [5]$$

$$+ m_{\prime} \left\{ - \frac{3 a}{2 a_{\prime}^3} + \frac{3 a}{2 a_{\prime}^2} b_{3,0} - \frac{a^2}{2 a_{\prime}^3} b_{3,1} - \frac{a}{4 a_{\prime}^2} b_{3,2} \right\} e \cos (n_{\prime} t + \varepsilon - \varpi) \qquad [6] \; [16]^*$$

$$+ m_{\prime} \left\{ - \frac{a^2}{a_{\prime}^3} b_{3,0} + \frac{a}{2 a_{\prime}^2} b_{3,1} \right\} e \cos (n t + \varepsilon - \varpi) \qquad [7] \; [15]$$

$$+ m_{\prime} \left\{ \frac{a}{2 a_{\prime}^2} - \frac{a}{2 a_{\prime}^2} b_{3,0} - \frac{a^2}{2 a_{\prime}^3} b_{3,1} + \frac{3}{4} \frac{a}{a_{\prime}^2} b_{3,2} \right\} e \cos (2 n t - n_{\prime} t + 2 \varepsilon - \varepsilon_{\prime} - \varpi) \qquad [8] \; [20]$$

$$+ m_{\prime} \left\{ - \frac{a}{4 a_{\prime}^2} b_{3,1} - \frac{a^2}{2 a_{\prime}^2} b_{3,2} + \frac{3}{4} \frac{a}{a_{\prime}^2} b_{3,3} \right\} e \cos (3 n t - 2 n_{\prime} t + 3 \varepsilon - 2 \varepsilon_{\prime} - \varpi) \qquad [9] \; [21]$$

$$+ m_{\prime} \left\{ - \frac{a}{4 a_{\prime}^2} b_{3,2} - \frac{a^2}{2 a_{\prime}^3} b_{3,3} + \frac{3}{4} \frac{a}{a_{\prime}^2} b_{3,4} \right\} e \cos (4 n t - 3 n_{\prime} t + 4 \varepsilon - 3 \varepsilon_{\prime} - \varpi) \qquad [10] \; [22]$$

$$+ m_{\prime} \left\{ - \frac{a}{4 a_{\prime}^2} b_{3,3} - \frac{a}{2 a_{\prime}^3} b_{3,4} + \frac{3}{4} \frac{a}{a_{\prime}^2} b_{3,5} \right\} e \cos (5 n t - 4 n_{\prime} t + 5 \varepsilon - 4 \varepsilon_{\prime} - \varpi) \qquad [11] \; [23]$$

$$+ m_{\prime} \left\{ \frac{3 a}{4 a_{\prime}^2} b_{3,1} - \frac{a^2}{2 a_{\prime}^3} b_{3,2} - \frac{a}{4 a_{\prime}^2} b_{3,3} \right\} e \cos (n t - 2 n_{\prime} t + \varepsilon - 2 \varepsilon_{\prime} + \varpi) \qquad [12] \; [17]$$

$$+ m_{\prime} \left\{ \frac{3 a}{4 a_{\prime}^2} b_{3,2} - \frac{a^2}{2 a_{\prime}^2} b_{3,3} - \frac{a}{4 a_{\prime}^2} b_{3,4} \right\} e \cos (2 n t - 3 n_{\prime} t + 2 \varepsilon - 3 \varepsilon_{\prime} + \varpi) \qquad [13] \; [18]$$

$$+ m_{\prime} \left\{ \frac{3 a}{4 a_{\prime}^2} b_{3,3} - \frac{a^2}{2 a_{\prime}^3} b_{3,4} - \frac{a}{4 a_{\prime}^2} b_{3,5} \right\} e_{\prime} \cos (3 n t - 4 n_{\prime} t + 3 \varepsilon - 4 \varepsilon_{\prime} + \varpi) \qquad [14] \; [19]$$

$$+ m_{\prime} \left\{ - \frac{a_{\prime}^2}{a_{\prime}^3} b_{3,0} + \frac{a}{2 a_{\prime}^2} b_{3,1} \right\} e_{\prime} \cos (n_{\prime} t + \varepsilon_{\prime} - \varpi_{\prime}) \qquad [15] \; [7]$$

* These numbers indicate the arguments which are symmetrical with regard to $n t$ and $n_{\prime} t$.

Development of R.

$$+ m_{,}\left\{\frac{3a}{2a_{,}^{2}}b_{3,0}-\frac{a_{,}^{2}}{2a_{,}^{3}}b_{3,1}-\frac{a}{4a_{,}^{2}}b_{3,2}\right\}e_{,}\cos(nt+\varepsilon-\varpi_{,}) \qquad [16] \quad [6]$$

$$+ m_{,}\left\{\frac{3a}{4a_{,}^{2}}b_{3,1}-\frac{a_{,}^{2}}{2a_{,}^{3}}b_{3,2}-\frac{a}{4a_{,}^{2}}b_{3,3}\right\}e_{,}\cos(2nt-n_{,}t+2\varepsilon-\varepsilon_{,}-\varpi_{,}) \qquad [17] \quad [12]$$

$$+ m_{,}\left\{\frac{3a}{4a_{,}^{2}}b_{3,2}-\frac{a_{,}^{2}}{2a_{,}^{3}}b_{3,3}-\frac{a}{4a_{,}^{2}}b_{3,4}\right\}e_{,}\cos(3nt-2n_{,}t+3\varepsilon-2\varepsilon_{,}-\varpi_{,}) \qquad [18] \quad [13]$$

$$+ m_{,}\left\{\frac{3a}{4a_{,}^{2}}b_{3,3}-\frac{a_{,}^{2}}{2a_{,}^{3}}b_{3,4}-\frac{a}{4a_{,}^{2}}b_{3,5}\right\}e_{,}\cos(4nt-3n_{,}t+4\varepsilon-3\varepsilon_{,}-\varpi_{,}) \qquad [19] \quad [14]$$

$$+ m_{,}\left\{\frac{a}{2a_{,}^{2}}-\frac{a}{2a_{,}^{2}}b_{3,0}-\frac{a_{,}^{2}}{2a^{3}}b_{3,1}+\frac{3a}{4a_{,}^{2}}b_{3,2}\right\}e_{,}\cos(nt-2n_{,}t+\varepsilon-2\varepsilon_{,}+\varpi_{,}) \qquad [20] \quad [8]$$

$$+ m_{,}\left\{-\frac{a}{4a_{,}^{2}}b_{3,1}-\frac{a_{,}^{2}}{2a_{,}^{3}}b_{3,2}+\frac{3a}{4a_{,}^{2}}b_{3,3}\right\}e_{,}\cos(2nt-3n_{,}t+2\varepsilon-3\varepsilon_{,}+\varpi_{,}) \qquad [21] \quad [9]$$

$$+ m_{,}\left\{-\frac{a}{4a_{,}^{2}}b_{3,2}-\frac{a_{,}^{2}}{2a_{,}^{3}}b_{3,3}+\frac{3a}{4a_{,}^{2}}b_{3,4}\right\}e_{,}\cos(3nt-4n_{,}t+3\varepsilon-4\varepsilon_{,}+\varpi_{,}) \qquad [22] \quad [10]$$

$$+ m_{,}\left\{-\frac{a}{4a_{,}^{2}}b_{3,3}-\frac{a_{,}^{2}}{2a_{,}^{3}}b_{3,4}+\frac{3a}{4a_{,}^{2}}b_{3,5}\right\}e_{,}\cos(4nt-5n_{,}t+4\varepsilon-5\varepsilon_{,}+\varpi_{,}) \qquad [23] \quad [11]$$

$$+ m_{,}\left\{-\frac{5}{8}\frac{a}{a_{,}^{2}}b_{3,1}+\frac{a^{2}}{2a_{,}^{3}}b_{3,2}\right\}e^{2}\cos(2n_{,}t+2\varepsilon_{,}-2\varpi) \qquad [24] \quad [59]$$

$$+ m_{,}\left\{\frac{a}{8a_{,}^{2}}-\frac{a}{8a_{,}^{2}}b_{3,0}-\frac{a}{16a_{,}^{2}}b_{3,2}\right\}e^{2}\cos(nt+n_{,}t+\varepsilon+\varepsilon_{,}-2\varpi) \qquad [25] \quad [58]$$

$$+ m_{,}\left\{-\frac{a^{2}}{a_{,}^{3}}b_{3,0}+\frac{3}{8}\frac{a}{a_{,}^{2}}b_{3,1}\right\}e^{2}\cos(2nt+2\varepsilon-2\varpi) \qquad [26] \quad [57]$$

$$+ m_{,}\left\{\frac{3}{8}\frac{a}{a_{,}^{2}}-\frac{3}{8}\frac{a}{a_{,}^{2}}b_{3,0}-\frac{a^{2}}{a_{,}^{3}}b_{3,1}+\frac{17}{16}\frac{a}{a_{,}^{2}}b_{3,2}\right\}e^{2}\cos(3nt-n_{,}t+3\varepsilon-\varepsilon_{,}-2\varpi) \qquad [27] \quad [63]$$

$$+ m_{,}\left\{-\frac{a}{4a_{,}^{2}}b_{3,1}-\frac{3}{2}\frac{a^{2}}{a_{,}^{3}}b_{3,2}+\frac{13}{8}\frac{a}{a_{,}^{2}}b_{3,3}\right\}e^{2}\cos(4nt-2n_{,}t+4\varepsilon-2\varepsilon_{,}-2\varpi) \qquad [28] \quad [64]$$

$$+ m_{,}\left\{-\frac{5}{16}\frac{a}{a_{,}^{2}}b_{3,2}-2\frac{a^{2}}{a_{,}^{3}}b_{3,3}+\frac{35}{16}\frac{a}{a_{,}^{2}}b_{3,4}\right\}e^{2}\cos(5nt-3n_{,}t+5\varepsilon-3\varepsilon_{,}-2\varpi) \qquad [29] \quad [65]$$

$$+ m_{,}\left\{-\frac{3}{8}\frac{a}{a_{,}^{2}}b_{3,3}-\frac{5}{2}\frac{a^{2}}{a_{,}^{3}}b_{3,4}+\frac{11}{4}\frac{a}{a_{,}^{2}}b_{3,5}\right\}e^{2}\cos(6nt-4n_{,}t+6\varepsilon-4\varepsilon_{,}-2\varpi) \qquad [30] \quad [66]$$

$$+ m_{,}\left\{-\frac{7}{16}\frac{a}{a_{,}^{2}}b_{3,4}-3\frac{a^{2}}{a_{,}^{3}}b_{3,5}+\frac{53}{16}\frac{a}{a_{,}^{2}}b_{3,6}\right\}e^{2}\cos(7nt-5n_{,}t+7\varepsilon-5\varepsilon_{,}-2\varpi) \qquad [31] \quad [67]$$

$$+ m_{,}\left\{-\frac{19}{16}\frac{a}{a_{,}^{2}}b_{3,2}+\frac{a^{2}}{a_{,}^{3}}b_{3,3}+\frac{a}{16a_{,}^{2}}b_{3,4}\right\}e^{2}\cos(nt-3n_{,}t+\varepsilon-3\varepsilon_{,}+2\varpi) \qquad [32] \quad [60]$$

$$+ m_{,}\left\{-\frac{7}{4}\frac{a}{a_{,}^{2}}b_{3,3}+\frac{3}{2}\frac{a^{2}}{a_{,}^{3}}b_{3,4}+\frac{1}{8}\frac{a}{a_{,}^{2}}b_{3,5}\right\}e^{2}\cos(2nt-4n_{,}t+2\varepsilon-4\varepsilon_{,}+2\varpi) \qquad [33] \quad [61]$$

Development of R.

$$+ m_{,}\left\{-\frac{37}{16}\frac{a}{a_{,}^{2}}b_{3,4} + 2\,\frac{a^{2}}{a_{,}^{3}}b_{3,5} + \frac{3}{16}\frac{a}{a_{,}^{2}}b_{3,6}\right\} e^{2}\cos\left(3nt - 5n_{,}t + 3\varepsilon - 5\varepsilon_{,} + 2\varpi\right) \qquad [34]\ [62]$$

$$+ m_{,}\frac{a}{4a_{,}^{2}}b_{3,1}\,ee_{,}\cos\left(nt - n_{,}t + \varepsilon - \varepsilon_{,} - \varpi + \varpi_{,}\right) \qquad [35]$$

$$+ m_{,}\left\{\frac{a}{a_{,}^{2}} - \frac{a}{a_{,}^{2}}b_{3,0} + \frac{3}{4}\frac{a}{a_{,}^{2}}b_{3,2}\right\} ee_{,}\cos\left(2nt - 2n_{,}t + 2\varepsilon - 2\varepsilon_{,} - \varpi + \varpi_{,}\right) \qquad [36]$$

$$- m_{,}\left\{\frac{7}{8}\frac{a}{a_{,}^{2}}b_{3,1} - \frac{9}{8}\frac{a}{a_{,}^{2}}b_{3,3}\right\} ee_{,}\cos\left(3nt - 3n_{,}t + 3\varepsilon - 3\varepsilon_{,} - \varpi + \varpi_{,}\right) \qquad [37]$$

$$- m_{,}\left\{\frac{5}{4}\frac{a}{a_{,}^{2}}b_{3,2} - \frac{3}{2}\frac{a}{a_{,}^{2}}b_{3,4}\right\} ee_{,}\cos\left(4nt - 4n_{,}t + 4\varepsilon - 4\varepsilon_{,} - \varpi + \varpi_{,}\right) \qquad [38]$$

$$- m_{,}\left\{\frac{13}{8}\frac{a}{a_{,}^{2}}b_{3,3} - \frac{15}{8}\frac{a}{a_{,}^{2}}b_{3,5}\right\} ee_{,}\cos\left(5nt - 5n_{,}t + 5\varepsilon - 5\varepsilon_{,} - \varpi + \varpi_{,}\right) \qquad [39]$$

$$- m_{,}\left\{2\,\frac{a}{a_{,}^{2}}b_{3,4} - \frac{9}{4}\frac{a}{a_{,}^{2}}b_{3,6}\right\} ee_{,}\cos\left(6nt - 6n_{,}t + 6\varepsilon - 6\varepsilon_{,} - \varpi - \varpi_{,}\right) \qquad [40]$$

$$+ m_{,}\frac{a}{4a_{,}^{2}}b_{3,2}\,ee_{,}\cos\left(\varpi - \varpi_{,}\right) \qquad [41]$$

$$- m_{,}\left\{\frac{3}{8}\frac{a}{a_{,}^{2}}b_{3,1} - \frac{5}{8}\frac{a}{a_{,}^{2}}b_{3,3}\right\} ee_{,}\cos\left(nt - n_{,}t + \varepsilon - \varepsilon_{,} + \varpi - \varpi_{,}\right) \qquad [42]$$

$$- m_{,}\left\{\frac{3}{4}\frac{a}{a_{,}^{2}}b_{3,2} - \frac{a}{a_{,}^{2}}b_{3,4}\right\} ee_{,}\cos\left(2nt - 2n_{,}t + 2\varepsilon - 2\varepsilon_{,} + \varpi - \varpi_{,}\right) \qquad [43]$$

$$- m_{,}\left\{\frac{9}{8}\frac{a}{a_{,}^{2}}b_{3,3} - \frac{11}{8}\frac{a}{a_{,}^{2}}b_{3,5}\right\} ee_{,}\cos\left(3nt - 3n_{,}t + 3\varepsilon - 3\varepsilon_{,} + \varpi - \varpi_{,}\right) \qquad [44]$$

$$- m_{,}\left\{\frac{3}{2}\frac{a}{a_{,}^{2}}b_{3,4} - \frac{7}{4}\frac{a}{a_{,}^{2}}b_{3,6}\right\} ee_{,}\cos\left(4nt - 4n_{,}t + 4\varepsilon - 4\varepsilon_{,} + \varpi - \varpi_{,}\right) \qquad [45]$$

$$+ m'\left\{-\frac{3a}{a_{,}^{2}} + \frac{1}{a_{,}}b_{3,1} - \frac{3}{4}\frac{a}{a_{,}^{2}}b_{3,2}\right\} ee_{,}\cos\left(2n_{,}t + 2\varepsilon_{,} - \varpi - \varpi_{,}\right) \qquad [46]\ [48]$$

$$+ m_{,}\frac{a}{4a_{,}^{2}}b_{3,1}\,ee_{,}\cos\left(nt + n_{,}t + \varepsilon + \varepsilon_{,} - \varpi - \varpi_{,}\right) \qquad [47]$$

$$+ m_{,}\left\{\frac{a^{2}}{a_{,}^{3}}b_{3,1} - \frac{3}{4}\frac{a}{a_{,}^{2}}b_{3,2}\right\} ee_{,}\cos\left(2nt + 2\varepsilon - \varpi - \varpi_{,}\right) \qquad [48]\ [46]$$

$$+ m_{,}\left\{\frac{21}{8}\frac{a}{a_{,}^{2}}b_{3,1} - \frac{2}{a_{,}}b_{3,2} - \frac{3}{8}\frac{a}{a_{,}^{2}}b_{3,3}\right\} ee_{,}\cos\left(3nt - n_{,}t + 3\varepsilon - \varepsilon_{,} - \varpi - \varpi\right) \qquad [49]\ [53]$$

$$+ m_{,}\left\{\frac{15}{4}\frac{a}{a_{,}^{2}}b_{3,2} - \frac{3}{a_{,}}b_{3,3} - \frac{1}{2}\frac{a}{a_{,}^{2}}b_{3,4}\right\} ee_{,}\cos\left(4nt - 2n_{,}t + 4\varepsilon - 2\varepsilon_{,} - \varpi - \varpi\right) \qquad [50]\ [54]$$

Development of R.

$$+ m_{\prime} \left\{ \frac{39}{8} \frac{a}{a_{\prime}^2} b_{3,3} - \frac{4}{a_{\prime}} b_{3,4} - \frac{5}{8} \frac{a}{a_{\prime}^2} b_{3,5} \right\} e e_{\prime} \cos (5 n t - 3 n_{\prime} t + 5 \varepsilon - 3 \varepsilon_{\prime} - \varpi - \varpi_{\prime}) \qquad [51]\ [55]$$

$$+ m_{\prime} \left\{ 6 \frac{a}{a_{\prime}^2} b_{3,4} - \frac{5}{a_{\prime}} b_{3,5} - \frac{3}{4} \frac{a}{a_{\prime}^2} b_{3,6} \right\} e e_{\prime} \cos (6 n t - 4 n_{\prime} t + 6 \varepsilon - 4 \varepsilon_{\prime} - \varpi - \varpi_{\prime}) \qquad [52]\ [56]$$

$$+ m_{\prime} \left\{ \frac{21}{8} \frac{a}{a_{\prime}^2} b_{3,1} - \frac{2 a^2}{a_{\prime}^3} b_{3,2} - \frac{3 a}{8 a_{\prime}^2} b_{3,3} \right\} e e_{\prime} \cos (n t - 3 n_{\prime} t + \varepsilon - 3 \varepsilon_{\prime} + \varpi + \varpi_{\prime}) \qquad [53]\ [49]$$

$$+ m_{\prime} \left\{ \frac{15}{4} \frac{a}{a_{\prime}^2} b_{3,2} - \frac{3 a^2}{a_{\prime}^3} b_{3,3} - \frac{a}{2 a_{\prime}^2} b_{3,4} \right\} e e_{\prime} \cos (2 n t - 4 n_{\prime} t + 2 \varepsilon - 4 \varepsilon_{\prime} + \varpi + \varpi_{\prime}) \qquad [54]\ [50]$$

$$+ m_{\prime} \left\{ \frac{39}{8} \frac{a}{a_{\prime}^2} b_{3,3} - \frac{4 a^2}{a_{\prime}^3} b_{3,4} - \frac{5}{8} \frac{a}{a_{\prime}^2} b_{3,5} \right\} e e_{\prime} \cos (3 n t - 5 n_{\prime} t + 3 \varepsilon - 5 \varepsilon_{\prime} - \varpi + \varpi_{\prime}) \qquad [55]\ [51]$$

$$+ m_{\prime} \left\{ 6 \frac{a}{a_{\prime}^2} b_{3,4} - \frac{5 a^2}{a_{\prime 3}} b_{3,5} - \frac{3}{4} \frac{a}{a_{\prime}^2} b_{3,6} \right\} e e_{\prime} \cos (4 n t - 6 n_{\prime} t + 4 \varepsilon - 6 \varepsilon_{\prime} - \varpi + \varpi_{\prime}) \qquad [56]\ [52]$$

$$+ m_{\prime} \left\{ - \frac{1}{a_{\prime}} b_{3,0} + \frac{3}{8} \frac{a}{a_{\prime}^2} b_{3,1} \right\} e_{\prime}^2 \cos (2 n_{\prime} t + 2 \varepsilon_{\prime} - 2 \varpi_{\prime}) \qquad [57]\ [26]$$

$$+ m_{\prime} \left\{ \frac{a}{8 a_{\prime}^2} - \frac{a}{8 a_{\prime}^2} b_{3,0} - \frac{a}{16 a_{\prime}^2} b_{3,2} \right\} e_{\prime}^2 \cos (n t + n_{\prime} t + \varepsilon + \varepsilon_{\prime} - 2 \varpi_{\prime}) \qquad [58]\ [25]$$

$$+ m_{\prime} \left\{ - \frac{5}{8} \frac{a}{a_{\prime}^2} b_{3,1} + \frac{1}{2 a_{\prime}} b_{3,2} \right\} e_{\prime}^2 \cos (2 n t + 2 \varepsilon - 2 \varpi_{\prime}) \qquad [59]\ [24]$$

$$+ m_{\prime} \left\{ - \frac{19}{16} \frac{a}{a_{\prime}^2} b_{3,2} + \frac{1}{a_{\prime}} b_{3,3} + \frac{a}{16 a_{\prime}^2} b_{3,4} \right\} e_{\prime}^2 \cos (3 n t - n_{\prime} t + 3 \varepsilon - \varepsilon_{\prime} - 2 \varpi_{\prime}) \qquad [60]\ [32]$$

$$+ m_{\prime} \left\{ - \frac{7}{4} \frac{a}{a_{\prime}^2} b_{3,3} + \frac{3}{2 a_{\prime}} b_{3,4} + \frac{a}{8 a_{\prime}^2} b_{3,5} \right\} e_{\prime} \cos (4 n t - 2 n_{\prime} t + 4 \varepsilon - 2 \varepsilon_{\prime} - 2 \varpi_{\prime}) \qquad [61]\ [33]$$

$$+ m_{\prime} \left\{ - \frac{37}{16} \frac{a}{a_{\prime}^2} b_{3,4} + \frac{2}{a_{\prime}} b_{3,5} + \frac{3}{16} \frac{a}{a_{\prime}^2} b_{3,6} \right\} e_{\prime}^2 \cos 5 n t - 3 n_{\prime} t + 5 \varepsilon - 3 \varepsilon_{\prime} - 2 \varpi_{\prime}) \qquad [62]\ [34]$$

$$+ m_{\prime} \left\{ \frac{27}{8} \frac{a}{a_{\prime}^2} - \frac{3}{8} \frac{a}{a_{\prime}^2} b_{3,0} - \frac{1}{a_{\prime}} b_{3,1} + \frac{17}{16} \frac{a}{a_{\prime}^2} b_{3,2} \right\} e_{\prime}^2 \cos (n t - 3 n_{\prime} t + \varepsilon - 3 \varepsilon_{\prime} + 2 \varpi_{\prime}) \qquad [63]\ [27]$$

$$+ m_{\prime} \left\{ - \frac{a}{4 a_{\prime}^2} b_{3,1} - \frac{3}{2 a_{\prime}} b_{3,2} + \frac{13}{8} \frac{a}{a_{\prime}^2} b_{3,3} \right\} e_{\prime}^2 \cos (2 n t - 4 n_{\prime} t + 2 \varepsilon - 4 \varepsilon_{\prime} + 2 \varpi_{\prime}) \qquad [64]\ [28]$$

$$+ m_{\prime} \left\{ - \frac{5}{16} \frac{a}{a_{\prime}^2} b_{3,2} - \frac{2}{a_{\prime}} b_{3,3} + \frac{35}{16} \frac{a}{a_{\prime}^2} b_{3,4} \right\} e_{\prime}^2 \cos (3 n t - 5 n_{\prime} t + 3 \varepsilon - 5 \varepsilon_{\prime} + 2 \varpi_{\prime}) \qquad [65]\ [29]$$

$$+ m_{\prime} \left\{ - \frac{3}{8} \frac{a}{a_{\prime}^2} b_{3,3} - \frac{5}{2 a_{\prime}} b_{3,4} + \frac{11}{4} \frac{a}{a_{\prime}^2} b_{3,5} \right\} e_{\prime}^2 \cos (4 n t - 6 n_{\prime} t + 4 \varepsilon - 6 \varepsilon_{\prime} + 2 \varpi) \qquad [66]\ [30]$$

$$+ m_{\prime} \left\{ - \frac{7}{16} \frac{a}{a_{\prime}^2} b_{3,4} - \frac{3}{a_{\prime}} b_{3,5} + \frac{53}{16} \frac{a}{a_{\prime}^2} b_{3,6} \right\} e_{\prime}^2 \cos (5 n t - 7 n_{\prime} t + 5 \varepsilon - 7 \varepsilon_{\prime} + 2 \varpi_{\prime}) \quad [67] \quad [31]$$

Development of R.

$$+ m_{\prime} \left\{ \frac{a}{a_{\prime}^2} - \frac{a}{a_{\prime}^2} b_{3,0} \right\} \sin^2 \frac{\iota_{\prime}}{2} (n t + n_{\prime} t + \varepsilon + \varepsilon_{\prime} - 2 \nu_{\prime}) \quad [68]$$

$$- \frac{m_{\prime}}{2} \frac{a}{a_{\prime}^2} b_{3,1} \sin^2 \frac{\iota_{\prime}}{2} \cos (2 n_{\prime} t + 2 \varepsilon_{\prime} - 2 \nu_{\prime}) \quad [69]$$

$$- \frac{m_{\prime}}{2} \frac{a}{a_{\prime}^2} b_{3,1} \sin^2 \frac{\iota_{\prime}}{2} \cos (2 n t + 2 \varepsilon - 2 \nu_{\prime}) \quad [70]$$

$$- \frac{m_{\prime}}{2} \frac{a}{a_{\prime}^2} b_{3,2} \sin^2 \frac{\iota_{\prime}}{2} \cos (n t + 3 n_{\prime} t + \varepsilon - 3 \varepsilon_{\prime} + 2 \nu_{\prime}) \quad [71]$$

$$- \frac{m_{\prime}}{2} \frac{a}{a_{\prime}^2} b_{3,2} \sin^2 \frac{\iota_{\prime}}{2} \cos (3 n t - n_{\prime} t + 3 \varepsilon - \varepsilon_{\prime} - 2 \nu_{\prime}) \quad [72]$$

$$- \frac{m_{\prime}}{2} \frac{a}{a_{\prime}^2} b_{3,3} \sin^2 \frac{\iota_{\prime}}{2} \cos (2 n t - 4 n_{\prime} t + 2 \varepsilon - 4 \varepsilon_{\prime} + 2 \nu_{\prime}) \quad [73]$$

$$- \frac{m_{\prime}}{2} \frac{a}{a_{\prime}^2} b_{3,3} \sin^2 \frac{\iota_{\prime}}{2} \cos (4 n t - 2 n_{\prime} t + 4 \varepsilon - 2 \varepsilon_{\prime} - 2 \nu_{\prime}) \quad [74]$$

In the lunar theory, the small value of the quantity $\frac{a}{a_{\prime}}$ makes it desirable to ordain the results according to powers of this quantity. Transforming therefore the preceding expression for R by means of the equations given in the former part of this paper, Phil. Trans. for 1830, p. 346, neglecting terms multiplied by $\frac{a^4}{a_{\prime}^5}$, and supposing $\iota_{\prime} = 0$,

$$R = m_{\prime} \left\{ - \frac{1}{a_{\prime}} \left(1 + \frac{a^2}{4 a_{\prime}^2} \right) + \frac{3}{2} \left(\sin^2 \frac{\iota}{2} - \frac{(e^2 + e_{\prime}^2)}{4} \right) \frac{a^2}{a_{\prime}^3} \right\} \quad [0]$$

$$+ m_{\prime} \left\{ \left(\cos^2 \frac{\iota}{2} - \frac{(e^2 + e_{\prime}^2)}{2} \right) \frac{a}{a_{\prime}^2} - \frac{a}{a_{\prime}^2} \left(1 + \frac{3}{8} \frac{a}{a_{\prime}^2} \right) + \sin^2 \frac{\iota}{2} \frac{a}{a_{\prime}^2} \left(1 + \frac{33}{8} \frac{a^2}{a_{\prime}^2} \right) \right.$$

$$\left. + \frac{(e^2 + e_{\prime}^2)}{2} \frac{a}{a_{\prime}^2} \left(1 - \frac{3}{2} \frac{a^2}{a_{\prime}^2} \right) \right\} \cos (n t - n_{\prime} t + \varepsilon - \varepsilon_{\prime}) \quad [1]$$

$$+ m_{\prime} \left\{ - \frac{3}{4} \frac{a^2}{a_{\prime}^3} + \frac{3}{2} \sin^2 \frac{\iota}{2} \frac{a^2}{a_{\prime}^3} + \frac{15}{8} (e^2 + e_{\prime}^2) \frac{a^2}{a_{\prime}^3} \right\} \cos (2 n t - 2 n_{\prime} t + 2 \varepsilon - 2 \varepsilon_{\prime}) \quad [2]$$

$$+ m_{\prime} \left\{ - \frac{5}{8} \frac{a^3}{a_{\prime}^4} + \frac{15}{8} \sin^2 \frac{\iota}{2} \frac{a^3}{a_{\prime}^4} + \frac{15}{4} (e^2 + e_{\prime}^2) \frac{a^3}{a_{\prime}^4} \right\} \cos (3 n t - 3 n_{\prime} t + 3 \varepsilon - 3 \varepsilon_{\prime}) \quad [3]$$

Development of R according to powers of $\frac{a}{a_{\prime}}$.

$$+ m_{\prime} \frac{15}{16} \frac{a^3}{a_{\prime}^4} e \cos (n_{\prime} t + \varepsilon_{\prime} - \varpi) \qquad [6]$$

$$+ \frac{m_{\prime}}{2} \frac{a^2}{a_{\prime}^3} e \cos (n t + \varepsilon - \varpi) \qquad [7]$$

$$+ m_{\prime} \frac{3}{16} \frac{a^3}{a_{\prime}^4} e \cos (2 n t - n_{\prime} t + 2 \varepsilon - \varepsilon_{\prime} - \varpi) \qquad [8]$$

$$- m_{\prime} \frac{3}{4} \frac{a^2}{a_{\prime}^3} e \cos (3 n t - 2 n_{\prime} t + 3 \varepsilon - 2 \varepsilon_{\prime} - \varpi) \qquad [9]$$

$$- m_{\prime} \frac{15}{16} \frac{a^3}{a_{\prime}^4} e \cos (4 n t - 3 n_{\prime} t + 4 \varepsilon - 3 \varepsilon_{\prime} - \varpi) \qquad [10]$$

$$+ m_{\prime} \frac{9}{4} \frac{a^2}{a_{\prime}^3} e \cos (n t - 2 n_{\prime} t + \varepsilon - 2 \varepsilon_{\prime} + \varpi) \qquad [12]$$

$$+ m_{\prime} \frac{45}{16} \frac{a^3}{a_{\prime}^4} e \cos (2 n t - 3 n_{\prime} t + 2 \varepsilon - 3 \varepsilon_{\prime} + \varpi) \qquad [13]$$

$$- m_{\prime} \left\{ \frac{1}{a_{\prime}} + \frac{3}{4} \frac{a^2}{a_{\prime}^3} \right\} e_{\prime} \cos (n_{\prime} t + \varepsilon_{\prime} - \varpi_{\prime}) \qquad [15]$$

$$- m_{\prime} \frac{3}{8} \frac{a^3}{a_{\prime}^4} e_{\prime} \cos (n t + \varepsilon_{\prime} - \varpi_{\prime}) \qquad [16]$$

$$+ m_{\prime} \frac{3}{8} \frac{a^2}{a_{\prime}^3} e_{\prime} \cos (2 n t - n_{\prime} t + 2 \varepsilon - \varepsilon_{\prime} - \varpi_{\prime}) \qquad [17]$$

$$+ m_{\prime} \frac{5}{8} \frac{a^3}{a_{\prime}^4} e_{\prime} \cos (3 n t - 2 n_{\prime} t + 3 \varepsilon - 2 \varepsilon_{\prime} - \varpi_{\prime}) \qquad [18]$$

$$- m_{\prime} \frac{9}{8} \frac{a^3}{a_{\prime}^4} e_{\prime} \cos (n t - 2 n_{\prime} t + \varepsilon - 2 \varepsilon_{\prime} + \varpi_{\prime}) \qquad [20]$$

$$- m_{\prime} \frac{21}{8} \frac{a^2}{a_{\prime}^3} e_{\prime} \cos (2 n t - 3 n_{\prime} t + 2 \varepsilon - 3 \varepsilon_{\prime} + \varpi_{\prime}) \qquad [21]$$

$$- m_{\prime} \frac{25}{8} \frac{a^3}{a_{\prime}^4} e_{\prime} \cos (3 n t - 4 n_{\prime} t + 3 \varepsilon - 4 \varepsilon_{\prime} + \varpi_{\prime}) \qquad [22]$$

$$- m_{\prime} \frac{15}{8} \frac{a^2}{a_{\prime}^3} e^2 \cos (2 n_{\prime} t + 2 \varepsilon_{\prime} - 2 \varpi) \qquad [24]$$

$$- m_{\prime} \frac{33}{64} \frac{a^3}{a_{\prime}^4} e^2 \cos (n t + n_{\prime} t + \varepsilon + \varepsilon_{\prime} - 2 \varpi) \qquad [25]$$

$$+ \frac{m_{\prime}}{8} \frac{a^2}{a_{\prime}^3} e^2 \cos (2 n t + 2 \varepsilon - 2 \varpi) \qquad [26]$$

$$+ m_{\prime} \frac{9}{64} \frac{a^3}{a_{\prime}^4} e^2 \cos (3 n t - n_{\prime} t + 3 \varepsilon - \varepsilon_{\prime} - 2 \varpi) \qquad [27]$$

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$$- m_{\prime} \frac{3}{4} \frac{a^2}{a_{\prime}^3} e^2 \cos (4 n t - 2 n_{\prime} t + 4 \varepsilon - 2 \varepsilon_{\prime} - 2 \varpi) \qquad [28]$$

$$- m_{\prime} \frac{75}{64} \frac{a^3}{a_{\prime}^4} e^2 \cos (5 n t - 3 n_{\prime} t + 5 \varepsilon - 3 \varepsilon_{\prime} - 2 \varpi) \qquad [29]$$

$$- m_{\prime} \frac{285}{64} \frac{a^3}{a_{\prime}^4} e^2 \cos (n t - 3 n_{\prime} t + \varepsilon - 3 \varepsilon_{\prime} + 2 \varpi) \qquad [32]$$

$$+ m_{\prime} \frac{3}{4} \frac{a^2}{a_{\prime}^3} e e_{\prime} \cos (n t - n_{\prime} t + \varepsilon - \varepsilon_{\prime} - \varpi + \varpi_{\prime}) \qquad [35]$$

$$+ m_{\prime} \frac{9}{16} \frac{a^3}{a_{\prime}^4} e e_{\prime} \cos (2 n t - 2 n_{\prime} t + 2 \varepsilon - 2 \varepsilon_{\prime} - \varpi + \varpi_{\prime}) \qquad [36]$$

$$- m_{\prime} \frac{21}{8} \frac{a^2}{a_{\prime}^3} e e_{\prime} \cos (3 n t - 3 n_{\prime} t + 3 \varepsilon - 3 \varepsilon_{\prime} - \varpi + \varpi_{\prime}) \qquad [37]$$

$$+ m_{\prime} \frac{15}{16} \frac{a^3}{a_{\prime}^4} e e_{\prime} \cos (\varpi - \varpi_{\prime}) \qquad [41]$$

$$- m_{\prime} \frac{9}{8} \frac{a^2}{a_{\prime}^3} e e_{\prime} \cos (n t - n_{\prime} t + \varepsilon - \varepsilon_{\prime} + \varpi - \varpi_{\prime}) \qquad [42]$$

$$- m_{\prime} \frac{45}{16} \frac{a^3}{a_{\prime}^4} e e_{\prime} \cos (2 n t - 2 n_{\prime} t + 2 \varepsilon - 2 \varepsilon_{\prime} + \varpi - \varpi_{\prime}) \qquad [43]$$

$$+ m_{\prime} \frac{45}{16} \frac{a^3}{a_{\prime}^4} e e_{\prime} \cos (2 n_{\prime} t + 2 \varepsilon_{\prime} - \varpi - \varpi_{\prime}) \qquad [46]$$

$$+ m_{\prime} \frac{3}{4} \frac{a^2}{a_{\prime}^3} e e_{\prime} \cos (n t + n_{\prime} t + \varepsilon + \varepsilon_{\prime} - \varpi - \varpi_{\prime}) \qquad [47]$$

$$+ m_{\prime} \frac{3}{16} \frac{a^3}{a_{\prime}^4} e e_{\prime} \cos (2 n t + 2 \varepsilon - \varpi - \varpi_{\prime}) \qquad [48]$$

$$+ m_{\prime} \frac{3}{8} \frac{a^2}{a_{\prime}^3} e e_{\prime} \cos (3 n t - n_{\prime} t + 3 \varepsilon - \varepsilon_{\prime} - \varpi - \varpi_{\prime}) \qquad [49]$$

$$+ m_{\prime} \frac{15}{16} \frac{a^3}{a_{\prime}^4} e e_{\prime} \cos (4 n t - 2 n_{\prime} t + 4 \varepsilon - 2 \varepsilon_{\prime} - \varpi - \varpi_{\prime}) \qquad [50]$$

$$+ m_{\prime} \frac{63}{8} \frac{a^2}{a_{\prime}^3} e e_{\prime} \cos (n t - 3 n_{\prime} t + \varepsilon - 3 \varepsilon_{\prime} + \varpi + \varpi_{\prime}) \qquad [53]$$

$$+ m_{\prime} \frac{225}{16} \frac{a^3}{a_{\prime}^4} e e_{\prime} \cos (2 n t - 4 n_{\prime} t + 2 \varepsilon - 4 \varepsilon_{\prime} + \varpi + \varpi_{\prime}) \qquad [54]$$

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$$+ m_{\prime} \left\{ \frac{1}{a_{\prime}} - \frac{9}{8} \frac{a^2}{a_{\prime}^3} \right\} e_{\prime}^2 \cos (2 n_{\prime} t + 2 \varepsilon_{\prime} - 2 \varpi_{\prime}) \qquad [57]$$

$$- m_{\prime} \frac{33}{64} \frac{a^3}{a_{\prime}^4} e_{\prime}^2 \cos (n t + n_{\prime} t + \varepsilon + \varepsilon_{\prime} - 2 \varpi_{\prime}) \qquad [58]$$

$$- m_{\prime} \frac{5}{64} \frac{a^3}{a_{\prime}^4} e_{\prime}^2 \cos (3 n t - n_{\prime} t + 3 \varepsilon - \varepsilon_{\prime} - 2 \varpi_{\prime}) \qquad [60]$$

$$- m_{\prime} \frac{159}{64} \frac{a^3}{a_{\prime}^4} e_{\prime}^2 \cos (n t - 3 n_{\prime} t + \varepsilon - 3 \varepsilon_{\prime} + 2 \varpi_{\prime}) \qquad [63]$$

$$- m_{\prime} \frac{51}{8} \frac{a^2}{a_{\prime}^3} e_{\prime}^2 \cos (2 n t - 4 n_{\prime} t + 2 \varepsilon - 4 \varepsilon_{\prime} + 2 \varpi_{\prime}) \qquad [64]$$

$$- m_{\prime} \frac{635}{64} \frac{a^3}{a_{\prime}^4} e_{\prime}^2 \cos (3 n t - 5 n_{\prime} t + 3 \varepsilon - 5 \varepsilon_{\prime} + 2 \varpi_{\prime}) \qquad [65]$$

$$- m_{\prime} \frac{9}{4} \frac{a^3}{a_{\prime}^4} \sin^2 \frac{\iota}{2} \cos (n t + n_{\prime} t + \varepsilon + \varepsilon_{\prime} - 2 \nu) \qquad [75]$$

$$- m_{\prime} \frac{3}{2} \frac{a^2}{a_{\prime}^3} \sin^2 \frac{\iota}{2} \cos (2 n t + 2 \varepsilon - 2 \nu) \qquad [76]$$

$$- m_{\prime} \frac{3}{2} \frac{a^2}{a_{\prime}^3} \sin^2 \frac{\iota}{2} \cos (2 n_{\prime} t + 2 \varepsilon_{\prime} - 2 \nu) \qquad [77]$$

$$- m_{\prime} \frac{15}{8} \frac{a^3}{a_{\prime}^4} \sin^2 \frac{\iota}{2} \cos (3 n t - n_{\prime} t + 3 \varepsilon - \varepsilon_{\prime} - 2 \nu) \qquad [78]$$

$$- m_{\prime} \frac{15}{8} \frac{a^3}{a_{\prime}^4} \sin^2 \frac{\iota}{2} \cos (n t - 3 n_{\prime} t + \varepsilon - 3 \varepsilon_{\prime} + 2 \nu) \qquad [79]$$

If according to the notation of M. Damoiseau, (Théorie Lunaire, p. 547, Mémoires des Savans Étrangers,) $n t - n_{\prime} t + \varepsilon - \varepsilon_{\prime} = t$, and x and z be put for the mean anomalies of m and $m_{\prime}$ respectively and y for the distance of the planet m from its node, or, what is the same in the Lunar Theory, the distance of the moon from her node reckoned on the ecliptic ($\iota_{\prime} = 0$),

$$R = m_{\prime} \Big\{ \underbrace{- \frac{1}{a_{\prime}} \left(1 + \frac{a^2}{4 a_{\prime}^2}\right) + \frac{3}{2} \left(\sin^2 \frac{\iota}{2} - \frac{(e^2 + e_{\prime}^2)}{4}\right) \frac{a^2}{a_{\prime}^3}}_{[0]}$$

$$+ \Big\{ \left(\cos^2 \frac{\iota_{\prime}}{2} - \frac{(e^2 + e_{\prime}^2)}{2}\right) \frac{a}{a_{\prime}^2} - \frac{a}{a_{\prime}^2} \left(1 + \frac{3}{8} \frac{a^2}{a_{\prime}^2}\right) + \sin^2 \frac{\iota}{2} \frac{a}{a_{\prime}^2} \left(1 + \frac{33}{8} \frac{a^2}{a_{\prime}^2}\right)$$

$$+ \frac{(e^2 + e_{\prime}^2)}{2} \frac{a}{a_{\prime}^2} \left(1 - \frac{3}{2} \frac{a^2}{a_{\prime}^2}\right) \Big\} \underset{[1]}{\cos t} + \Big\{ - \frac{3}{4} \frac{a^2}{a_{\prime}^3} + \frac{3}{2} \sin^2 \frac{\iota}{2} \frac{a^2}{a_{\prime}^3} + \frac{15}{8} (e^2 + e_{\prime}^2) \frac{a^2}{a_{\prime}^3} \Big\} \cos 2 t \qquad [2]$$

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$$+\left\{-\frac{5}{8}\frac{a^3}{a_{\prime}^4}+\frac{15}{8}\sin^2\frac{\iota}{2}\frac{a^3}{a_{\prime}^4}+\frac{15}{4}(e^2+e_{\prime}^2)\frac{a^3}{a_{\prime}^4}\right\}\underset{[3]}{\cos 3t}+\underset{[6]}{\frac{15}{16}\frac{a^3}{a_{\prime}^4}\cos(t-x)}$$

$$+\underset{[7]}{\frac{a^3}{2a_{\prime}^3}e\cos x}+\underset{[8]}{\frac{3}{16}\frac{a^3}{a_{\prime}^4}e\cos t+x)}-\underset{[9]}{\frac{3}{4}\frac{a^2}{a_{\prime}^3}e\cos(2t+x)}-\underset{[10]}{\frac{15}{16}\frac{a^3}{a_{\prime}^4}e\cos(3t+x)}$$

$$+\underset{[12]}{\frac{9}{4}\frac{a^2}{a_{\prime}^3}e\cos(2t-x)}+\underset{[13]}{\frac{45}{16}\frac{a^3}{a_{\prime}^4}e\cos(3t-x)}-\underset{[15]}{\left\{\frac{1}{a_{\prime}}+\frac{3}{4}\frac{a^2}{a_{\prime}^3}\right\}e_{\prime}\cos z}$$

$$-\underset{[16]}{\frac{3}{8}\frac{a^3}{a_{\prime}^4}e_{\prime}\cos(t+z)}+\underset{[17]}{\frac{3}{8}\frac{a^2}{a_{\prime}^3}e_{\prime}\cos(2t+z)}+\underset{[18]}{\frac{5}{8}\frac{a_{\prime}^3}{a_{\prime}^4}e_{\prime}\cos(3t+z)}$$

$$-\underset{[20]}{\frac{9}{8}\frac{a^3}{a_{\prime}^4}e_{\prime}\cos(t-z)}-\underset{[21]}{\frac{21}{8}\frac{a^2}{a_{\prime}^3}e_{\prime}\cos(2t-z)}-\underset{[22]}{\frac{25}{8}\frac{a^3}{a_{\prime}^4}e_{\prime}\cos(3t-z)}$$

$$-\underset{[24]}{\frac{15}{8}\frac{a^2}{a_{\prime}^3}e^2\cos(2t-2x)}-\underset{[25]}{\frac{33}{64}\frac{a^3}{a_{\prime}^4}e^2\cos(t-2x)}+\underset{[26]}{\frac{a^2}{8a_{\prime}^3}e^2\cos 2x}$$

$$+\underset{[27]}{\frac{9}{64}\frac{a^3}{a_{\prime}^4}e^2\cos(t+2x)}-\underset{[28]}{\frac{3}{4}\frac{a^2}{a_{\prime}^3}e^2\cos(2t+2x)}-\underset{[29]}{\frac{75}{64}\frac{a^3}{a_{\prime}^4}e^2\cos(3t+2x)}$$

$$-\underset{[32]}{\frac{285}{64}\frac{a^3}{a_{\prime}^4}e^2\cos(3t-2x)}+\underset{[35]}{\frac{3}{4}\frac{a^2}{a_{\prime}^3}ee_{\prime}\cos(x-z)}+\underset{[36]}{\frac{9}{16}\frac{a^3}{a_{\prime}^4}ee_{\prime}\cos(t-z+x)}$$

$$-\underset{[37]}{\frac{27}{8}\frac{a^2}{a_{\prime}^3}ee_{\prime}\cos(2t-z+x)}+\underset{[41]}{\frac{15}{16}\frac{a^3}{a_{\prime}^4}ee_{\prime}\cos(t+z-x)}-\underset{[42]}{\frac{9}{8}\frac{a^2}{a_{\prime}^3}ee_{\prime}\cos(2t+z-x)}$$

$$-\underset{[43]}{\frac{45}{16}\frac{a^3}{a_{\prime}^4}ee_{\prime}\cos(3t+z-x)}+\underset{[46]}{\frac{45}{16}\frac{a^3}{a_{\prime}^4}ee_{\prime}\cos(t-z-x)}+\underset{[47]}{\frac{3}{4}\frac{a^2}{a_{\prime}^3}ee_{\prime}\cos(x+z)}$$

$$+\underset{[48]}{\frac{3}{16}\frac{a^3}{a_{\prime}^4}ee_{\prime}\cos(t+z+x)}+\underset{[49]}{\frac{3}{8}\frac{a^2}{a_{\prime}^3}ee_{\prime}\cos(2t+z+x)}+\underset{[50]}{\frac{15}{16}\frac{a^3}{a_{\prime}^4}ee_{\prime}\cos(3t+z+x)}$$

$$+\underset{[53]}{\frac{63}{8}\frac{a^2}{a_{\prime}^3}ee_{\prime}\cos(2t-z-x)}-\underset{[54]}{\frac{225}{16}\frac{a^3}{a_{\prime}^4}ee_{\prime}\cos(3t-z-x)}+\underset{[57]}{\left\{\frac{1}{a_{\prime}}-\frac{9}{8}\frac{a^2}{a_{\prime}^3}\right\}e_{\prime}^2\cos 2z}$$

$$-\underset{[58]}{\frac{33}{64}\frac{a^3}{a_{\prime}^4}e_{\prime}^2\cos(t+2z)}-\underset{[60]}{\frac{5}{64}\frac{a^3}{a_{\prime}^4}e_{\prime}^2\cos(3t+2z)}-\underset{[63]}{\frac{159}{64}\frac{a^3}{a_{\prime}^4}e_{\prime}^2\cos(t-2z)}$$

$$-\underset{[64]}{\frac{51}{8}\frac{a^2}{a_{\prime}^3}e_{\prime}^2(2t-2z)}-\underset{[65]}{\frac{635}{64}\frac{a^3}{a_{\prime}^4}e_{\prime}^2\cos(3t-2z)}-\underset{[75]}{\frac{9}{4}\frac{a^3}{a_{\prime}^4}\sin^2\frac{\iota}{2}\cos(t-2y)}$$

$$- \frac{3}{2}\frac{a^2}{a_{\prime}^3}\sin^2\frac{\iota}{2}\cos 2y - \frac{3}{2}\frac{a^2}{a_{\prime}^3}\sin^2\frac{\iota}{2}\cos(2t-2y) - \frac{15}{8}\frac{a^3}{a_{\prime}^4}\cos(t+2y)$$

[76] [77] [78]

$$\left.- \frac{15}{8}\frac{a^3}{a_{\prime}^4}\sin^2\frac{\iota}{2}\cos(3t-2y)\right\}$$

[79]

Let $\frac{a}{r} = 1 + e\cos\left(n(1+k)t + \varepsilon - \varpi\right) + e^2(1+r_{48})\cos\left(2n(1+k_2)t + 2\varepsilon - 2\varpi\right)$

$$+ e_{\prime}^2 r_{59}\cos\left(2n(1+k_{\prime 2})t + 2\varepsilon - 2\varpi_{\prime}\right) + r_0 + r_1\cos(nt + n_{\prime}t + \varepsilon - \varepsilon_{\prime})$$

$$+ r_2\cos(2nt - 2n_{\prime}t + 2\varepsilon - 2\varepsilon_{\prime}) + \&c. + e r_6\cos(n_{\prime}t + \varepsilon_{\prime} - \varpi) + \&c.$$

$$\frac{d^2 r^2}{2\,dt^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int dR + r\left(\frac{dR}{dr}\right) = 0$$

Let $\delta\,.\,\frac{1}{r}$ denote that part of $\frac{1}{r}$ which depends on the first power of the disturbing force. It is more simple to obtain $\delta\,.\,\frac{1}{r}$ from the above differential equation than δr; and the circumstance that the elliptic value of r^3 does not contain any term $e^2\cos(2nt + 2\varepsilon - 2\varpi)$, gives an additional facility.

$$-\frac{d^2.\,r^3\,\delta.\frac{1}{r}}{dt^2} - \mu\,\delta.\frac{1}{r} + 2\int dR + r\left(\frac{dR}{dr}\right) = 0$$

When the disturbing force is neglected

$$r^3 = a^3\left\{1 + 3e^2\left(1 + \frac{e^2}{12}\right) - 3e\left(1 + \frac{3}{8}e^2\right)\cos(nt + \varepsilon - \varpi) - \frac{3}{8}e^4\cos(2nt + 2\varepsilon - 2\varpi)\right.$$

$$\left.+ \frac{e^3}{8}\cos(3nt + 3\varepsilon - 3\varpi) + \frac{e^4}{8}\cos(4nt + 4\varepsilon - 4\varpi)\right\}$$

Integrating the above differential equation by the method of indeterminate co-efficients, q_n being the co-efficient of the nth argument in the development of $2\int dR + r\left(\frac{dR}{dr}\right)$,

$$-r_0 + \frac{m_{\prime}}{\mu}a\,q_0 = 0$$

$$\frac{(n-n_{\prime})^2}{n^2}\left\{(1+3e^2)\,r_1 - \frac{3e^2}{2}(r_6 + r_8)\right\} - r_1 + \frac{m_{\prime}}{\mu}a\,q_1 = 0$$

$$\frac{(2n-2n_{\prime})^2}{n^2}\left\{(1+3e^2)\,r_2 - \frac{3e^2}{2}(r_9 + r_{12})\right\} - r_2 + \frac{m_{\prime}}{\mu}a\,q_2 = 0$$

Equations which serve to determine the coefficients of the inequalities of the reciprocal of the radius vector.

$$\frac{(3n-3n_{\prime})^2}{n^2}\left\{(1+3e^2)r_3-\frac{3e^2}{2}(r_{10}+r_{13})\right\}-r_3+\frac{m_{\prime}}{\mu}a\,q_3=0$$

$$\frac{(4n-4n_{\prime})}{n^2}\left\{(1+3e^2)r_4-\frac{3e^2}{2}(r_{11}+r_{14})\right\}-r_4+\frac{m_{\prime}}{\mu}a\,q_4=0$$

$$\frac{(5n-5n_{\prime})^2}{n^2}(1+3e^2)r_5-r_5+\frac{m_{\prime}}{\mu}a\,q_5=0$$

$$\frac{n^2}{n_{\prime}^2}\left\{r_6-\frac{3}{2}r_1\right\}-r_6+\frac{m_{\prime}}{\mu}a\,q_6=0$$

$$*\frac{(2n-n_{\prime})^2}{n^2}\left\{r_8-\frac{3}{2}r_1\right\}-r_8+\frac{m_{\prime}}{\mu}a\,q_8=0$$

$$\frac{(3n-2n_{\prime})^2}{n^2}\left\{r_9-\frac{3}{2}r_2\right\}-r_9+\frac{m_{\prime}}{\mu}a\,q_9=0$$

$$\frac{(4n-3n_{\prime})^2}{n^2}\left\{r_{10}-\frac{3}{2}r_3\right\}-r_{10}+\frac{m_{\prime}}{\mu}a\,q_{10}=0$$

$$\frac{(5n-4n_{\prime})^2}{n^2}\left\{r_{11}-\frac{3}{2}r_4\right\}-r_{11}+\frac{m_{\prime}}{\mu}a\,q_{11}=0$$

$$\frac{(n-2n_{\prime})^2}{n^2}\left\{r_{12}-\frac{3}{2}r_2\right\}-r_{12}+\frac{m_{\prime}}{\mu}a\,q_{12}=0$$

$$\frac{(2n-3n_{\prime})^2}{n^2}\left\{r_{13}-\frac{3}{2}r_3\right\}-r_{13}+\frac{m_{\prime}}{\mu}a\,q_{13}=0$$

$$\frac{(3n-4n_{\prime})^2}{n^2}\left\{r_{14}-\frac{3}{2}r_4\right\}-r_{14}+\frac{m_{\prime}}{\mu}a\,q_{14}=0$$

$$\frac{n_{\prime}^2}{n^2}r_{15}-r_{15}+\frac{m_{\prime}}{\mu}a\,q_{15}=0$$

$$\frac{(2n-n_{\prime})^2}{n^2}r_{17}-r_{17}+\frac{m_{\prime}}{\mu}a\,q_{17}=0$$

$$\frac{(3n-2n_{\prime})^2}{n^2}r_{18}-r_{18}+\frac{m_{\prime}}{\mu}a\,q_{18}=0$$

$$\frac{(4n-3n_{\prime})^2}{n^2}r_{19}-r_{19}+\frac{m_{\prime}}{\mu}a\,q_{19}=0$$

$$\frac{(n-3n_{\prime})^2}{n^2}r_{20}-r_{20}+\frac{m_{\prime}}{\mu}a\,q_{20}-0$$

$$\frac{(2n-3n_{\prime})^2}{n^2}r_{21}-r_{21}+\frac{m_{\prime}}{\mu}a\,q_{21}=0$$

$$\frac{(3n-4n_{\prime})^2}{n^2}r_{22}-r_{22}+\frac{m_{\prime}}{\mu}a\,q_{22}=0$$

* For the determination of the quantity k, see p. 50.

Equations which serve to determine the coefficients of the inequalities of the reciprocal of the radius vector.

$$\frac{(4n-5n_{,})^2}{n^2} r_{23} - r_{23} + \frac{m_{,}}{\mu} a q_{23} = 0$$

$$\frac{4n_{,}^2}{n^2}\left\{r_{24} - \frac{3}{2} r_{12}\right\} - r_{24} + \frac{m_{,}}{\mu} a q_{24} = 0$$

$$\frac{(n+n_{,})^2}{n^2}\left\{r_{25} - \frac{3}{2} r_{6}\right\} - r_{25} + \frac{m_{,}}{\mu} a q_{25} = 0$$

$$\frac{(3n-n_{,})^2}{n^2}\left\{r_{27} - \frac{3}{2} r_{8}\right\} - r_{27} + \frac{m_{,}}{\mu} a q_{27} = 0$$

$$\frac{(4n-2n_{,})^2}{n^2}\left\{r_{28} - \frac{3}{2} r_{9}\right\} - r_{28} + \frac{m_{,}}{\mu} a q_{28} = 0$$

$$\frac{(5n-3n_{,})^2}{n^2}\left\{r_{29} - \frac{3}{2} r_{10}\right\} - r_{29} + \frac{m_{,}}{\mu} a q_{29} = 0$$

$$\frac{(6n-4n_{,})^2}{n^2}\left\{r_{30} - \frac{3}{2} r_{11}\right\} - r_{30} + \frac{m_{,}}{\mu} a q_{30} = 0$$

$$\frac{(7n-5n_{,})^2}{n^2} r_{31} - r_{31} + \frac{m_{,}}{\mu} a q_{31} = 0$$

$$\frac{(n-3n_{,})^2}{n^2}\left\{r_{32} - \frac{3}{2} r_{13}\right\} - r_{32} + \frac{m_{,}}{\mu} a q_{32} = 0$$

$$\frac{(2n-4n_{,})^2}{n^2}\left\{r_{33} - \frac{3}{2} r_{14}\right\} - r_{33} + \frac{m_{,}}{\mu} a q_{33} = 0$$

$$\frac{(3n-5n_{,})^2}{n^2} r_{34} - r_{34} + \frac{m_{,}}{\mu} a q_{34} = 0$$

$$\frac{(n-n_{,})^2}{n^2}\left\{r_{35} - \frac{3}{2} r_{15}\right\} - r_{35} + \frac{m_{,}}{\mu} a q_{35} = 0$$

$$\frac{(2n-2n_{,})^2}{n^2}\left\{r_{36} - \frac{3}{2} r_{20}\right\} - r_{36} + \frac{m_{,}}{\mu} a q_{36} = 0$$

$$\frac{(3n-3n_{,})^2}{n^2}\left\{r_{37} - \frac{3}{2} r_{21}\right\} - r_{37} + \frac{m_{,}}{\mu} a q_{37} = 0$$

$$\frac{(4n-4n_{,})^2}{n^2}\left\{r_{38} - \frac{3}{2} r_{22}\right\} - r_{38} + \frac{m_{,}}{\mu} a q_{38} = 0$$

$$\frac{(5n-5n_{,})^2}{n^2}\left\{r_{39} - \frac{3}{2} r_{23}\right\} - r_{39} + \frac{m_{,}}{\mu} a q_{39} = 0$$

$$\frac{(6n-6n_{,})^2}{n^2} r_{40} - r_{40} + \frac{m_{,}}{\mu} a q_{40} = 0$$

$$- r_{41} + \frac{m_{,}}{\mu} a q_{41} = 0$$

$$\frac{(n-n_{,})^2}{n^2}\left\{r_{42} - \frac{3}{2} r_{17}\right\} - r_{42} + \frac{m_{,}}{\mu} a q_{42} = 0$$

Equations which serve to determine the coefficients of the inequalities of the reciprocal of the radius vector.

$$\frac{(2n-2n_{\prime})^2}{n^2}\left\{r_{43}-\frac{3}{2}r_{18}\right\}-r_{43}+\frac{m_{\prime}}{\mu}a\,q_{43}=0$$

$$\frac{(3n-3n_{\prime})^2}{n^2}\left\{r_{44}-\frac{3}{2}r_{19}\right\}-r_{44}+\frac{m_{\prime}}{\mu}a\,q_{44}=0$$

$$\frac{(4n-4n_{\prime})^2}{n^2}\left\{r_{45}-r_{45}+\frac{m_{\prime}}{\mu}a\,q_{45}=0\right.$$

$$\frac{4n_{\prime}^2}{n^2}\left\{r_{46}-\frac{3}{2}r_{20}\right\}-r_{46}+\frac{m_{\prime}}{\mu}a\,q_{46}=0$$

$$\frac{(n+n_{\prime})^2}{n^2}\left\{r_{47}-\frac{3}{2}r_{15}\right\}-r_{47}+\frac{m_{\prime}}{\mu}a\,q_{47}=0$$

$$\frac{(3n-3n_{\prime})^2}{n^2}\left\{r_{49}-\frac{3}{2}r_{17}\right\}-r_{49}+\frac{m_{\prime}}{\mu}a\,q_{45}=0$$

$$\frac{(4n-2n_{\prime})^2}{n^2}\left\{r_{50}-\frac{3}{2}r_{18}\right\}-r_{50}+\frac{m_{\prime}}{\mu}a\,q_{50}=0$$

$$\frac{(5n-5n_{\prime})^2}{n^2}\left\{r_{51}-\frac{3}{2}r_{19}\right\}-r_{51}+\frac{m_{\prime}}{\mu}a\,q_{51}=0$$

$$\frac{(6n-4n_{\prime})^2}{n^2}\left\{r_{52}-r_{52}+\frac{m_{\prime}}{\mu}a\,q_{52}=0\right.$$

$$\frac{(n-n_{\prime})^2}{n^2}\left\{r_{53}-\frac{3}{2}r_{21}\right\}-r_{53}+\frac{m_{\prime}}{\mu}a\,q_{53}=0$$

$$\frac{(2n-4n_{\prime})^2}{n}\left\{r_{54}-\frac{3}{2}r_{22}\right\}-r_{54}+\frac{m_{\prime}}{\mu}a\,q_{54}=0$$

$$\frac{(3n-5n_{\prime})^2}{n^2}\left\{r_{55}-\frac{3}{2}r_{23}\right\}-r_{55}+\frac{m_{\prime}}{\mu}a\,q_{55}=0$$

$$\frac{(4n-6n_{\prime})^2}{n^2}r_{56}-r_{56}+\frac{m_{\prime}}{\mu}a\,q_{56}=0$$

$$\frac{4n_{\prime}^2}{n^2}r_{57}-r_{57}+\frac{m_{\prime}}{\mu}a\,q_{57}=0$$

$$\frac{(m+n_{\prime})^2}{n^2}r_{58}-r_{58}+\frac{m_{\prime}}{\mu}a\,q_{58}=0$$

$$\frac{(3n-n_{\prime})^2}{n^2}r_{60}-r_{60}+\frac{m_{\prime}}{\mu}a\,q_{60}=0$$

$$\frac{(4n-2n_{\prime})^2}{n^2}r_{61}-r_{61}+\frac{m_{\prime}}{\mu}a\,q_{61}=0$$

$$\frac{(5n-3n_{\prime})^2}{n^2}r_{62}-r_{62}+\frac{m_{\prime}}{\mu}a\,q_{62}=0$$

$$\frac{(n-3n_{\prime})^2}{n^2}r_{65}-r_{63}+\frac{m_{\prime}}{\mu}a\,q_{62}=0$$

Equations which serve to determine the coefficients of the inequalities of the reciprocal of the radius vector.

$$\frac{(2n-4n_{\prime})^2}{n^2} r_{64} - r_{64} + \frac{m_{\prime}}{\mu} a q_{64} = 0$$

$$\frac{(3n-5n_{\prime})^2}{n^2} r_{65} - r_{65} + \frac{m_{\prime}}{\mu} a q_{65} = 0$$

$$\frac{(4n-6n_{\prime})^2}{n^2} r_{66} - r_{66} + \frac{m_{\prime}}{\mu} a q_{66} = 0$$

$$\frac{(5n-7n_{\prime})^2}{n^2} r_{67} - r_{67} + \frac{m_{\prime}}{\mu} a q_{67} = 0$$

$$\frac{(n+n_{\prime})^2}{n^2} r_{68} - r_{68} + \frac{m_{\prime}}{\mu} a q_{68} = 0$$

In order to obtain the values of the coefficients of the inequalities from these equations when the cubes of the eccentricities are neglected, as has been the case throughout, the values of $r_0, r_1, r_2, r_3, r_4, r_5$, found from the first six equations by neglecting the terms multiplied by e^2, may be substituted in the succeeding equations, which will then serve to determine r_6, r_8, r_9, &c. and these values of r_6, r_8, r_9, &c. being substituted in the terms multiplied by e^2, of the equations which determine r_1, r_2, r_3, r_4, &c. more accurate values of those quantities may be obtained. All the other coefficients of which the general symbol is r with a numerical index at foot, may then be obtained in succession without any difficulty.

$$\frac{d\lambda}{dt} = \frac{h}{r^2} - \frac{1}{r^2}\int \frac{dR}{d\lambda} dt$$

Let r denote the elliptic value of r, then

$$\frac{d\lambda}{dt} = \frac{h}{\mathrm{r}^2} + \frac{2h}{\mathrm{r}} \delta . \frac{1}{r} - \frac{1}{\mathrm{r}^2}\int \frac{dR}{d\lambda} dt$$

$$\frac{a^2}{\mathrm{r}^2} = 1 + \frac{e^2}{2} + \frac{3}{8} e^4 + 2e\left(1 + \frac{3e^2}{8}\right)\cos(nt - \varpi) + \frac{5e^2}{2}\left(1 + \frac{2}{15}e^2\right)\cos(2nt - 2\varpi)$$
$$+ \frac{13}{4} e^3 \cos(3nt - 3\varpi) + \frac{103}{24} e^4 \cos(4nt - 4\varpi)$$

$$\frac{a}{\mathrm{r}} = 1 + e\left(1 - \frac{e^2}{8}\right)\cos(nt - \varpi) + e^2\left(1 - \frac{e^2}{3}\right)\cos(2nt - 2\varpi)$$
$$+ \frac{9}{8} e^3 \cos(3nt - 3\varpi) + \frac{4}{3} e^4 \cos(4nt - 4\varpi)$$

Let R_n denote the coefficient of the cosine in the development of R which corresponds to the number n multiplied by e or $e_{,}$, &c. Thus

$$R_6 = \left\{ -\frac{3a}{a_{,}^2} + \frac{3a}{2a_{,}^2} b_{3,0} - \frac{a^2}{2a_{,}^3} b_{3,1} - \frac{a}{4a_{,}^2} b_{3,2} \right\}$$ See p. 31

Since $\frac{dR}{d\lambda} = \frac{dR}{d\varepsilon}$, $\frac{dR}{d\varepsilon}$ being the differential of R with respect to ε, considering $\varepsilon - \varpi$ and $\varepsilon - \nu$ constant

$$\lambda = n \left\{ 1 + 2r_0 \right\} t + \varepsilon$$

$$+ \left\{ 2\left\{ r_1\left(1 - \frac{e^2}{2}\right) + \frac{e^2}{2}(r_6 + r_8) \right\} - \frac{m_{,}}{\mu}\left\{ \left(1 + \frac{e^2}{2}\right)\frac{aR_1 n}{(n - n_{,})} - \frac{e^2}{n_{,}} a R_6 n + \frac{e^2 a R_8 n}{(2n - n_{,})} \right\} \right\}$$

$$\frac{n}{(n - n_{,})} \sin (nt - n_{,}t + \varepsilon - \varepsilon_{,})$$

$$+ \left\{ 2\left\{ r_2\left(1 - \frac{e^2}{2}\right) + \frac{e^2}{2}(r_9 + r_{12}) \right\} - \frac{m_{,}}{\mu}\left\{ \left(1 + \frac{e^2}{2}\right)\frac{aR_2 n}{(n - n_{,})} + \frac{2e^2 a R_9 n}{(3n - 2n_{,})} + \frac{2e^2 a R_{12} n}{(n - 2n_{,})} \right\} \right\}$$

$$\frac{n}{(2n - 2n_{,})} \sin (2nt - 2n_{,}t + 2\varepsilon - 2\varepsilon_{,})$$

$$+ \left\{ 2\left\{ r_3\left(1 - \frac{e^2}{2}\right) + \frac{e^2}{2}(r_{10} + r_{13}) \right\} - \frac{m_{,}}{\mu}\left\{ \left(1 + \frac{e^2}{2}\right)\frac{aR_3 n}{(n - n_{,})} + \frac{3e^2 a R_{10} n}{(4n - 3n_{,})} + \frac{3e^2 a R_{13} n}{(2n - 3n_{,})} \right\} \right\}$$

$$\frac{n}{(3n - 3n_{,})} \sin (3nt - 3n_{,}t + 3\varepsilon - 3\varepsilon_{,})$$

$$+ \left\{ 2\left\{ r_4\left(1 - \frac{e^2}{2}\right) + \frac{e^2}{2}(r_{11} + r_{14}) \right\} - \frac{m_{,}}{\mu}\left\{ \left(1 + \frac{e^2}{2}\right)\frac{aR_4 n}{(n - n_{,})} + \frac{4e^2 a R_{11} n}{(5n - 4n_{,})} + \frac{4e^2 a R_{14} n}{(3n - 4n_{,})} \right\} \right\}$$

$$\frac{n}{(4n - 4n_{,})} \sin (4nt - 4n_{,}t + 4\varepsilon - 4\varepsilon_{,})$$

$$+ \left\{ 2\left(r_6 + \frac{r_1}{2}\right) - \frac{m_{,}}{\mu}\left\{ -\frac{aR_6 n}{n_{,}} + \frac{aR_1 n}{(n - n_{,})} \right\} \right\} \frac{e}{n_{,}} \sin (n_{,}t + \varepsilon_{,} - \varpi)$$

$$+ \left\{ 2\left(r_8 + \frac{r_1}{2}\right) - \frac{m_{,}}{\mu}\left\{ \frac{aR_8 n}{(2n - n_{,})} + \frac{aR_1 n}{(n - n_{,})} \right\} \right\} \frac{ne}{(n - n)} \sin (2nt - n_{,}t + 2\varepsilon - \varepsilon_{,} - \varpi)$$

$$+ \left\{ 2\left(r_9 + \frac{r_2}{2}\right) - \frac{m_{,}}{\mu}\left\{ \frac{2aR_9 n}{(3n - 2n_{,})} + \frac{aR_2 n}{(n - n_{,})} \right\} \right\} \frac{ne}{(3n - 2n_{,})} \sin (3nt - 2nt + 3\varepsilon - 2\varepsilon_{,} - \varpi)$$

$$+ \left\{ 2\left(r_{10} + \frac{r_3}{2}\right) - \frac{m_{,}}{\mu}\left\{ \frac{3aR_{10} n}{(4n - 3n_{,})} + \frac{aR_3 n}{(n - n_{,})} \right\} \right\} \frac{ne}{(4n - 3n_{,})} \sin (4nt - 3n_{,}t + 4\varepsilon - 3\varepsilon_{,} - \varpi)$$

Expression for the longitude.

$$+\left\{2\left(r_{11}+\frac{r_4}{2}\right)-\frac{m_{\prime}}{\mu}\left\{\frac{4\,a\,R_{11}\,n}{(5\,n-4\,n_{\prime})}+\frac{a\,R_4\,n}{(n-n_{\prime})}\right\}\right\}\frac{n\,e}{(5\,n-4\,n_{\prime})}\sin\left(5\,n\,t-4\,n_{\prime}\,t+5\,\varepsilon-4\,\varepsilon_{\prime}-\varpi\right)$$

$$+\left\{2\left(r_{12}+\frac{r_2}{2}\right)-\frac{m_{\prime}}{\mu}\left\{\frac{2\,a\,R_{12}\,n}{(n-2\,n_{\prime})}+\frac{a\,R_2\,n}{(n-n_{\prime})}\right\}\right\}\frac{n\,e}{(n-2\,n_{\prime})}\sin\left(n\,t-2\,n_{\prime}\,t+\varepsilon-2\,\varepsilon_{\prime}+\varpi\right)$$

$$+\left\{2\left(r_{13}+\frac{r_3}{2}\right)-\frac{m_{\prime}}{\mu}\left\{\frac{3\,a\,R_{13}\,n}{(2\,n-3\,n_{\prime})}+\frac{a\,R_3\,n}{(n-n_{\prime})}\right\}\right\}\frac{n\,e}{(2\,n-3\,n_{\prime})}\sin\left(2\,n\,t-3\,n_{\prime}\,t+2\,\varepsilon-3\,\varepsilon_{\prime}+\varpi\right)$$

$$+\left\{2\left(r_{14}+\frac{r_4}{2}\right)-\frac{m_{\prime}}{\mu}\left\{\frac{4\,a\,R_{14}\,n}{(3\,n-4\,n_{\prime})}+\frac{a\,R_4\,n}{(n-n_{\prime})}\right\}\right\}\frac{n\,e}{(3\,n-4\,n_{\prime})}\sin\left(3\,n\,t-4\,n_{\prime}\,t+3\,\varepsilon-4\,\varepsilon_{\prime}+\varpi\right)$$

$$+\,2\,r_{15}\frac{n\,e_{\prime}}{n_{\prime}}\sin\left(n_{\prime}\,t+\varepsilon_{\prime}-\varpi_{\prime}\right)$$

$$+\left\{2\,r_{17}-\frac{2\,m_{\prime}\,a\,R_{17}\,n}{\mu\,(2\,n-n_{\prime})}\right\}\frac{n\,e_{\prime}}{(2\,n-n_{\prime})}\sin\left(2\,n\,t-n_{\prime}\,t+2\,\varepsilon-\varepsilon_{\prime}-\varpi_{\prime}\right)$$

$$+\left\{2\,r_{18}-\frac{3\,m_{\prime}\,a\,R_{18}\,n}{\mu\,(3\,n-2\,n_{\prime})}\right\}\frac{n\,e_{\prime}}{(3\,n-2\,n_{\prime})}\sin\left(3\,n\,t-2\,n_{\prime}\,t+3\,\varepsilon-2\,\varepsilon_{\prime}-\varpi_{\prime}\right)$$

$$+\left\{2\,r_{19}-\frac{4\,a\,R_{19}\,n}{\mu\,(4\,n-3\,n_{\prime})}\right\}\frac{n\,e_{\prime}}{(4\,n-3\,n_{\prime})}\sin\left(4\,n\,t-3\,n_{\prime}\,t+4\,\varepsilon-3\,\varepsilon_{\prime}-\varpi_{\prime}\right)$$

$$+\left\{2\,r_{20}-\frac{m_{\prime}\,a\,R_{20}\,n}{\mu\,(n-2\,n_{\prime})}\right\}\frac{n\,e_{\prime}}{(n-2\,n_{\prime})}\sin\left(n\,t-2\,n_{\prime}\,t+\varepsilon-2\,\varepsilon_{\prime}+\varpi_{\prime}\right)$$

$$+\left\{2\,r_{21}-\frac{2\,m_{\prime}\,a\,R_{21}\,n}{\mu\,(2\,n-3\,n_{\prime})}\right\}\frac{n\,e_{\prime}}{(2\,n-3\,n_{\prime})}\sin\left(2\,n\,t-3\,n_{\prime}\,t+2\,\varepsilon-2\,\varepsilon_{\prime}+\varpi_{\prime}\right)$$

$$+\left\{2\,r_{22}-\frac{3\,m_{\prime}\,a\,R_{22}\,n}{\mu\,(3\,n-4\,n_{\prime})}\right\}\frac{n\,e_{\prime}}{(3\,n-4\,n_{\prime})}\sin\left(3\,n\,t-4\,n_{\prime}\,t+3\,\varepsilon-4\,\varepsilon_{\prime}+\varpi_{\prime}\right)$$

$$+\left\{2\,r_{23}-\frac{4\,m_{\prime}\,a\,R_{23}\,n}{\mu\,(4\,n-5\,n_{\prime})}\right\}\frac{n\,e_{\prime}}{(4\,n-5\,n_{\prime})}\sin\left(4\,n\,t-5\,n_{\prime}\,t-4\,\varepsilon-5\,\varepsilon_{\prime}+\varpi_{\prime}\right)$$

$$+\left\{2\left(r_{24}+\frac{r_{12}}{2}\right)\frac{m_{\prime}}{\mu}\left\{\frac{a\,R_{24}\,n}{m_{\prime}}+\frac{2\,a\,R_{12}\,n}{(n-2\,n_{\prime})}\right\}\right\}\frac{n\,e^2}{n_{\prime}}\sin\left(2\,n_{\prime}\,t+2\,\varepsilon_{\prime}-2\,\varpi\right)$$

$$+\left\{2\left(r_{25}+\frac{r_6}{2}+\frac{r_{\prime}}{2}\right)-\frac{m_{\prime}}{\mu}\left\{\frac{a\,R_{25}\,n}{(n+n_{\prime})}-\frac{a\,R_6\,n}{n_{\prime}}+\frac{5\,a\,R_1\,n}{4\,(n-n_{\prime})}\right\}\right\}\frac{n\,e^2}{n_{\prime}}\sin\left(nt+n_{\prime}t+\varepsilon+\varepsilon_{\prime}-2\,\varpi\right)$$

$$+\left\{2\left(r_{27}+\frac{r_8}{2}+\frac{r_{\prime}}{2}\right)-\frac{m_{\prime}}{\mu}\left\{\frac{a\,R_{27}\,n}{(3\,n-n_{\prime})}+\frac{a\,R_8\,n}{(2\,n-n_{\prime})}+\frac{5\,a\,R_1\,n}{4\,(n-n_{\prime})}\right\}\right\}$$

$$\frac{n\,e^2}{(3\,n-n_{\prime})}\sin\left(3\,n\,t-n_{\prime}t+3\,\varepsilon-\varepsilon_{\prime}-2\varpi\right)$$

Expression for the longitude.

$$+\left\{2\left(r_{28}+\frac{r_9}{2}+\frac{r_2}{2}\right)-\frac{m_{,}}{\mu}\left\{\frac{2\,a\,R_{28}\,n}{(4\,n-2\,n_{,})}+\frac{2\,a\,R_9\,n}{(3\,n-2\,n_{,})}+\frac{5\,a\,R_2\,n}{4\,(n-n_{,})}\right\}\right\}$$

$$\frac{n\,e^2}{(4\,n-2\,n_{,})}\sin\,(4\,n\,t-2\,n_{,}\,t+4\,\varepsilon-2\,\varepsilon_{,}-2\,\varpi)$$

$$+\left\{2\left(r_{29}+\frac{r_{10}}{2}+\frac{r_3}{2}\right)-\frac{m_{,}}{\mu}\left\{\frac{3\,a\,R_{29}\,n}{(5\,n-3\,n_{,})}+\frac{3\,a\,R_{10}\,n}{(4\,n-3\,n_{,})}+\frac{5\,a\,R_3\,n}{4\,(n-n_{,})}\right\}\right\}$$

$$\frac{n\,e^2}{(5\,n-3\,n_{,})}\sin\,(5\,n\,t-3\,n_{,}t+5\,\varepsilon-3\,\varepsilon_{,}-2\,\varpi)$$

$$+\left\{2\left(r_{30}+\frac{r_{11}}{2}+\frac{r_4}{2}\right)-\frac{m_{,}}{\mu}\left\{\frac{4\,a\,R_{30}\,n}{(6\,n-4\,n_{,})}+\frac{4\,a\,R_{11}\,n}{(5\,n-4\,n_{,})}+\frac{5\,a\,R_4\,n}{4\,(n-n_{,})}\right\}\right\}$$

$$\frac{n\,e^2}{(6\,n-4\,n_{,})}\sin\,(6\,n\,t-4\,n_{,}t+6\,\varepsilon-4\,\varepsilon_{,}-2\,\varpi)$$

$$+\left\{2\left(r_{32}+\frac{r_{13}}{2}+\frac{r_3}{2}\right)-\frac{m_{,}}{\mu}\left\{\frac{3\,a\,R_{32}\,n}{(n-3\,n_{,})}+\frac{3\,a\,R_{13}\,n}{(2\,n-3\,n_{,})}+\frac{5\,a\,R_3\,n}{4\,(n-n_{,})}\right\}\right\}$$

$$\frac{n\,e^2}{(n-3\,n_{,})}\sin\,(n\,t-3\,n_{,}t+\varepsilon-3\,\varepsilon_{,}+2\,\varpi)$$

$$+\left\{2\left(r_{33}+\frac{r_{14}}{2}+\frac{r_4}{2}\right)-\frac{m_{,}}{\mu}\left\{\frac{4\,a\,R_{33}\,n}{(2\,n-4\,n_{,})}+\frac{4\,a\,R_{14}\,n}{(3\,n-4\,n_{,})}+\frac{5\,a\,R_4\,n}{4\,(n-n_{,})}\right\}\right\}$$

$$\frac{n\,e^2}{(2\,n-4\,n_{,})}\sin\,(2\,n\,t-4\,n_{,}t+2\,\varepsilon-4\,\varepsilon_{,}+2\,\varpi)$$

$$+\left\{2\left(r_{36}+\frac{r_{20}}{2}\right)-\frac{m_{,}}{\mu}\left\{\frac{a\,R_{36}\,n}{(2\,n-2\,n_{,})}+\frac{a\,R_{20}\,n}{(n-2\,n_{,})}\right\}\right\}\frac{n\,e\,e_{,}}{(2\,n-2\,n_{,})}\sin\,(2nt-2n_{,}t+2\varepsilon-2\varepsilon_{,}-\varpi+\varpi_{,})$$

$$+\left\{2\left(r_{37}+\frac{r_{21}}{2}\right)-\frac{m_{,}}{\mu}\left\{\frac{2\,a\,R_{37}\,n}{(3n-3n_{,})}+\frac{2\,a\,R_{21}\,n}{(2\,n-3n_{,})}\right\}\right\}\frac{n\,e\,e_{,}}{(3n-3n_{,})}\sin\,(3nt-3n_{,}t+3\varepsilon-3\varepsilon_{,}-\varpi+\varpi_{,})$$

$$+\left\{2\left(r_{38}+\frac{r_{22}}{2}\right)-\frac{m_{,}}{\mu}\left\{\frac{3\,a\,R_{38}\,n}{(4n-4n_{,})}+\frac{3\,a\,R_{22}\,n}{(3n-4n_{,})}\right\}\right\}\frac{n\,e\,e_{,}}{(4n-4n_{,})}\sin\,(4nt-4n_{,}t+4\varepsilon-4\varepsilon_{,}-\varpi+\varpi_{,})$$

$$+\left\{2\left(r_{39}+\frac{r_{23}}{2}\right)-\frac{m_{,}}{\mu}\left\{\frac{4\,a R_{39}\,n}{(5n-5n_{,})}+\frac{4\,a\,R_{23}\,n}{(4n-5n_{,})}\right\}\right\}\frac{n\,e\,e_{,}}{(5\,n-5\,n_{,})}\sin\,(5nt-5\,n_{,}t+5\,\varepsilon-5\varepsilon_{,}-\varpi+\varpi_{,})$$

$$+\left\{2\left(r_{43}+\frac{r_{18}}{2}\right)-\frac{m_{,}}{\mu}\left\{\frac{3\,a\,R_{43}n}{(2\,n-2\,n_{,})}+\frac{3\,a\,R_{18}\,n}{(3\,n-2\,n_{,})}\right\}\right\}\frac{n\,e\,e_{,}}{(2\,n-2\,n_{,})}\sin(2nt-2n_{,}t+2\varepsilon-2\varepsilon_{,}+\varpi-\varpi_{,})$$

$$+\left\{2\left(r_{44}+\frac{r_{19}}{2}\right)-\frac{m_{,}}{\mu}\left\{\frac{4\,a\,R_{44}\,n}{(3\,n-3\,n_{,})}+\frac{4\,a\,R_{19}\,n}{(4\,n-3\,n_{,})}\right\}\right\}\frac{n\,e\,e_{,}}{(3\,n-3\,n_{,})}\sin(3nt-3n_{,}t+3\varepsilon-3\varepsilon_{,}+\varpi-\varpi_{,})$$

$$+\left\{2\left(r_{46}+\frac{r_{20}}{2}\right)-\frac{m_{,}}{\mu}\left\{-\frac{a\,R_{46}\,n}{2\,n_{,}}+\frac{a\,R_{20}\,n}{(n-2\,n_{,})}\right\}\right\}\frac{n\,e\,e_{,}}{2\,n_{,}}\sin\,(2\,n_{,}t+2\,\varepsilon-\varpi-\varpi_{,})$$

Expression for the longitude.

$$+\left\{2\left(r_{49}+\frac{r_{17}}{2}\right)-\frac{m_{\prime}}{\mu}\left\{\frac{2\,a\,R_{49}\,n}{(3\,n-n_{\prime})}+\frac{2\,a\,R_{17}\,n}{(2\,n-n_{\prime})}\right\}\right\}\frac{n\,e\,e_{\prime}}{(3\,n-n_{\prime})}\sin\left(3\,n\,t-n_{\prime}t+3\,\varepsilon-\varepsilon_{\prime}-\varpi-\varpi_{\prime}\right)$$

$$+\left\{2\left(r_{50}+\frac{r_{18}}{2}\right)-\frac{m_{\prime}}{\mu}\left\{\frac{3\,a\,R_{50}\,n}{(4\,n-2\,n_{\prime})}+\frac{3\,a\,R_{18}\,n}{(3\,n-2\,n_{\prime})}\right\}\right\}\frac{n\,e\,e_{\prime}}{(4\,n-2\,n_{\prime})}\sin(4nt-2n_{\prime}t+4\varepsilon-2\varepsilon_{\prime}-\varpi-\varpi_{\prime})$$

$$+\left\{2\left(r_{51}+\frac{r_{19}}{2}\right)-\frac{m_{\prime}}{\mu}\left\{\frac{4\,a\,R_{51}\,n}{(5\,n-3\,n_{\prime})}+\frac{4\,a\,R_{19}\,n}{(4\,n-3\,n_{\prime})}\right\}\right\}\frac{n\,e\,e_{\prime}}{(5\,n-3\,n_{\prime})}\sin\left(5\,nt-3\,n_{\prime}t+5\varepsilon-3\varepsilon_{\prime}-\varpi-\varpi_{\prime}\right)$$

$$+\left\{2\left(r_{53}+\frac{r_{21}}{2}\right)-\frac{m_{\prime}}{\mu}\left\{\frac{2\,a\,R_{53}\,n}{(n-3\,n_{\prime})}+\frac{2\,a\,R_{21}\,n}{(2\,n-3\,n_{\prime})}\right\}\right\}\frac{n\,e\,e_{\prime}}{(n-3\,n_{\prime})}\sin\left(n\,t-3n_{\prime}t+e-3\varepsilon_{\prime}+\varpi+\varpi_{\prime}\right)$$

$$+\left\{2\left(r_{54}+\frac{r_{22}}{2}\right)-\frac{m_{\prime}}{\mu}\left\{\frac{3\,a\,R_{54}\,n}{(2n-4\,n_{\prime})}+\frac{3\,a\,R_{22}\,n}{(3\,n-4\,n_{\prime})}\right\}\right\}\frac{n\,e\,e_{\prime}}{(2\,n-4\,n_{\prime})}\sin(2nt-4n_{\prime}t+2\varepsilon-4\varepsilon_{\prime}+\varpi+\varpi_{\prime})$$

$$+\left\{2\left(r_{55}+\frac{r_{23}}{2}\right)-\frac{m_{\prime}}{\mu}\left\{\frac{4\,a\,R_{55}\,n}{(3\,n-5\,n_{\prime})}+\frac{4\,a\,R_{23}\,n}{(4\,n-5\,n_{\prime})}\right\}\right\}\frac{n\,e\,e_{\prime}}{(3\,n-5\,n_{\prime})}\sin(3nt-5n_{\prime}t+3\varepsilon-5\varepsilon_{\prime}+\varpi+\varpi_{\prime})$$

$$+\left\{2\,r_{58}-\frac{m_{\prime}}{\mu}\frac{a\,R_{58}\,n}{(n+n_{\prime})}\right\}\frac{n\,e_{\prime}^{2}}{(n+n_{\prime})}\sin\left(n\,t+n_{\prime}t+\varepsilon+\varepsilon_{\prime}-2\,\varpi_{\prime}\right)$$

$$+\left\{2\,r_{59}-\frac{m_{\prime}\,a\,R_{59}}{\mu}\right\}e_{\prime}^{2}\sin\left(2\,n\,t+2\,\varepsilon-2\,\varpi_{\prime}\right)$$

$$+\left\{2\,r_{60}-\frac{m_{\prime}}{\mu}\frac{3\,a\,R_{60}\,n}{(3\,n-n_{\prime})}\right\}\frac{n\,e_{\prime}^{2}}{(3\,n-n_{\prime})}\sin\left(3\,n\,t-n_{\prime}t+3\,\varepsilon-\varepsilon_{\prime}-2\,\varpi_{\prime}\right)$$

$$+\left\{2\,r_{61}-\frac{m_{\prime}}{\mu}\frac{4\,a\,R_{61}\,n}{(4\,n-2\,n_{\prime})}\right\}\frac{n\,e_{\prime}^{2}}{(4\,n-2\,n_{\prime})}\sin\left(4\,n\,t-2\,n_{\prime}t+4\,\varepsilon-2\,\varepsilon_{\prime}-2\,\varpi_{\prime}\right)$$

$$+\left\{2\,r_{62}-\frac{m_{\prime}}{\mu}\frac{5\,a\,R_{62}\,n}{(5\,n-3\,n_{\prime})}\right\}\frac{n\,e_{\prime}^{2}}{(5\,n-3\,n_{\prime})}\sin 5\,n\,t-3\,n_{\prime}t+5\,\varepsilon-3\,\varepsilon_{\prime}-2\,\varpi_{\prime})$$

$$+\left\{2\,r_{63}-\frac{m_{\prime}}{\mu}\frac{a\,R_{63}\,n}{(n-3\,n_{\prime})}\right\}\frac{n\,e_{\prime}^{2}}{(n-3\,n_{\prime})}\sin\left(n\,t-3\,n_{\prime}t+\varepsilon-3\,\varepsilon_{\prime}+2\,\varpi_{\prime}\right)$$

$$+\left\{2\,r_{64}-\frac{m_{\prime}}{\mu}\frac{2\,a\,R_{64}\,n}{(2\,n-4\,n_{\prime})}\right\}\frac{n\,e_{\prime}^{2}}{(2\,n-4\,n_{\prime})}\sin\left(2\,n\,t-4\,n_{\prime}t+2\,\varepsilon-4\,\varepsilon_{\prime}+2\,\varpi_{\prime}\right)$$

$$+\left\{2\,r_{65}-\frac{m_{\prime}}{\mu}\frac{3\,a\,R_{65}\,n}{(3\,n-5\,n_{\prime})}\right\}\frac{n\,e_{\prime}^{2}}{(3\,n-5\,n_{\prime})}\sin\left(3\,n\,t-5\,n_{\prime}t+3\,\varepsilon-5\,\varepsilon_{\prime}+2\,\varpi_{\prime}\right)$$

$$+\left\{2\,r_{66}-\frac{m_{\prime}}{\mu}\frac{4\,a\,R_{66}\,n}{(4\,n-6\,n_{\prime})}\right\}\frac{n\,e_{\prime}^{2}}{(4\,n-6\,n_{\prime})}\sin\left(4\,n\,t-6\,n_{\prime}t+4\,\varepsilon-6\,\varepsilon_{\prime}+2\,\varpi_{\prime}\right)$$

$$+\left\{2\,r_{67}-\frac{m_{\prime}}{\mu}\frac{5\,a\,R_{67}\,n}{(5\,n-7\,n_{\prime})}\right\}\frac{n\,e_{\prime}^{2}}{(5\,n-7\,n_{\prime})}\sin\left(5\,n\,t-7\,n_{\prime}t+5\,\varepsilon-7\,\varepsilon_{\prime}+2\,\varpi_{\prime}\right)$$

In order to convert the coefficients of the inequalities of the longitude into sexagesimal seconds, they must be multiplied by $\frac{1296000}{2\pi}$, the logarithm of which number is 5·3144251, the corresponding logarithm for centesimal seconds is 5·8038801.

$$\frac{\mathrm{d}R}{\mathrm{d}z} = m_{,} \left\{ \frac{z_{,}}{r_{,}^{3}} + \frac{z - z_{,}}{\{r^2 - 2rr_{,}' \cos(\lambda - \lambda'_{,}) + r_{,}^2\}^{\frac{3}{2}}} \right\}$$

If $z = 0$, and the products $e \tan \iota_{,}$, $e_{,} \tan \iota_{,}$ be neglected,

$$\frac{\mathrm{d}R}{\mathrm{d}z} = \frac{m_{,}}{a_{,}^2} \tan \iota_{,} \sin(n_{,}t + \varepsilon_{,} - \nu_{,}) \{1 - b_{3,0} - b_{3,1} \cos(nt - nt + \varepsilon - \varepsilon_{,}) - b_{3,2} \cos(2nt - 2n_{,}t + 2\varepsilon - 2\varepsilon_{,})\}$$

$$\frac{a^3 \mathrm{d}^2 s}{\mathrm{d}t^2} + \mu s + \frac{m_{,}}{a_{,}^2} a^2 \tan \iota_{,} \sin(n_{,}t + \varepsilon_{,} - \nu_{,}) \{1 - b_{3,0} - b_{3,1} \cos(nt - n_{,}t + \varepsilon - \varepsilon_{,}) - \&c.\} = 0$$

Expression for the tangent of the latitude.

$$s = -\frac{m_{,}}{\mu} \frac{n^2}{(n - n_{,})(n + n_{,})} \frac{a^2}{a_{,}^2} \tan \iota_{,} \{1 - b_{3,0}\} \sin(n_{,}t + \varepsilon_{,} - \nu_{,})$$

$$+ i \sin\Big((1 + l)nt + \varepsilon - \nu_{,}\Big)$$

$$- \frac{m_{,}}{\mu} \frac{n^2}{4n_{,}(2n - 2n_{,})} \frac{a^2}{a_{,}^2} \tan \iota_{,} b_{3,1} \sin(nt - 2n_{,}t + \varepsilon - 2\varepsilon_{,} + \nu_{,})$$

$$- \frac{m_{,}}{\mu} \frac{n^2}{2(n - n_{,})(3n - n_{,})} \frac{a^2}{a_{,}^2} \tan \iota_{,} b_{3,2} \sin(2nt - n_{,}t + 2\varepsilon - \varepsilon_{,} - \nu_{,})$$

$$+ \frac{m_{,}}{\mu} \frac{n^2}{2(n - 3n_{,})(3n - 3n_{,})} \frac{a^2}{a_{,}^2} \tan \iota_{,} b_{3,2} \sin(2nt - 3n_{,}t + 2\varepsilon - 3\varepsilon_{,} + \nu_{,})$$

$$- \frac{m_{,}}{\mu} \frac{n^2}{2(2n - 2n_{,})(4n - 2n_{,})} \frac{a^2}{a_{,}^2} \tan \iota_{,} b_{3,3} \sin(3nt - 2n_{,}t + 3\varepsilon - 2\varepsilon_{,} - \nu_{,})$$

$$+ \frac{m_{,}}{\mu} \frac{n^2}{2(2n - 4n_{,})(4n - 4n_{,})} \frac{a^2}{a_{,}^2} \tan \iota_{,} b_{3,4} \sin(3nt - 4n_{,}t + 3\varepsilon - 4\varepsilon_{,} + \nu_{,})$$

$$- \frac{m_{,}}{\mu} \frac{n^2}{2(3n - 3n_{,})(5n - 3n_{,})} \frac{a^2}{a_{,}^2} \tan \iota_{,} b_{3,4} \sin(4nt \quad 3n_{,}t + 4\varepsilon - 3\varepsilon_{,} - \nu_{,})$$

$$l(2 + l)i = \frac{m}{2\mu_{,}} \frac{a^2}{a_{,}^2} \tan \iota_{,} b_{3,1}$$

$$R_6 = -\frac{3a}{2a_{,}^2} + \frac{3a}{2a_{,}^2} b_{3,0} - \frac{a^2}{2a_{,}^3} b_{3,1} - \frac{a}{4a_{,}^2} b_{3,2}$$ (See p. 31.)

$$q_6 = -\frac{3a}{2a_{,}^2} + \frac{3}{2} \frac{a}{a_{,}^2} b_{3,0} - \frac{a^2}{2a_{,}^3} b_{3,1} - \frac{a}{4a_{,}^2} b_{3,2} - \frac{3 \,.\, 3a^2}{2a_{,}^3} \left\{ \frac{a}{a_{,}} b_{5,0} - \tfrac{1}{2} b_{5,1} \right\}$$

$$+ \frac{3}{2} \frac{a^3}{a_{,}^4} \left\{ \frac{a}{a_{,}} b_{5,1} - b_{5,0} - \tfrac{1}{2} b_{5,2} \right\} + \frac{3}{4} \frac{a^2}{a_{,}^3} \left\{ \frac{a}{a_{,}} b_{5,2} - \tfrac{1}{2} b_{5,1} - \tfrac{1}{2} b_{5,3} \right\}$$

$$q_6 = -\frac{3\,a}{2\,a_{\prime}^2} + \frac{3}{2}\frac{a}{a_{\prime}^2} b_{3,0} - \frac{a^2}{2\,a_{\prime}^3} b_{3,1} - \frac{a}{4\,a_{\prime}^2} b_{3,2} + \frac{3}{2}\frac{a^2}{a_{\prime}^3}\left\{\left(1 + \frac{a^2}{a_{\prime}^2}\right) b_{5,1} - 2\frac{a}{a_{\prime}} b_{5,0} - \frac{a}{a_{\prime}^2} b_{5,2}\right\}$$

$$- \frac{3\,a^3}{a_{\prime}^4}\left\{b_{5,0} - \tfrac{1}{2} b_{5,2}\right\} + \frac{3}{8}\frac{a^2}{a_{\prime}^3}\left\{b_{5,1} - b_{5,3}\right\}$$

$$= -\frac{3\,a}{2\,a_{\prime}^2} + \frac{3}{2}\frac{a}{a_{\prime}^2} b_{3,0} - \frac{a^2}{2\,a_{\prime}^3} b_{3,1} - \frac{a}{4\,a_{\prime}^2} b_{3,2} + \frac{3}{2}\frac{a^2}{a_{\prime 3}} b_{3,1} - \frac{a^2}{a_{\prime 3}} b_{3,1} + \frac{2}{4}\frac{a}{a_{\prime}^2} b_{3,2}$$

$$= -\frac{3\,a}{2\,a_{\prime}^2} + \frac{3}{2}\frac{a}{a_{\prime}^2} b_{3,0} - \frac{2\,a^2}{2\,a_{\prime}^3} b_{3,1} - \frac{a}{4\,a_{\prime}^2} b_{3,2}$$

The quantities of which the general symbol is q, and which refer to the terms in the development of R multiplied by the eccentricities, admit of similar reductions; so that

$$q_7 = -\frac{a^2}{a_{\prime}^3} b_{3,0} + \frac{a}{a_{\prime}^2} b_{3,1} \qquad\qquad q_{16} = -\frac{a}{2\,a_{\prime}^2} b_{3,2}$$

Considering only the terms in $2\int \mathrm{d}R + r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)$, of which the arguments are $n\,t + \varepsilon - \varpi$, and $n\,t + \varepsilon - \varpi_{\prime}$

$$2\int \mathrm{d}R + r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right) = m_{\prime}\, q_7\, e \cos\left(n\,t + \varepsilon - \varpi\right) + m_{\prime}\, q_{16}\, e_{\prime} \cos\left(n\,t + \varepsilon - \varpi_{\prime}\right)$$

$$= m_{\prime}\, \mathrm{q} \cos\left(n\,t + \varepsilon - \varpi_1\right)$$

provided

$$\mathrm{q} \cos \varpi_1 = q_7\, e \cos \varpi + q_{16}\, e_{\prime} \cos \varpi_{\prime}$$

$$\mathrm{q} \cos \varpi_1 = q_7\, e \sin \varpi + q_{16}\, e_{\prime} \sin \varpi_{\prime}$$

And if

$$\frac{a}{r} = 1 + r_0 + e \cos\left(n\,(1 + k)\,t + \varepsilon - \varpi_1\right) + \&c.$$

$$(1 + k)^2 (1 - 3\,r_0) - 1 + \frac{m_{\prime}\, a}{\mu\, e}\mathrm{q} = 0 \qquad\qquad k = \frac{3}{2} r_0 - \frac{m_{\prime}\, a}{2\,\mu\, e}\mathrm{q}$$

$$r_0 = \frac{m_{\prime}}{\mu}\left\{\frac{a^3}{a_{\prime}^3} b_{3,0} - \frac{a^2}{2\,a_{\prime}^2} b_{3,1}\right\}$$

$$\mathrm{q}^2 = q_7^2 e^2 + 2\,q_7\, q_{16}\, e\, e_{\prime} \cos(\varpi - \varpi_{\prime}) + q_{16}^2\, e_{\prime}^2$$

$$\mathrm{q} = q_7\, e + q_{16}\, e_{\prime} \cos(\varpi - \varpi_{\prime}) \text{ nearly}$$

$$\frac{\mathrm{q}}{e} = -\frac{a^2}{a_{\prime}^3} b_{3,0} + \frac{a}{a_{\prime}^2} b_{3,1} - \frac{a}{2\,a_{\prime}^2}\frac{e_{\prime}}{e} b_{3,2} \cos(\varpi - \varpi_{\prime})$$

$$k = \frac{m_{\prime}}{\mu}\left\{\frac{3\,a^3}{2\,a_{\prime}^3} b_{3,0} - \frac{3\,a^2}{4\,a_{\prime}^2} b_{3,1} + \frac{a^3}{2\,a_{\prime}^3} b_{3,0} - \frac{a^2}{2\,a_{\prime}^2} b_{3,1} + \frac{a^2}{4\,a_{\prime}^2} b_{3,2}\frac{e_{\prime}}{e}\cos(\varpi - \varpi_{\prime})\right\}$$

$$= \frac{m_{\prime}}{\mu}\left\{\frac{2\,a^3}{a_{\prime}^3} b_{3,0} - \frac{5}{4}\frac{a^2}{a_{\prime}^2} b_{3,1} + \frac{a^2}{4\,a_{\prime}^2} b_{3,2}\frac{e_{\prime}}{e}\cos(\varpi - \varpi_{\prime})\right\}$$

$$\frac{d\lambda}{dt} = n(1 + 2r_0)t + \varepsilon + 2(1 + r_0)e\cos\left(n(1 + k)t + \varepsilon - \varpi_1\right)$$

$$- \frac{m_{\prime}}{\mu} a R_{16} e_{\prime} \cos(nt + \varepsilon - \varpi_{\prime})$$

and neglecting the square of the disturbing force

$$\frac{d\lambda}{dt} = n(1 + 2r_0)t + \varepsilon + 2(1 + r_0)e\cos\left(n(1 + k)t + \varepsilon - \varpi_{\prime}\right)$$

$$- \frac{m_{\prime}}{\mu} a R_{16} e_{\prime} \cos\left(n(1 + k)t + \varepsilon - \varpi_{\prime}\right)$$

If

$$\mathrm{e}(1 + k)\cos\varpi_2 = (1 + r_0)e\cos\varpi_1 - \frac{m_{\prime}}{2\mu} a R_{16} e_{\prime}\cos\varpi_{\prime}$$

$$\mathrm{e}(1 + k)\sin\varpi_2 = (1 + r_0)e\sin\varpi_1 - \frac{m_{\prime}}{2\mu} a R_{16} e_{\prime}\sin\varpi_{\prime}$$

$$\lambda = n(1 + 2r_0)t + \varepsilon\, 2\,\mathrm{e}\sin\left(n(1 + k)t + \varepsilon - \varpi_2\right)$$

$$e(1 + r_0)\cos\varpi_1 = \mathrm{e}(1 + k)\cos\varpi_2 + \frac{m_{\prime}}{2\mu} a R_{16} e_{\prime}\cos\varpi_{\prime}$$

$$e(1 + r_0)\sin\varpi_1 = \mathrm{e}(1 + k)\sin\varpi_2 + \frac{m_{\prime}}{2\mu} a R_{16} e_{\prime}\sin\varpi_{\prime}$$

$$e(1 + r_0) = \mathrm{e}(1 + k)\left\{1 + \frac{m_{\prime}}{2\mu} a R_{16}\frac{e_{\prime}}{e}\cos(\varpi_2 - \varpi_{\prime})\right.$$

$$\cos\varpi_1 = \cos\varpi_2\left\{1 - \frac{m_{\prime}}{2\mu} a R_{16}\frac{e_{\prime}}{e}\cos(\varpi_2 - \varpi_{\prime})\right\} + \frac{m_{\prime}}{2\mu} a R_{16}\frac{e_{\prime}}{e}\cos\varpi_{\prime}$$

$$\sin\varpi_1 = \sin\varpi_2\left\{1 - \frac{m_{\prime}}{2\mu} a R_{16}\frac{e_{\prime}}{e}\cos(\varpi_2 - \varpi_{\prime})\right\} + \frac{m_{\prime}}{2\mu} a R_{16}\frac{e_{\prime}}{e}\sin\varpi_{\prime}$$

$$\sin(\varpi_2 - \varpi_1) = \frac{m_{\prime}}{2\mu} a R_{16}\frac{e_{\prime}}{e}\cos\varpi_{\prime},$$

therefore neglecting the square of the disturbing force

$$\frac{e}{a} = \frac{\mathrm{e}}{a}\left\{1 + k - r_0 + \frac{m_{\prime}}{2\mu} a R_{16}\frac{e_{\prime}}{e}\cos(\varpi - \varpi_{\prime})\right\}$$

$$= \frac{\mathrm{e}}{a}\left\{1 + \frac{m_{\prime}}{\mu}\left\{\frac{2a^3}{a_{\prime}^3} b_{3,0} - \frac{5}{4}\frac{a^2}{a_{\prime}^2} b_{3,1} - \frac{a^3}{a_{\prime}^3} b_{3,0} + \frac{a^2}{2a_{\prime}^2} b_{3,1}\right\}\right.$$

$$\left. + \frac{m_{\prime}}{\mu}\left\{\frac{a^2}{4a_{\prime}^2} b_{3,2} + \frac{3a^2}{4a_{\prime}^2} b_{3,0} - \frac{a}{4a_{\prime}} b_{3,1} - \frac{a^2}{8a_{\prime}^2} b_{3,2}\right\}\frac{e_{\prime}}{e}\cos(\varpi - \varpi_{\prime})\right\}$$

$$= \frac{\mathrm{e}}{a}\left\{1 + \frac{m_{\prime}}{\mu}\left\{\frac{a^3}{a_{\prime}^3} b_{3,0} - \frac{3}{4}\frac{a^2}{a_{\prime}^2} b_{3,1} - \frac{a^3}{4a_{\prime}^3} b_{3,1}\frac{e_{\prime}}{e}\cos(\varpi - \varpi_{\prime})\right\}\right\}$$

Let

$$n\left\{1+\frac{2\,m_{\prime}}{\mu}\left(\frac{a^3}{a_{\prime}^3}b_{3,0}-\frac{a^2}{2\,a_{\prime}^2}b_{3,1}\right)\right\}=\mathrm{n}$$

$$n\,(1+k)=\mathrm{n}\left\{1-\frac{m\;a^2}{4\,\mu_{\prime}a_{\prime}^2}b_{3,1}+\frac{m_{\prime}\,a^2\,e_{\prime}}{4\,\mu\,a_{\prime}^2\,e}b_{3,2}\cos\left(\varpi-\varpi_{\prime}\right)\right\}$$

$$\text{Let }\frac{\mu}{\mathrm{a}^3}=\mathrm{n}^2,\text{ then }a^3=\mathrm{a}^3\left\{1+\frac{4\,m_{\prime}}{\mu}\left(\frac{a^3}{a_{\prime}^3}b_{3,0}-\frac{a^2}{a_{\prime}^2}b_{3,1}\right)\right\}$$

$$e=\mathrm{a}\left\{1+\frac{4\,m_{\prime}}{3\,\mu}\left(\frac{a^3}{a_{\prime}^3}b_{3,0}-\frac{a^2}{a_{\prime}^2}b_{3,1}\right)\right\}$$

$$\lambda=\mathrm{n}\,t+\varepsilon+2\,\mathrm{e}\sin\left(\mathrm{n}\left\{1-\frac{m_{\prime}\,a^2}{4\,\mu\,a_{\prime}^2}b_{3,1}+\frac{m_{\prime}}{4\,\mu_{\prime}}\frac{a^2}{a_{\prime}^2}\frac{e_{\prime}}{e}b_{3,2}\cos\left(\varpi-\varpi_{\prime}\right)\right\}t+\varepsilon-\varpi_2\right)+\&c.$$

$$\frac{\mathrm{a}}{r}=1-\frac{m_{\prime}}{3\,\mu}\left\{\frac{a^3}{a_{\prime}^3}b_{3,0}-\frac{a^2}{2\,a_{\prime}^2}b_{3,1}\right\}$$

$$+\,\mathrm{e}\left\{1-\frac{m_{\prime}}{\mu}\left\{\frac{a^3}{a_{\prime}^3}b_{3,0}+\frac{a^2}{12\,a_{\prime}^2}b_{3,1}-\frac{a^2}{4\,a_{\prime}^2}b_{3,2}\frac{e_{\prime}}{e}\cos\left(\varpi-\varpi_{\prime}\right)\right\}\right\}$$

$$\cos\left(\mathrm{n}\left\{1-\frac{m_{\prime}}{4\,\mu}\frac{a^2}{a_{\prime}^2}b_{3,1}+\frac{m_{\prime}}{4\,\mu}\frac{a^2}{a_{\prime}^2}\frac{e_{\prime}}{e}b_{3,2}\cos\left(\varpi-\varpi_{\prime}\right)\right\}t+\varepsilon-\varpi_2\right)$$

$$+\frac{m_{\prime}}{\mu}\left\{\frac{3}{4}\frac{a^2}{a_{\prime}^2}b_{3,0}-\frac{a}{4\,a_{\prime}}b_{3,1}-\frac{a^2}{8\,a_{\prime}^2}b_{3,2}\right\}e_{\prime}\cos\left(n\,t+\varepsilon-\varpi_{\prime}\right)$$

$$\frac{r}{\mathrm{a}}=1+\frac{m_{\prime}}{3\,\mu}\left\{\frac{a^3}{a_{\prime}^3}b_{3,0}-\frac{a^2}{2\,a_{\prime}^2}b_{3,1}\right\}$$

$$-\,e\left\{1+\frac{m_{\prime}}{\mu}\left\{\frac{a^3}{a_{\prime}^3}b_{3,0}-\frac{5}{12}\frac{a^2}{a_{\prime}^2}b_{3,1}+\frac{a^2}{a_{\prime}^2}b_{3,2}\frac{e_{\prime}}{e}\cos\left(\varpi-\varpi_{\prime}\right)\right\}\right\}$$

$$\cos\left(\mathrm{n}\left\{1-\frac{m_{\prime}}{4\,\mu}\frac{a^2}{a_{\prime}^2}b_{3,1}+\frac{m_{\prime}}{4\,\mu}\frac{a^2}{a_{\prime}^2}\frac{e_{\prime}}{e}b_{3,2}\cos\left(\varpi-\varpi_{\prime}\right)\right\}t+\varepsilon-\varpi_2\right)$$

$$-\frac{m_{\prime}}{\mu}\left\{\frac{3}{4}\frac{a^2}{a_{\prime}^2}b_{3,0}-\frac{a}{4\,a_{\prime}}b_{3,1}-\frac{a^2}{8\,a_{\prime}^2}b_{3,2}\right\}e_{\prime}\cos\left(n\,t+\varepsilon-\varpi_{\prime}\right)+\&c.$$

If

$$\frac{a}{r}=1+r_0+e(1+f)\cos\left(n\,(1+k^{*})\,t+\varepsilon-\varpi\right)+e_{\prime}f_{\prime}\cos\left(n(1+k_{\prime})\,t+\varepsilon-\varpi_{\prime}\right)+\&c.$$

$$(1+f)\left\{(1+k)^2(1-3\,r_0)-1\right\}+\frac{m_{\prime}}{\mu}a\,q_7=0$$

* This quantity k must not be confounded with the quantity k above.

whence neglecting k^2, kf, &c.

$$k = \frac{3}{2} r_0 - \frac{m_{\prime}}{2\mu} a\, q_7$$

$$= \frac{m_{\prime}}{\mu} \left\{ \frac{a^3}{a_{\prime}^3} b_{3,0} - \frac{a^2}{2\, a_{\prime}^2} b_{3,1} \right\}$$

similarly

$$f_{\prime} \{(1 + k_{\prime})^2 - 1\} + \frac{m_{\prime}}{\mu} a\, q_{16} = 0$$

$$k_{\prime}^2 + 2\, k_{\prime} + \frac{m_{\prime}}{\mu} \frac{a}{f_{\prime}} q_{16} = 0, \qquad q_{16} = -\frac{a}{2\, a_{\prime}^2} b_{3,2}$$

$$k_{\prime} = -2 - \frac{m_{\prime}}{4\mu} \frac{a^2}{f_{\prime} a_{\prime}^2} b_{3,2} \qquad \text{or} + \frac{m_{\prime}}{4\mu} \frac{a^2}{a_{\prime}^2 f_{\prime}} b_{3,2}$$

$$\lambda = n\{1 + 2\, r_0\}\, t + \varepsilon + \frac{2\, e\,(1 + f)(1 + r_0)}{(1 + k)} \sin\left(n\,(1 + k)\, t + \varepsilon - \varpi\right)$$

$$+ 2\, e_{\prime} \left\{ \frac{f_{\prime}}{1 + k_{\prime}} - \frac{m_{\prime} a}{2\mu} R_{16} \right\} \sin\left(n\,(1 + k_{\prime})\, t + \varepsilon - \varpi_{\prime}\right) + \text{\&c. nearly}$$

If

$$\frac{e(1 + f)(1 + r_0)}{1 + k} = \mathrm{e}$$

$$(1 + f) = \mathrm{e}\,(1 + k - r_0)$$

If $\frac{f_{\prime}}{1 + k_{\prime}} - \frac{m_{\prime}}{2\mu} a\, R_{16} = 0$ and if $k_{\prime}^2$ be neglected

$$f_{\prime} = \frac{m_{\prime}}{4\mu} \frac{a^2}{a_{\prime}^2} b_{3,2} + \frac{m_{\prime}}{2\mu} a\, R_{16}$$

If $n\,(1 + 2\, r_0) = \mathrm{n}$, and $\mathrm{n}^2 = \frac{\mu}{\mathrm{a}^3}$

$$a = \mathrm{a} \left\{ 1 + \frac{4}{3} r_0 \right\}$$

$$\frac{\mathrm{a}}{r} = 1 - \frac{1}{3} r_0 + \mathrm{e} \left\{ 1 + k - \frac{7}{3} r_0 \right\} \cos\left(n\,(1 + k)\, t + \varepsilon - \varpi\right) + e_{\prime} f_{\prime} \cos\left(n\,(1 + k_{\prime})\, t + \varepsilon - \varpi_{\prime}\right)$$

$$= 1 - \frac{1}{3} r_0 + \mathrm{e} \left\{ 1 - \frac{5}{6} r_0 - \frac{m_{\prime}}{2\mu} a\, q_7 \right\} \cos\left(\mathrm{n}\,(1 + \frac{1}{2} r_0 - \frac{m_{\prime}}{2\mu} a\, q_7)\, t + \varepsilon - \varpi\right)$$

$$- e_{\prime} \left\{ \frac{m_{\prime}}{2\mu} a\, q_{16} - \frac{m_{\prime}}{2\mu} a\, R_{16} \right\} \cos\left(n\,(1 + k_{\prime})\, t + \varepsilon - \varpi_{\prime}\right)$$

$$\frac{r}{\mathrm{a}} = 1 + \frac{1}{3} r_0 - \mathrm{e} \left\{ 1 - \frac{1}{6} r_0 - \frac{m_{\prime}}{2\mu} a\, q_7 \right\} \cos\left(\mathrm{n}\,(1 - \frac{1}{2} r_0 - \frac{m_{\prime}}{2\mu} a\, q_7)\, t + \varepsilon - \varpi\right)$$

$$+ \mathrm{e} \left\{ \frac{m_{\prime}}{2\mu} a\, q_{16} - \frac{m_{\prime}}{2\mu} a\, R_{16} \right\} \cos\left(n\,(1 + k_{\prime})\, t + \varepsilon - \varpi_{\prime}\right)$$

$$\frac{r}{a}=1+\frac{1}{3}r_0-\mathrm{e}\left\{1-\frac{1}{6}r_0-\frac{m_{\prime}}{2\mu}aq_7\right\}\cos(\mathrm{n}t+\varepsilon-\varpi)+e_{\prime}\left\{\frac{m_{\prime}}{2\mu}aq_{16}-\frac{m_{\prime}}{2\mu}a\,R_{16}\right\}\cos(nt+\varepsilon-\varpi_{\prime})$$

$$-\mathrm{e}\left\{\frac{1}{2}r_0+\frac{m_{\prime}}{2\mu}aq_7\right\}n\,t\sin(\mathrm{n}\,t+\varepsilon-\varpi)-e_{\prime}\frac{m_{\prime}}{2\mu}a\,q_{16}\,n\,t\sin(n\,t+\varepsilon-\varpi_{\prime})$$

In the notation of the Mécanique Céleste

$$r_0=\frac{a^2}{2}\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right)\qquad a\,q_7=-\frac{a^3}{2}\left(\frac{\mathrm{d}^2\,A^{(0)}}{\mathrm{d}\,a^2}\right)-\frac{3\,a^2}{2}\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right)$$

$$a\,q_{16}=-\frac{1}{2}\left\{a^2\,a'\left(\frac{\mathrm{d}^2\,A^{(1)}}{\mathrm{d}\,a\,\mathrm{d}\,a'}\right)+2\,a^2\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right)+2\,a'a\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a'}\right)+4\,a\,A^{(1)}\right\}$$

$$=-\frac{1}{2}\left\{2\,a\,A-2\,a^2\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right)-a^3\left(\frac{\mathrm{d}^2\,A^{(1)}}{\mathrm{d}\,a^2}\right)\right\}$$

$$R_{16}=-\frac{1}{2}a'\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a'}\right)-A^{(1)}$$

$$=\frac{a}{2}\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right)-\frac{1}{2}\,A^{(1)}$$

$$\frac{\mu}{6\,m_{\prime}}r_0+\frac{a}{2}q_7=\frac{a^2}{12}\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right)-\frac{a^3}{4}\left(\frac{\mathrm{d}^2\,A^{(0)}}{\mathrm{d}\,a^2}\right)-\frac{3}{4}a^2\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right)$$

$$=-\frac{2\,a^2}{3}\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right)-\frac{a^3}{4}\left(\frac{\mathrm{d}^2\,A^{(0)}}{\mathrm{d}\,a^2}\right)=-f$$

$$\frac{a}{2}q_{16}-\frac{a}{2}R_{16}=-\frac{a\,A^{(1)}}{2}+\frac{a^2}{2}\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right)+\frac{a^3}{4}\left(\frac{\mathrm{d}^2\,A^{(1)}}{\mathrm{d}\,a^2}\right)+\frac{a\,A^{(1)}}{4}-\frac{a^2}{4}\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right)$$

$$=-\frac{a\,A^{(1)}}{4}+\frac{a^2}{4}\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right)\overset{*}{+}\frac{a^3}{4}\left(\frac{\mathrm{d}^2\,A^{(1)}}{\mathrm{d}\,a^2}\right)=-f'$$

$$\frac{\mu}{2\,m_{\prime}}r_0+a\,q_7=\frac{a^2}{4}\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right)-\frac{a^3}{4}\left(\frac{\mathrm{d}^2\,A^{(0)}}{\mathrm{d}\,a^2}\right)-\frac{3}{4}a^2\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right)$$

$$=-\frac{a^2}{2}\left(\frac{\mathrm{d}\,A^{(0)}}{\mathrm{d}\,a}\right)-\frac{a^3}{4}\left(\frac{\mathrm{d}^2\,A^{(0)}}{\mathrm{d}\,a^2}\right)=-\frac{C}{2}=\frac{a^2}{4\,a_{\prime}^2}b_{3,1}$$

$$-\frac{1}{2}a\,q_{16}=\frac{1}{2}\left\{a\,A-a^2\left(\frac{\mathrm{d}\,A^{(1)}}{\mathrm{d}\,a}\right)-\frac{a^3}{2}\left(\frac{\mathrm{d}^2\,A^{(1)}}{\mathrm{d}\,a^2}\right)\right\}=-\frac{D}{2}=\frac{a^2}{2\,a_{\prime}^2}b_{3,2}$$

These results evidently agree with those given in the Mécanique Céleste, vol. i. p. 279, with the exception of the sign in the value of f' marked with an asterisk, which I think requires alteration in that work.

Finally, neglecting the quantities multiplied by t, which may be made to de-

pend upon the secular inequalities of the constants ε, ϖ, &c., and the squares of the eccentricities.

$$\frac{r}{a} = 1 + \frac{m_{,}}{3\mu}\left\{\frac{a^3}{a_{,}^3}b_{3,0} - \frac{a^2}{2a_{,}^2}b_{3,1}\right\} - e\left\{1 + \frac{m_{,}}{\mu}\left\{\frac{a^3}{a_{,}^3}b_{3,0} - \frac{5}{12}\frac{a^2}{a_{,}^2}b_{3,1}\right\}\right\}\cos(nt + \varepsilon - \varpi)$$

$$- \left\{\frac{3}{4}\frac{a^2}{a_{,}^2}b_{3,0} - \frac{a}{4a_{,}}b_{3,1} + \frac{a^2}{8a_{,}^2}b_{3,2}\right\}e_{,}\cos(nt + \varepsilon - \varpi_{,})$$

$$-r_1\cos(nt - n_{,}t + \varepsilon - \varepsilon_{,}) - r_2\cos(2nt - 2n_{,}t + 2\varepsilon - 2\varepsilon_{,}) - r_3\cos(3nt - 3n_{,}t + 3\varepsilon - 3\varepsilon_{,}) - \text{\&c.}$$

$$- \{r_6 - r_1\}\, e\cos(n_{,}t + \varepsilon_{,} - \varpi) - \{r_8 - r_1\}\, e\cos(2nt - n_{,}t + 2\varepsilon - \varepsilon_{,} - \varpi)$$

$$- \{r_9 - r_2\}\, e\cos(3nt - 2n_{,}t + 3\varepsilon - 2\varepsilon_{,} - \varpi)$$

$$- \{r_{10} - r_3\}\, e\cos(4nt - 3n_{,}t + 4\varepsilon - 3\varepsilon_{,} - \varpi) - \{r_{11} - r_4\}\, e\cos(5nt - 4n_{,}t + 5\varepsilon - 4\varepsilon_{,} - \varpi)$$

$$- \{r_{12} - r_2\}\, e\cos(nt - 2n_{,}t + \varepsilon - 2\varepsilon_{,} + \varpi) - \{r_{13} - r_3\}\, e\cos(2nt - 3n_{,}t + 2\varepsilon - 3\varepsilon_{,} + \varpi)$$

$$- \{r_{14} - r_4\}\, e\cos(3nt - 4n_{,}t + 3\varepsilon - 4\varepsilon_{,} + \varpi) - r_{16}e_{,}\cos(n_{,}t + e_{,} - \varpi_{,}) - r_{17}e_{,}\cos(nt + \varepsilon - \varpi_{,}) - \text{\&c.}$$

The constant part of R

$$= m_{,}\left\{-\frac{b_{1,0}}{a_{,}} + \frac{a}{2a_{,}^2}\left(\sin^2\frac{\iota_{,}}{2} - \frac{e^2 + e_{,}^2}{4}\right)b_{3,1} + \frac{a^2}{4a_{,}^2}b_{3,2}\cos(\varpi - \varpi_{,})\right\}$$ See p. 29.

If this quantity $= - F$ according to the notation of the Théor. Anal. vol. i. p. 336.

$$d\varepsilon = (1 - \sqrt{1 - e^2})\, d\varpi - \frac{2a^2 n}{\mu}\left(\frac{dF}{da}\right)dt$$

$$d\varpi = an\frac{\sqrt{1 - e^2}}{\mu e}\left(\frac{dF}{de}\right)dt$$

$$\frac{d\varepsilon - d\varpi}{dt} = \frac{m_{,}}{\mu}\left\{\frac{2a^2}{a_{,}^3}b_{3,0} - \frac{5}{4}\frac{a^2}{a_{,}^2}b_{3,1} + \frac{a^2}{4a_{,}^2}b_{3,2}\frac{e_{,}}{e}\cos(\varpi - \varpi_{,})\right\}$$

Let, as hitherto, (Phil. Trans. for 1830. p. 336.)

$$r = \frac{h^2\sqrt{1 + s^2}}{\mu\cos\iota^2\{\sqrt{1 + s^2} + e\cos(\lambda' - \varpi)\}} = a\{1 - e'\cos(v - a)\}$$

Fig. 1.

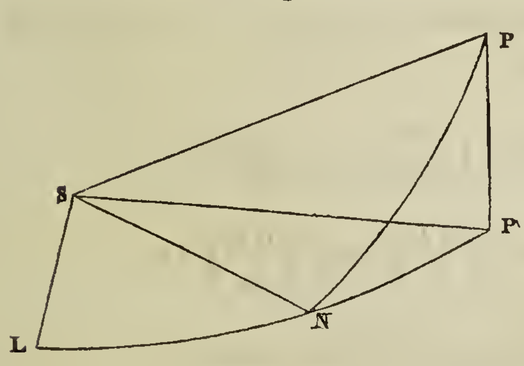
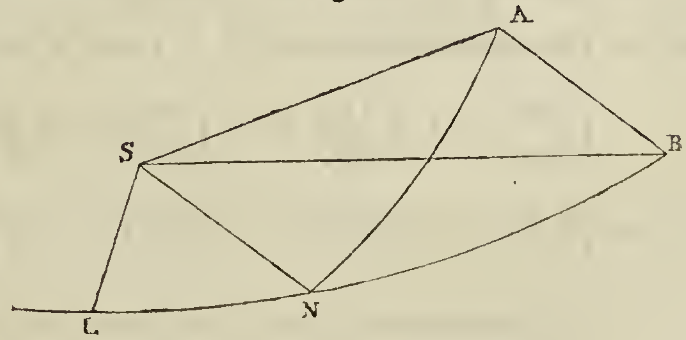


Fig. 2.



Let P be the place of the planet, P' its projection on the fixed plane L N P' (fig. 1 & 2.) S N the line of nodes, S L the line from which longitudes are reckoned. The angle L S P' $= \lambda'$. Let S A be the line of apsides. (fig. 2.)

In the notation of M. de Pontécoulant, vol. i. p. 316, the angle A S N $= g, \frac{\varepsilon - \varpi}{n} = l,$ M. de Pontécoulant has given expressions for the variations of the constants a, g, e', l, ι and ν in terms of the partial differences of the quantity R with regard to these quantities. It is easy from these to find similar expressions for the variations or differentials with regard to the time of the constants $a, \varpi, e, \varepsilon, \iota$ and ν.

Let S A B be a plane cutting the plane of the orbit at right angles, so that the angle S A B $= 90°$, A N B $= \iota$, B S N $= \varpi - \nu$

$$\frac{\mathrm{d}\, r^2}{\mathrm{d}\, t^2} + \frac{h^2}{r^2 \cos \iota^2} - \frac{2\mu}{r} + \frac{\mu}{a} = 0$$

$$r = a \{1 - e' \cos (v - \alpha)\}$$

When r is a maximum or minimum $\frac{\mathrm{d}\, r}{\mathrm{d}\, t} = 0$,

$$\frac{a\, h^2}{\mu \cos \iota^2} - 2\, a\, r + r^2 = 0, \quad \text{whence } r = a \pm \sqrt{a - \frac{h^2}{\mu \cos \iota^2}}$$

$$r = a\,(1 \pm e')$$

$$\frac{h^2}{\mu \cos \iota^2} = a\,(1 - e'^2)$$

By the equation of p. 336, line 12, (Phil. Trans. 1830.)

$$\frac{h^2}{\mu \cos \iota^2} = a \left(1 - e^2 + e^2 \sin^2 \iota \sin^2 (\nu - \varpi)\right)$$

$$e'^2 = e^2 \{1 - \sin^2 \iota \sin^2 (\nu - \varpi)\} * = e^2 \cos^2 \text{A S B}$$

Considering R first as a function of the quantities a, g, e', l, ι and ν, and then of the quantities $a, \varpi, e, \varepsilon, \iota$ and ν, we have

$$\left(\frac{\mathrm{d}\, R}{\mathrm{d}\, a}\right) \mathrm{d}\, a + \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, g}\right) \mathrm{d}\, g + \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, e'}\right) \mathrm{d}\, e' + \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, l}\right) \mathrm{d}\, l + \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, \iota}\right) \mathrm{d}\, \iota + \frac{\mathrm{d}\, R}{\mathrm{d}\, \nu} \mathrm{d}\, \nu$$

$$= \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, a}\right) \mathrm{d}\, a + \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, \varpi}\right) \mathrm{d}\, \varpi + \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, e}\right) \mathrm{d}\, e + \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, \varepsilon}\right) \mathrm{d}\, \varepsilon + \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, \iota}\right) \mathrm{d}\, \iota + \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, \nu}\right) \mathrm{d}\, \nu$$

* The equation I gave, Phil. Trans. for 1830, p. 336, line 17, is not correct.

By means of this equation, the equations

$$\varepsilon - \varpi = n\,l, \quad e'^2 = e^2\{1 - \sin^2\iota \sin^2(\nu - \varpi)\}$$

and the equations given by M. DE PONTÉCOULANT, vol. i. p. 328, the values of $\frac{d\,a}{d\,t}$, $\frac{d\,\varpi}{d\,t}$, $\frac{d\,e}{d\,t}$, $\frac{d\,\varepsilon}{d\,t}$, $\frac{d\,\iota}{d\,t}$ and $\frac{d\,\nu}{d\,t}$ may be easily obtained in terms of the quantities a, ϖ, e, ε, ι, ν, and the partial differential coefficients $\left(\frac{d\,R}{d\,a}\right)$, $\left(\frac{d\,R}{d\,\varpi}\right)$, $\left(\frac{d\,R}{d\,e}\right)$, $\left(\frac{d\,R}{d\,\varepsilon}\right)$, $\left(\frac{d\,R}{d\,\iota}\right)$ and $\left(\frac{d\,R}{d\,\nu}\right)$.

Substituting in the equations of p. 40, for q_n their values and neglecting e^2, $e_{\prime}^2$, $e\,e_{\prime}$, and $\sin^2\frac{\iota_{\prime}}{2}$, when a is less than $a_{\prime}$;

Expression for the reciprocal of the radius vector, when $a < a_{\prime}$.

$$\frac{\mathrm{a}}{r} = 1 - \frac{m_{\prime}}{\mu}\left\{\frac{a^3}{a_{\prime}^3}b_{3,0} - \frac{a^2}{a_{\prime}^2}b_{3,1}\right\} + \mathrm{e}\left\{1 - \frac{m_{\prime}}{\mu}\left(\frac{a^3}{a_{\prime}^3}b_{3,0} + \frac{a^2}{12\,a_{\prime}^2}b_{3,1}\right)\right\}\cos(n\,t + \varepsilon - \varpi)$$

$$+ \frac{m_{\prime}}{\mu}\frac{n^2}{(2n - n_{\prime})\,n_{\prime}}\left\{\frac{2\,n}{(n - n_{\prime})}\left(\frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}}b_{1,1}\right) + \frac{a^2}{a_{\prime}^2}\right.$$

$$\left. + \frac{a^2}{a_{\prime}^2}\left(\frac{a}{a_{\prime}}b_{3,1} - b_{3,0} - \tfrac{1}{2}b_{3,2}\right)\right\}\cos(n\,t - n_{\prime}\,t + \varepsilon - \varepsilon_{\prime}) \qquad [1]$$

$$+ \frac{m_{\prime}}{\mu}\frac{n^2}{(3\,n - 2\,n_{\prime})\,(n - 2\,n_{\prime})}\left\{\frac{2n}{(n - n_{\prime})}\frac{a}{a_{\prime}}b_{1,2}\right.$$

$$\left. - \frac{a^2}{a_{\prime}^2}\left(\frac{a}{a_{\prime}}b_{3,2} - \tfrac{1}{2}b_{3,1} - \tfrac{1}{2}b_{3,3}\right)\right\}\cos(2\,n\,t - 2\,n_{\prime}\,t + 2\,\varepsilon - 2\,\varepsilon_{\prime}) \qquad [2]$$

$$+ \frac{m_{\prime}}{\mu}\frac{n^2}{(4\,n - 3\,n_{\prime})\,(2\,n - 3\,n_{\prime})}\left\{\frac{2\,n}{(n - n_{\prime})}\frac{a}{a_{\prime}}b_{1,3}\right.$$

$$\left. - \frac{a^2}{a_{\prime}^2}\left(\frac{a}{a_{\prime}}b_{3,3} - \tfrac{1}{2}b_{3,2} - b_{3,4}\right)\right\}\cos(3\,n\,t - 3\,n_{\prime}\,t + 3\,\varepsilon - 3\,\varepsilon_{\prime}) \qquad [3]$$

$$+ \frac{m_{\prime}}{\mu}\frac{n^2}{(5\,n - 4\,n_{\prime})\,(3\,n - 4\,n_{\prime})}\left\{\frac{2\,n}{(n - n_{\prime})}\frac{a}{a_{\prime}}b_{1,4}\right.$$

$$\left. - \frac{a^2}{a_{\prime}^2}\left(\frac{a}{a_{\prime}}b_{3,4} - \tfrac{1}{2}b_{3,3} - \tfrac{1}{2}b_{3,5}\right)\right\}\cos(4\,n\,t - 4\,n_{\prime}\,t + 4\,\varepsilon + 4\,\varepsilon_{\prime}) \qquad [4]$$

$$+ \frac{m_{\prime}}{\mu}\frac{n^2}{(6\,n - 5\,n_{\prime})\,(4\,n - 5\,n_{\prime})}\left\{\frac{2\,n}{(n - n_{\prime})}\frac{a}{a_{\prime}}b_{1,5}\right.$$

$$\left. - \frac{a^2}{a_{\prime}^2}\left(\frac{a}{a_{\prime}}b_{3,5} - \tfrac{1}{2}b_{3,4} - \tfrac{1}{2}b_{3,6}\right)\right\}\cos(5\,n\,t - 5\,n_{\prime}\,t + 5\,\varepsilon - 5\,\varepsilon_{\prime}) \qquad [5]$$

* In the terms multiplied by $\frac{m_{\prime}}{\mu}$ the quantities a & a, e & e may be used indifferently.

Expression for the reciprocal of the radius vector, when $a < a_{\prime}$.

$$-\frac{n^2}{(n-n_{\prime})(n+n_{\prime})}\left\{\frac{3n_{\prime}^2}{2n^2}r_1+\frac{m_{\prime}}{\mu}\left\{\frac{3a^2}{2a_{\prime}^2}-\frac{3a^2}{2a_{\prime}^2}b_{3,0}+\frac{a^3}{2a_{\prime}^3}b_{3,1}-\frac{a^2}{4a_{\prime}^2}b_{3,2}\right\}\right\}e\cos(n_{\prime}t+\varepsilon_{\prime}-\varpi)\quad[6]$$

$$\begin{aligned}&+\frac{n^2}{(3n-n_{\prime})(n-n_{\prime})}\left\{\frac{3(2n-n_{\prime})^2}{2n^2}r_1-\frac{m_{\prime}}{\mu}\left\{\frac{4n}{(2n-n_{\prime})}\left\{\frac{a^2}{2a_{\prime}^2}-\frac{a^2}{2a_{\prime}^2}b_{3,0}-\frac{a^3}{2a_{\prime}^3}b_{3,1}+\frac{3a^2}{4a_{\prime}^2}b_{3,2}\right\}\right.\right.\\&\left.\left.+\frac{a^2}{2a_{\prime}^2}-\frac{a^2}{2a_{\prime}^2}b_{3,0}+\frac{3a^3}{2a_{\prime}^3}b_{3,1}-\frac{3a^2}{4a_{\prime}^2}b_{3,2}\right\}\right\}e\cos(2nt-n_{\prime}t+2\varepsilon-\varepsilon_{\prime}-\varpi)\quad[8]\end{aligned}$$

$$\begin{aligned}&+\frac{n^2}{(4n-2n_{\prime})(2n-2n_{\prime})}\left\{\frac{3(3n-2n_{\prime})^2}{2n^2}r_2-\frac{m_{\prime}}{\mu}\left\{\frac{6n}{(3n-2n_{\prime})}\left\{-\frac{a^2}{4a_{\prime}^2}b_{3,1}-\frac{a^3}{2a_{\prime}^3}b_{3,2}+\frac{3a^2}{4a_{\prime}^2}b_{3,3}\right\}\right.\right.\\&\left.\left.-\frac{a^2}{2a_{\prime}^2}b_{3,1}+\frac{5}{2}\frac{a^3}{a_{\prime}^3}b_{3,2}-\frac{3}{2}\frac{a^2}{a_{\prime}^2}b_{3,3}\right\}\right\}e\cos(3nt-2n_{\prime}t+3\varepsilon-2\varepsilon_{\prime}-\varpi)\quad[9]\end{aligned}$$

$$\begin{aligned}&+\frac{n^2}{(5n-3n_{\prime})(3n-3n_{\prime})}\left\{\frac{3(4n-3n_{\prime})^2}{2n^2}r_3-\frac{m_{\prime}}{\mu}\left\{\frac{8n}{(4n-3n_{\prime})}\left\{-\frac{a^2}{4a_{\prime}^2}b_{3,2}-\frac{a^3}{2a_{\prime}^3}b_{3,3}+\frac{3a^2}{4a_{\prime}^2}b_{3,4}\right\}\right.\right.\\&\left.\left.-\frac{3a^2}{4a_{\prime}^2}b_{3,2}+\frac{7}{2}\frac{a^3}{a_{\prime}^3}b_{3,3}-\frac{9}{4}\frac{a^2}{a_{\prime}^2}b_{3,4}\right\}\right\}e\cos(4nt-3n_{\prime}t+4\varepsilon-3\varepsilon_{\prime}-\varpi)\quad[10]\end{aligned}$$

$$\begin{aligned}&+\frac{n^2}{(6n-4n_{\prime})(4n-4n_{\prime})}\left\{\frac{3(5n-4n_{\prime})^2}{2n^2}r_4-\frac{m_{\prime}}{\mu}\left\{\frac{10n}{(5n-4n_{\prime})}\left\{-\frac{a^2}{4a_{\prime}^2}b_{3,3}-\frac{a^3}{2a_{\prime}^3}b_{3,4}+\frac{3a^2}{4a_{\prime}^2}b_{3,5}\right\}\right.\right.\\&\left.\left.-\frac{a^2}{a_{\prime}^2}b_{3,3}+\frac{9}{2}\frac{a^3}{a_{\prime}^3}b_{3,4}-\frac{3a^2}{a_{\prime}^2}b_{3,5}\right\}\right\}e\cos(5nt-4n_{\prime}t+5\varepsilon-4\varepsilon_{\prime}-\varpi)\quad[11]\end{aligned}$$

$$\begin{aligned}&-\frac{n^2}{2n_{\prime}(2n-2n_{\prime})}\left\{\frac{3(n-2n_{\prime})^2}{2n^2}r_2-\frac{m_{\prime}}{\mu}\left\{\frac{2n}{(n-2n_{\prime})}\left\{\frac{3a^2}{4a_{\prime}^2}b_{3,1}-\frac{a^3}{2a_{\prime}^3}b_{3,2}-\frac{a^2}{4a_{\prime}^2}b_{3,3}\right\}\right.\right.\\&\left.\left.+\frac{3a^2}{2a_{\prime}^2}b_{3,1}-\frac{a^3}{a_{\prime}^3}b_{3,2}+\frac{a^2}{2a_{\prime}^2}b_{3,3}\right\}\right\}e\cos(nt-2n_{\prime}t+\varepsilon-2\varepsilon_{\prime}+\varpi)\quad[12]\end{aligned}$$

$$\begin{aligned}&+\frac{n^2}{(n-3n_{\prime})(3n-3n_{\prime})}\left\{\frac{3(2n-3n_{\prime})^2}{2n^2}r_3-\frac{m_{\prime}}{\mu}\left\{\frac{4n}{(2n-3n_{\prime})}\left\{\frac{3a^2}{4a_{\prime}^2}b_{3,2}-\frac{a^3}{2a_{\prime}^3}b_{3,3}-\frac{a^2}{4a_{\prime}^2}b_{3,4}\right\}\right.\right.\\&\left.\left.+\frac{9a^2}{4a_{\prime}^2}b_{3,2}-2\frac{a^3}{a_{\prime}^3}b_{3,3}+\frac{3a^2}{4a_{\prime}^2}b_{3,4}\right\}\right\}e\cos(2nt-3n_{\prime}t+2\varepsilon-3\varepsilon_{\prime}+\varpi)\quad[13]\end{aligned}$$

$$\begin{aligned}&+\frac{n^2}{(2n-4n_{\prime})(4n-4n_{\prime})}\left\{\frac{3(3n-4n_{\prime})^2}{2n^2}r_4-\frac{m_{\prime}}{\mu}\left\{\frac{6n}{(3n-4n_{\prime})}\left\{\frac{3a^2}{4a_{\prime}^2}b_{3,3}-\frac{a^3}{2a_{\prime}^3}b_{3,4}-\frac{a^2}{4a_{\prime}^2}b_{3,5}\right\}\right.\right.\\&\left.\left.+3\frac{a^3}{a_{\prime}^3}b_{3,3}-3\frac{a^3}{a_{\prime}^3}b_{3,4}+\frac{a^2}{a_{\prime}^2}b_{3,5}\right\}\right\}e\cos(3nt-4n_{\prime}t+3\varepsilon-4\varepsilon_{\prime}+\varpi)\quad[14]\end{aligned}$$

$$-\frac{m_{\prime}}{\mu}\,\frac{n^2}{(n-n_{\prime})\,(n+n_{\prime})2\,a_{\prime}^2}\,a^2\,b_{3,1}\,e_{\prime}\cos\,(n_{\prime}t+\varepsilon_{\prime}-\varpi_{\prime}) \qquad [15]$$

Expression for the reciprocal of the radius vector, when $a < a_{\prime}$.

$$-\frac{m_{\prime}}{\mu}\,\frac{n^2}{(n-n_{\prime})\,(3\,n-n_{\prime})}\left\{\frac{4\,n}{(2\,n-n_{\prime})}\left\{\frac{3\,a^2}{4\,a_{\prime}^2}b_{3,1}-\frac{a}{2\,a_{\prime}}b_{3,2}-\frac{a^2}{4\,a_{\prime}^2}b_{3,3}\right\}\right.$$

$$\left.-\frac{9}{4}\,\frac{a^2}{a_{\prime}^2}b_{3,1}+\frac{2\,a}{a_{\prime}}b_{3,2}-\frac{a^2}{4\,a_{\prime}^2}b_{3,3}\right\}e_{\prime}\cos\,(2\,n\,t-n_{\prime}t+2\,\varepsilon-\varepsilon_{\prime}-\varpi_{\prime}) \qquad [17]$$

$$-\frac{m_{\prime}}{\mu}\,\frac{n^2}{(2\,n-n_{\prime})\,(4\,n-2\,n_{\prime})}\left\{\frac{6\,n}{(3\,n-2\,n_{\prime})}\left\{\frac{3\,a^2}{4\,a_{\prime}^2}b_{3,2}-\frac{a}{2\,a_{\prime}}b_{3,3}-\frac{a^2}{4\,a_{\prime}^2}b_{3,4}\right\}\right.$$

$$\left.-\frac{3\,a^2}{a_{\prime}^2}b_{3,2}+\frac{3\,a}{a_{\prime}}b_{3,3}-\frac{a^2}{2\,a_{\prime}^2}b_{3,4}\right\}e_{\prime}\cos\,(3\,n\,t-2\,n_{\prime}t+3\,\varepsilon-2\,\varepsilon_{\prime}-\varpi_{\prime}) \qquad [18]$$

$$-\frac{m_{\prime}}{\mu}\,\frac{n^3}{(3\,n-2\,n_{\prime})\,(5\,n-3\,n_{\prime})}\left\{\frac{8\,n}{(4\,n-3\,n_{\prime})}\left\{\frac{3\,a^2}{4\,a_{\prime}^2}b_{3,3}-\frac{a}{2\,a_{\prime}}b_{3,4}-\frac{a^2}{4\,a_{\prime}^2}b_{3,5}\right\}\right.$$

$$\left.-\frac{15}{4}\,\frac{a^2}{a_{\prime}^2}b_{3,3}+\frac{4\,a}{a_{\prime}}b_{3,4}-\frac{3\,a^2}{4\,a_{\prime}^2}b_{3,5}\right\}e_{\prime}\cos\,(4\,n\,t-3\,n_{\prime}t+4\,\varepsilon-3\,\varepsilon_{\prime}-\varpi_{\prime}) \qquad [19]$$

$$+\frac{m_{\prime}}{\mu}\,\frac{n^2}{2\,n_{\prime}\,(2\,n-2\,n_{\prime})}\left\{\frac{2\,n}{(n-2\,n_{\prime})}\left\{\frac{2\,a^2}{a_{\prime}^2}-\frac{a^2}{2\,a_{\prime}^2}b_{3,0}-\frac{a}{2\,a_{\prime}}b_{3,1}+\frac{3\,a^2}{4\,a_{\prime}^2}b_{3,2}\right\}\right.$$

$$\left.+\frac{2\,a^2}{a_{\prime}^2}+\frac{a^2}{a_{\prime}^2}b_{3,0}-\frac{a}{a_{\prime}}b_{3,1}\right\}e_{\prime}\cos\,(n\,t-2\,n_{\prime}t+\varepsilon-\varepsilon_{\prime}+\varpi_{\prime}) \qquad [20]$$

$$-\frac{m_{\prime}}{\mu}\,\frac{n^2}{(n-3\,n_{\prime})\,(3\,n-3\,n_{\prime})}\left\{\frac{4\,n}{(2\,n-3\,n_{\prime})}\left\{-\frac{a^2}{4\,a_{\prime}^2}b_{3,1}-\frac{a}{2\,a_{\prime}}b_{3,2}+\frac{3\,a^2}{4\,a_{\prime}^2}b_{3,3}\right\}\right.$$

$$\left.+\frac{3\,a^2}{4\,a_{\prime}^2}b_{3,1}-\frac{2\,a}{a_{\prime}}b_{3,2}+\frac{3\,a^2}{4\,a_{\prime}^2}b_{3,3}\right\}e_{\prime}\cos\,(2\,n\,t-3\,n_{\prime}t+2\,\varepsilon-3\,\varepsilon_{\prime}+\varpi_{\prime}) \qquad [21]$$

$$-\frac{m_{\prime}}{\mu}\,\frac{n^2}{(2\,n-4\,n_{\prime})\,(4\,n-4\,n_{\prime})}\left\{\frac{6\,n}{(3\,n-4\,n_{\prime})}\left\{-\frac{a^2}{4\,a_{\prime}^2}b_{3,2}-\frac{a}{2\,a_{\prime}}b_{3,3}+\frac{3\,a^2}{4\,a_{\prime}^2}b_{3,4}\right\}\right.$$

$$\left.+\frac{a^2}{a_{\prime}^2}b_{3,2}-\frac{3\,a}{a_{\prime}}b_{3,3}+\frac{3\,a^2}{2\,a_{\prime}^2}b_{3,4}\right\}e_{\prime}\cos\,(3\,n\,t-4\,n_{\prime}t+3\,\varepsilon-4\,\varepsilon_{\prime}+\varpi_{\prime}) \qquad [22]$$

$$-\frac{m_{\prime}}{\mu}\,\frac{n^2}{(3\,n-5\,n_{\prime})\,(5\,n-5\,n_{\prime})}\left\{\frac{8\,n}{(4\,n-5\,n_{\prime})}\left\{-\frac{a^2}{4\,a_{\prime}^2}b_{3,3}-\frac{a}{2\,a_{\prime}}b_{3,4}+\frac{3\,a^2}{4\,a_{\prime}^2}b_{3,5}\right\}\right.$$

$$\left.+\frac{5}{4}\,\frac{a^2}{a_{\prime}^2}b_{3,3}-\frac{4\,a}{a_{\prime}}b_{3,4}+\frac{9}{4}\,\frac{a^2}{a_{\prime}^2}b_{3,5}\right\}e_{\prime}\cos\,(4\,n\,t-5\,n_{\prime}t+4\,\varepsilon-5\,\varepsilon_{\prime}+\varpi_{\prime}) \qquad [23]$$

Substituting in the equations of p. 45, for R_n their values and neglecting e^2, $e\,e_{\prime}$, $e_{\prime}^2$, and $\sin^2\frac{\iota_{\prime}}{2}$,

Expression for the longitude when $a < a_{\prime}$.

$$\lambda = \mathrm{n}\,t + \varepsilon + \mathrm{e}\sin(\mathrm{n}\,t + \varepsilon - \varpi)$$

$$+ \frac{n}{(n - n_{\prime})}\left\{2\,r_1 - \frac{m_{\prime}\,n}{\mu\,(n - n_{\prime})}\left(\frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}}b_{1,1}\right)\right\}\sin(n\,t - n_{\prime}t + \varepsilon - \varepsilon_{\prime}) \qquad [1]$$

$$+ \frac{n}{(2\,n - 2\,n_{\prime})}\left\{2\,r_2 + \frac{m_{\prime}\,n}{\mu\,(n - n_{\prime})}\,\frac{a}{a_{\prime}}\,b_{1,2}\right\}\sin(2\,n\,t - 2\,n_{\prime}t + 2\,\varepsilon - 2\,\varepsilon_{\prime}) \qquad [2]$$

$$+ \frac{n}{(3\,n - 3\,n_{\prime})}\left\{2\,r_3 + \frac{m_{\prime}\,n\,a}{\mu\,(n - n_{\prime})\,a_{\prime}}\,b_{1,3}\right\}\sin(3\,n\,t - 3\,n_{\prime}\,t + 3\,\varepsilon - 3\,\varepsilon_{\prime}) \qquad [3]$$

$$+ \frac{n}{(4\,n - 4\,n_{\prime})}\left\{2\,r_4 + \frac{m_{\prime}\,n\,a}{\mu\,(n - n_{\prime})\,a_{\prime}}\,b_{1,4}\right\}\sin(4\,n\,t - 4\,n_{\prime}\,t + 4\,\varepsilon - 4\,\varepsilon_{\prime}) \qquad [4]$$

$$+ \frac{n}{(5\,n - 5\,n_{\prime})}\left\{2\,r_5 + \frac{m_{\prime}\,n\,a}{\mu\,(n - n_{\prime})\,a_{\prime}}\,b_{1,5}\right\}\sin(5\,n\,t - 5\,n_{\prime}\,t + 5\,\varepsilon - 5\,\varepsilon_{\prime}) \qquad [5]$$

$$+ \frac{n}{n_{\prime}}\left\{2\left(r_6 + \frac{r_1}{2}\right) + \frac{m_{\prime}\,n}{\mu\,n_{\prime}}\left(-\frac{3\,a^2}{2\,a_{\prime}^2} + \frac{3\,a^2}{2\,a_{\prime}^2}\,b_{3,0} - \frac{a^3}{2\,a_{\prime}^3}\,b_{3,1} - \frac{a^2}{4\,a_{\prime}^2}\,b_{3,2}\right)\right.$$

$$\left. - \frac{m_{\prime}\,n}{\mu\,(n - n_{\prime})}\left(\frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}}\,b_{1,1}\right)\right\} e\sin(n_{\prime}\,t + \varepsilon - \varpi) \qquad [6]$$

$$+ \frac{n}{(2\,n - n_{\prime})}\left\{2\left(r_8 + \frac{r_1}{2}\right) - \frac{m_{\prime}\,n}{\mu\,(2\,n - n_{\prime})}\left(\frac{a^2}{2\,a_{\prime}^2} - \frac{a^2}{2\,a_{\prime}^2}\,b_{3,0} - \frac{a^3}{2\,a_{\prime}^3}\,b_{3,1} + \frac{3\,a^2}{4\,a_{\prime}^2}\,b_{3,2}\right)\right.$$

$$\left. - \frac{m_{\prime}\,n}{\mu\,(n - n_{\prime})}\left(\frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}}\,b_{1,1}\right)\right\} e\sin(2\,n\,t - n_{\prime}\,t + 2\,\varepsilon - \varepsilon_{\prime} - \varpi) \qquad [8]$$

$$+ \frac{n}{(3\,n - 2\,n_{\prime})}\left\{2\left(r_9 + \frac{r_2}{2}\right) - \frac{2\,m_{\prime}\,n}{\mu\,(3\,n - 2\,n_{\prime})}\left(-\frac{a^2}{4\,a_{\prime}^2}\,b_{3,1} - \frac{a^3}{2\,a_{\prime}^3}\,b_{3,2} + \frac{3\,a^2}{4\,a_{\prime}^2}\,b_{3,3}\right)\right.$$

$$\left. + \frac{m_{\prime}\,n\,a}{\mu\,(n - n_{\prime})\,a_{\prime}}\,b_{1,2}\right\} e\sin(3\,n\,t - 2\,n_{\prime}\,t + 3\,\varepsilon - 2\,\varepsilon_{\prime} - \varpi) \qquad [9]$$

$$+ \frac{n}{(4\,n - 3\,n_{\prime})}\left\{2\left(r_{10} + \frac{r_3}{2}\right) - \frac{3\,m_{\prime}\,n}{\mu\,(4\,n - 3\,n_{\prime})}\left(-\frac{a^2}{4\,a_{\prime}^2}\,b_{3,2} - \frac{a^3}{2\,a_{\prime}^3}\,b_{3,3} + \frac{3\,a^2}{4\,a_{\prime}^2}\,b_{3,4}\right)\right.$$

$$\left. + \frac{m_{\prime}\,n\,a}{\mu\,(n - n_{\prime})\,a_{\prime}}\,b_{1,3}\right\} e\sin(4\,n\,t - 3\,n_{\prime}\,t + 4\,\varepsilon - 3\,\varepsilon_{\prime} - \varpi) \qquad [10]$$

$$+ \frac{n}{(5\,n - 4\,n_{\prime})}\left\{2\left(r_{11} + \frac{r_4}{2}\right) - \frac{4\,m_{\prime}\,n}{\mu\,(5\,n - 4\,n_{\prime})}\left(-\frac{a^2}{4\,a_{\prime}^2}\,b_{3,3} - \frac{a^3}{2\,a_{\prime}^3}\,b_{3,4} + \frac{3\,a^2}{4\,a_{\prime}^2}\,b_{3,5}\right)\right.$$

$$\left. + \frac{m_{\prime}\,n\,a}{\mu\,(n - n_{\prime})\,a_{\prime}}\,b_{1,4}\right\} e\sin(5\,n\,t - 4\,n_{\prime}\,t + 5\,\varepsilon - 4\,\varepsilon_{\prime} - \varpi) \qquad [11]$$

$$+ \frac{n}{(n - 2\,n_{\prime})}\left\{2\left(r_{12} + \frac{r_2}{2}\right) - \frac{2\,m_{\prime}\,n}{\mu\,(n - 2\,n_{\prime})}\left(\frac{3\,a^2}{4\,a_{\prime}^2}\,b_{3,1} - \frac{a^3}{2\,a_{\prime}^3}\,b_{3,2} - \frac{a^2}{4\,a_{\prime}^2}\,b_{3,3}\right)\right.$$

$$\left. + \frac{m_{\prime}\,n\,a}{\mu\,(n - n_{\prime})\,a_{\prime}}\,b_{1,2}\right\} e\sin(n\,t - 2\,n_{\prime}\,t + \varepsilon - 2\,\varepsilon_{\prime} + \varpi) \qquad [12]$$

Expression for the longitude when $a < a_,$.

$$+\frac{n}{(3n-3n_,)}\left\{2\left(r_{13}+\frac{r_3}{2}\right)-\frac{3m_,n}{\mu(2n-3n_,)}\left(\frac{3a^2}{4a_,^2}b_{3,2}-\frac{a^3}{2a_,^3}b_{3,3}-\frac{a^2}{4a_,^2}b_{3,4}\right)\right.$$

$$\left.+\frac{m_,na}{\mu(n-n_,)a_,}b_{1,3}\right\}e\sin(2nt-3n_,t+2\varepsilon-3\varepsilon_,+\varpi) \qquad [13]$$

$$+\frac{n}{(3n-4n_,)}\left\{2\left(r_{14}+\frac{r_4}{2}\right)-\frac{4m_,n}{\mu(3n-4n_,)}\left(\frac{3a^2}{4a_,^2}b_{3,3}-\frac{a^3}{2a_,^3}b_{3,4}-\frac{a^2}{4a_,^2}b_{3,5}\right)\right.$$

$$\left.+\frac{m_,na}{\mu(n-n_,)a_,}b_{1,4}\right\}e\sin(3nt-4n_,t+3\varepsilon-4\varepsilon_,+\varpi) \qquad [14]$$

$$+\frac{2n}{n_,}r_{15}e_,\sin(n_,t+\varepsilon_,-\varpi_,) \qquad [15]$$

$$+\frac{n}{(2n-n_,)}\left\{2r_{17}-\frac{2m_,n}{\mu(2n-n_,)}\left(\frac{3a^2}{4a_,^2}b_{3,1}-\frac{a}{2a_,}b_{3,2}\right.\right.$$

$$\left.\left.-\frac{a^2}{4a_,^2}b_{3,3}\right)\right\}e_,\sin(2nt-n_,t+2\varepsilon-\varepsilon_,-\varpi_,) \qquad [17]$$

$$+\frac{n}{(3n-2n_,)}\left\{2r_{18}-\frac{3m_,n}{\mu(3n-2n_,)}\left(\frac{3a^2}{4a_,^2}b_{3,2}-\frac{a}{2a_,}b_{3,3}\right.\right.$$

$$\left.\left.-\frac{a^2}{4a_,^2}b_{3,4}\right)\right\}e_,\sin(3nt-2n_,t+3\varepsilon-2\varepsilon_,-\varpi_,) \qquad [18]$$

$$+\frac{n}{(4n-3n_,)}\left\{2r_{19}-\frac{4m_,n}{\mu(4n-3n_,)}\left(\frac{3a^2}{4a_,^2}b_{3,3}-\frac{a}{2a_,}b_{3,4}\right.\right.$$

$$\left.\left.-\frac{a^2}{4a_,^2}b_{3,3}\right)\right\}e_,\sin(4nt-3n_,t+4\varepsilon-3\varepsilon_,-\varpi_, \qquad [19]$$

$$+\frac{n}{(n-2n_,)}\left\{2r_{20}-\frac{m_,n}{\mu(n-2n_,)}\left(\frac{2a^2}{a_,^2}-\frac{a^2}{2a_,^2}b_{3,0}-\frac{a}{2a_,}b_{3,1}\right.\right.$$

$$\left.\left.+\frac{3}{4}\frac{a^2}{a_,^2}b_{3,2}\right)\right\}e_,\sin(nt-2n_,t+\varepsilon-2\varepsilon_,+\varpi_,) \qquad [20]$$

$$+\frac{n}{(2n-3n_,)}\left\{2r_{21}-\frac{m_,n}{\mu(2n-3n_,)}\left(-\frac{a^2}{4a_,^2}b_{3,1}-\frac{a}{2a_,}b_{3,2}\right.\right.$$

$$\left.\left.+\frac{3a^2}{4a_,^2}b_{3,3}\right)\right\}e_,\sin(2nt-3n_,t+2\varepsilon-3\varepsilon_,+\varpi_,) \qquad [21]$$

$$+\frac{n}{(3n-4n_,)}\left\{2r_{22}-\frac{m_,n}{\mu(3n-4n_,)}\left(-\frac{a^2}{4a_,^2}b_{3,2}-\frac{a}{2a_,}b_{3,3}\right.\right.$$

$$\left.\left.+\frac{3a^2}{4a_,^2}b_{3,4}\right)\right\}e_,\sin(3nt-4n_,t+3\varepsilon-4\varepsilon_,+\varpi_,) \qquad [22]$$

The expression for the tangent of the latitude has already been given, p. 49.

When the latitude is reckoned from the plane of the orbit of the planet P, the following terms must be added, in the general case where ι is not equal to zero, to that expression;

$$-\frac{n^2}{2(n-n_{\prime})(3n-n_{\prime})}\frac{a^2}{a_{\prime}^2}\tan\iota\, b_{3,1}\sin(2nt-n_{\prime}t+2\varepsilon-\varepsilon_{\prime}-\nu)$$

$$+\frac{n^2}{2(n-n_{\prime})(n+n_{\prime})}\frac{a^2}{a_{\prime}^2}\tan\iota\, b_{3,1}\sin(n_{\prime}t+\varepsilon_{\prime}-\nu_{\prime})$$

$$-\frac{n^2}{2(2n-2n_{\prime})(4n-2n_{\prime})}\frac{a^2}{a_{\prime}^2}\tan\iota\, b_{3,2}\sin(3nt-2n_{\prime}t+3\varepsilon-2\varepsilon_{\prime}-\nu)$$

$$-\frac{n^2}{4n_{\prime}(2n-2n_{\prime})}\frac{a^2}{a_{\prime}^2}\tan\iota\, b_{3,2}\sin(nt-2n_{\prime}t+\varepsilon-2\varepsilon_{\prime}+\nu)$$

$$-\frac{n^2}{2(3n-3n_{\prime})(5n-3n_{\prime})}\frac{a^2}{a_{\prime}^2}\tan\iota\, b_{3,3}\sin(4nt-3n_{\prime}t+4\varepsilon-3\varepsilon_{\prime}-\nu)$$

$$+\frac{n^2}{2(n-3n_{\prime})(3n-3n_{\prime})}\frac{a^2}{a_{\prime}^2}\tan\iota\, b_{3,3}\sin(2nt-3n_{\prime}t+2\varepsilon-3\varepsilon_{\prime}+\nu)$$

All the equations hitherto given apply to the case of an inferior disturbed by a superior planet, or when $a_{\prime} > a$, in order to render them applicable to the case when $a_{\prime} < a$ it is necessary to write a instead of $a_{\prime}$ in the denominator of the terms multiplied by $b_{1,n}$, and a^3 instead of $a_{\prime}^3$ in the denominator of the terms multiplied by $b_{3,n}$, in the disturbing function R, but the expressions for the quantities q are not the same in this case.

It will I think be admitted that the expressions which occur in the theory of the disturbances of the planets are more simple in terms of the quantities of which the general symbol is b, than in terms of the partial differential coefficients of the quantity called A in the notation of the Mécanique Céleste. The development of the disturbing function R in terms of the differential coefficients $\frac{dA}{da}$, $\frac{dA}{da_{\prime}}$, &c. admits of reductions, so that it may be expressed in terms of the differential coefficients of A with respect to a only. In this state it has been left by Laplace as may be seen, vol. ii. p. 12, but the coefficients of the terms multiplied by the squares and products of the eccentricities may be expressed very simply in terms of the quantities of which the general symbol is b, by means of reductions, of which two exam-

ples are given in the Théor. Anal. vol. i. p. 362. Similar reductions are applicable to the terms in $r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)$ multiplied by the first power of the eccentricities.

In Laplace's notation

$$b_{1,0} = \tfrac{1}{2} b_{\frac{1}{2}}^{(0)}, \quad b_{1,1} = b_{\frac{1}{2}}^{(1)}, \quad b_{1,2} = b_{\frac{1}{2}}^{(2)}, \quad b_{1,3} = b_{\frac{1}{2}}^{(3)}$$

$$b_{3,0} = \tfrac{1}{2} b_{\frac{3}{2}}^{(0)}, \quad b_{3,1} = b_{\frac{3}{2}}^{(1)}, \quad b_{3,2} = b_{\frac{3}{2}}^{(2)}, \quad b_{3,3} = b_{\frac{3}{2}}^{(3)}$$

$$b_{5,0} = \tfrac{1}{2} b_{\frac{5}{2}}^{(0)}, \quad b_{5,0} = b_{\frac{5}{2}}^{(1)}, \quad b_{5,2} = b_{\frac{5}{2}}^{(2)}, \quad b_{5,3} = b_{\frac{5}{2}}^{(3)}$$

$$\frac{a}{a_{\prime}} b_{3,0} - \tfrac{1}{2} b_{3,1} = -\tfrac{1}{2} \frac{\mathrm{d} b_{\frac{1}{2}}^{(0)}}{\mathrm{d}\alpha}, \qquad \frac{a}{a_{\prime}} b_{3,1} - b_{3,0} - \tfrac{1}{2} b_{3,2} = -\frac{\mathrm{d} b_{\frac{1}{2}}^{(1)}}{\mathrm{d}\alpha}$$

$$\frac{a}{a_{\prime}} b_{3,2} - \tfrac{1}{2} b_{3,1} - \tfrac{1}{2} b_{3,3} = -\frac{\mathrm{d} b_{\frac{1}{2}}^{(2)}}{\mathrm{d}\alpha}, \qquad \frac{a}{a_{\prime}} b_{3,3} - \tfrac{1}{2} b_{3,2} - \tfrac{1}{2} b_{3,4} = -\frac{\mathrm{d} b_{\frac{1}{2}}^{(3)}}{\mathrm{d}\alpha}$$

$$3\left\{\frac{a}{a_{\prime}} b_{5,0} - \tfrac{1}{2} b_{5,1}\right\} = -\tfrac{1}{2} \frac{\mathrm{d} b_{\frac{3}{2}}^{(0)}}{\mathrm{d}\alpha}, \qquad 3\left\{\frac{a}{a_{\prime}} b_{5,1} - b_{5,0} - \tfrac{1}{2} b_{5,2}\right\} = -\frac{\mathrm{d}. b_{\frac{3}{2}}^{(1)}}{\mathrm{d}\alpha}$$

$$3\left\{\frac{a}{a_{\prime}} b_{5,2} - \tfrac{1}{2} b_{5,1} - \tfrac{1}{2} b_{5,3}\right\} = -\frac{\mathrm{d}. b_{\frac{3}{2}}^{(2)}}{\mathrm{d}\alpha}, \qquad 3\left\{\frac{a}{a_{\prime}} b_{5,3} - \tfrac{1}{2} b_{5,2} - \tfrac{1}{2} b_{5,4}\right\} = -\frac{\mathrm{d}. b_{\frac{3}{2}}^{(3)}}{\mathrm{d}\alpha}$$

The numerical values of these quantities are given for the principal planets in the third volume of the Mécanique Céleste.

The following numerical examples will serve to explain the expressions given above, and to show their accuracy, the results agreeing exactly with those given in the Mécanique Céleste.

$$\frac{r}{\mathrm{a}} = 1 + \frac{m_{\prime}}{\mu}\left\{\frac{a^3}{a_{\prime}^3} b_{3,0} - \frac{a^2}{2a_{\prime}^2} b_{3,1}\right\} - \mathrm{e}\left\{1 + \frac{m_{\prime}}{\mu}\left\{\frac{{}^{*}a^3}{3a_{\prime}^3} b_{3,0} - \frac{5}{12}\frac{a^2}{a_{\prime}^2} b_{3,1}\right\}\right\}\cos(nt + \varepsilon - \varpi)$$

$$- r_1 \cos(nt - n_{\prime}t + \varepsilon - \varepsilon_{\prime}) \quad \text{See p. 55.}$$

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* The coefficient of $\cos(nt + \varepsilon - \varpi)$ p. 52 line 11, and p. 55 line 3, should be

$$-\mathrm{e}\left\{1 + \frac{m_{\prime}}{\mu}\left\{\frac{a^3}{3a_{\prime}^3} b_{3,0} - \frac{5}{12}\frac{a^2}{a_{\prime}^2} b_{3,1}\right\}\right\}.$$

$$r_1 = \frac{m_{\prime}}{\mu}\,\frac{n^2}{(2\,n - n_{\prime})\,n_{\prime}}\left\{\frac{2\,n}{(n - n_{\prime})}\left(\frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}}\,b_{1,1}\right) + \frac{a^2}{a_{\prime}^2} + \frac{a^2}{a_{\prime}^2}\left(\frac{a}{a_{\prime}}\,b_{3,1} - b_{3,0} - \frac{1}{2}\,b_{3,2}\right)\right\}$$ See p. 57.

$$\lambda = \mathrm{n}\,t + \varepsilon + \frac{n}{(n - n_{\prime})}\left\{2\,r_1 - \frac{m_{\prime}n}{\mu\,(n - n_{\prime})}\left(\frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}}\,b_{1,1}\right)\right\}\sin\,(n\,t - n_{\prime}t + \varepsilon - \varepsilon_{\prime})$$

In the theory of Jupiter disturbed by Saturn,

$$\frac{a}{a_{\prime}} = \cdot 54531725, \qquad b_{1,1} = \cdot 6206406, \qquad b_{3,0} = \frac{4\cdot 358387}{2} = 2\cdot 179193$$

$$b_{3,1} = 3\cdot 185493 \qquad b_{3,2} = 2\cdot 082131 \qquad \frac{m_{\prime}}{\mu} = \frac{1}{3359\cdot 4}$$

$$n = 337210\cdot 78 \qquad n_{\prime} = 135792\cdot 34$$

$$a = 5\cdot 20116636 \qquad e = \cdot 0480767$$

See Méc. Cél. vol. iii. p. 61 & 82.

Whence

$$\frac{a^3}{a_{\prime}^3}b_{3,0} = \cdot 353381, \qquad \frac{a^3}{2\,a_{\prime}^2}b_{3,1} = \cdot 473636 \qquad \frac{a^3}{a_{\prime}^3}b_{3,0} - \frac{a^2}{2\,a_{\prime}^2}b_{3,1} = -\cdot 120255$$

log. ·120255	= 9·0801033
log. 5·20116636	= 0·7161007
	9·7962040
log. $\frac{3\,\mu}{m_{\prime}}$	= 4·0033829
	5·7928211 = log. ·0000620613 minus

Laplace has . . . . . . ·0000620566 minus

See Méc. Cél. vol. iii. p. 121, line 5.

log. $\left\{\frac{5}{12}\,\frac{a^2}{a_{\prime}^2}b_{3,1} - \frac{a^3}{3\,a_{\prime}^3}b_{3,0}\right\}$	= 9·4423277
log. e	= 8·6819347
log. a	= 0·7161007
	8·8403631
log. $\frac{\mu}{m_{\prime}}$	= 3·5262617
	5·3141014 = log. ·0000206111 +

Laplace has . . . . ·0000206111 the sign omitted, Méc. Cél. vol. iii. p. 122, line 28.

Calculation of r_1, see p. 57.

$\log. n_{\prime} = 5{\cdot}1328751$

$\log. n = 5{\cdot}5279013$

$\underline{9{\cdot}6049738} = \log. {\cdot}4026928$

$1 - \frac{n_{\prime}}{n} = \quad {\cdot}5973072$

$\frac{a}{a_{\prime}} b_{3,1} - b_{3,0} - \frac{1}{2} b_{3,2} = - 1{\cdot}48315$

$\log. 1{\cdot}48315 = \quad {\cdot}1711851$

$\log. \frac{a^2}{a_{\prime}^2} \qquad = 9{\cdot}4732982$

$\underline{9{\cdot}6444333} = \log. {\cdot}441045$

$\log. 1{\cdot}5973072 = \quad {\cdot}2033884$

$\log. \ {\cdot}4026928 = 9{\cdot}6049738$

$\log. \ 3359{\cdot}4 \quad = 3{\cdot}5262617$

$\underline{3{\cdot}3346239}$

$r_1 = - {\cdot}0001301383$

$\frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}} b_{1,1} = - {\cdot}041075$

$\log. {\cdot}041075 = 8{\cdot}6135776$

$\log. \frac{n - n_{\prime}}{n} \quad = 0{\cdot}7761940$

$8{\cdot}8373836 = \log. {\cdot}0687676$

$\quad 2$

$- {\cdot}1375352$

$- {\cdot}441045$

$- {\cdot}5785802$

$\frac{a^2}{a_{\prime}^2} = {\cdot}297371$

$- \ {\cdot}281209$

$\log. {\cdot}281209 = 9{\cdot}4490293$

$3{\cdot}3346259$

$6{\cdot}1144054 = \log. {\cdot}0001301383$

$\log. a = \ {\cdot}7161007$

$\underline{6{\cdot}8305061} = \log. {\cdot}00067687\mathrm{l} +$

Laplace has . . . . ${\cdot}000676876$ + p. 120.

similarly $r_2 = {\cdot}000556924$, $\quad r_3 = {\cdot}00005809$, $\quad r_4 = {\cdot}000015045$, $\quad r_5 = {\cdot}0000049785$

Calculation of the coefficient of sin $(n\,t - n_{\prime}t + \varepsilon - \varepsilon_{\prime})$ in the value of λ. See p. 59.

$2\,r_1 = - {\cdot}0002602766$

$\qquad {\cdot}0000204702$

$\log. \quad {\cdot}0002398064 = 6{\cdot}3798601$

$\log. \frac{n - n_{\prime}}{n} \qquad = 9{\cdot}7761940$

$6{\cdot}6036661$

$5{\cdot}8038801$ see p. 49.

$\underline{2{\cdot}4075461} = \log. 255{\cdot}591$ minus.

$\log. \frac{n}{n - n_{\prime}} \left\{ \frac{a^2}{a_{\prime}^2} - \frac{a}{a_{\prime}} b_{1,1} \right\} = 8{\cdot}8373836$

$\log. 3359{\cdot}4 = 3{\cdot}5262617$

$\underline{5{\cdot}3111219} = \log. {\cdot}0000204702$

Laplace has . . . 255·5917 the sign omitted, p. 120.

In the notation of the Méc. Cél. vol. iii. p. 120.

$n = \overset{\text{iv}}{n}, \qquad n_{\prime} = \overset{\text{v}}{n}.$

The reader is requested to make the following corrections.

The expression for r^{-3} Phil. Trans. 1830, p. 345, should be

$$r^{-3} = a^{-3}\left\{1 + \frac{3}{2}e^2\left(1 + \frac{5}{4}e^2\right) + 3e\left(1 + \frac{9}{8}e^2\right)\cos(nt - \varpi)\right.$$

$$+ \frac{9}{2}e^2\left(1 + \frac{7}{9}e^2\right)\cos(2nt - 2\varpi) + \frac{53}{8}e^3\cos(3nt - 3\varpi)$$

$$\left. + \frac{77}{8}e^4\cos(4nt - 4\varpi)\right.$$

The third term (multiplied by d t) in the equations of p. 23, line 4 and 6, and the second term in the equation of the same page, line 11, must be suppressed.

p. 30, line 21, read $+ m_{\prime}\left\{\frac{a}{a_{\prime}^2}\left(\cos^2\frac{\iota_{\prime}}{2} - \frac{e^2 + e_{\prime}^2}{2}\right)\right.$ for $m_{\prime}\left\{-\frac{a}{a_{\prime}^2}\left(\cos^2\frac{\iota_{\prime}}{2} - \frac{e^2 + e_{\prime}^2}{2}\right)\right.$

p. 32, line 5, read $+ m_{\prime}\left\{\frac{2a}{a_{\prime}^2} - \frac{a}{2a_{\prime}^2}b_{3,0} - \frac{a_{\prime}^2}{2a^3}b_{3,1}\right.$ &c. for $+ m_{\prime}\left\{\frac{a}{2a_{\prime}^2} - \frac{a}{2a_{\prime}^2}b_{3,0} - \frac{a_{\prime}^2}{2a_{\prime}^3}b_{3,1}\right.$ &c.

p. 39, line 9, read $-\frac{21}{8}\frac{a^2}{a_{\prime}^3}ee_{\prime}\cos(2t - z + x)$ for $-\frac{27}{8}\frac{a^2}{a_{\prime}^3}ee_{\prime}\cos(2t - z + x)$

p. 41, line 4, read $\frac{n_{\prime}^2}{n^2}$ for $\frac{n^2}{n_{\prime}^2}$.

IV. *On the Transient Magnetic State of which various Substances are susceptible. By* WILLIAM SNOW HARRIS, *Esq. Communicated by* DAVIES GILBERT, *Esq. V.P.R.S.*

Read June 17, 1830.

1. THE influence of bodies, not permanently magnetic, on the compass needle, has led to some further researches in magnetism of singular importance. The valuable papers of M. ARAGO and other eminent philosophers on this subject, communicated to the Royal Academy of Sciences, together with those of our no less talented countrymen which have appeared in the Royal Society's Transactions, are calculated to excite a deep interest in physics. In presenting to the Royal Society, after such valuable contributions to science, an account of some inquiries which I have myself been led to make in this branch of natural knowledge, I am encouraged by the belief, that whilst the deductions are for the most part derived from simple and direct experiments, the investigation has been carried on under new conditions; and that although some facts already made known, have been again referred to, it will nevertheless be found to contain results of sufficient consequence to render it not unworthy of notice.

I have been particularly led to persevere in this investigation, at various times, from having observed (Edinburgh Journal of Science, vol. v. p. 325.) that the reciprocal action of a rotating metallic disc, and a magnetic bar, is not considered by M. ARAGO as the result of any magnetism induced in the disc, but is attributed by him to some new force as yet undiscovered:—and also, from the circumstance, that the magnetic susceptibility of non-metallic bodies, seems not to have been in many instances satisfactorily established:—two inquiries of considerable importance.

2. When these researches were first commenced*, a curious effect was observed, evidently resulting from vibration, which it may be of consequence to mention.

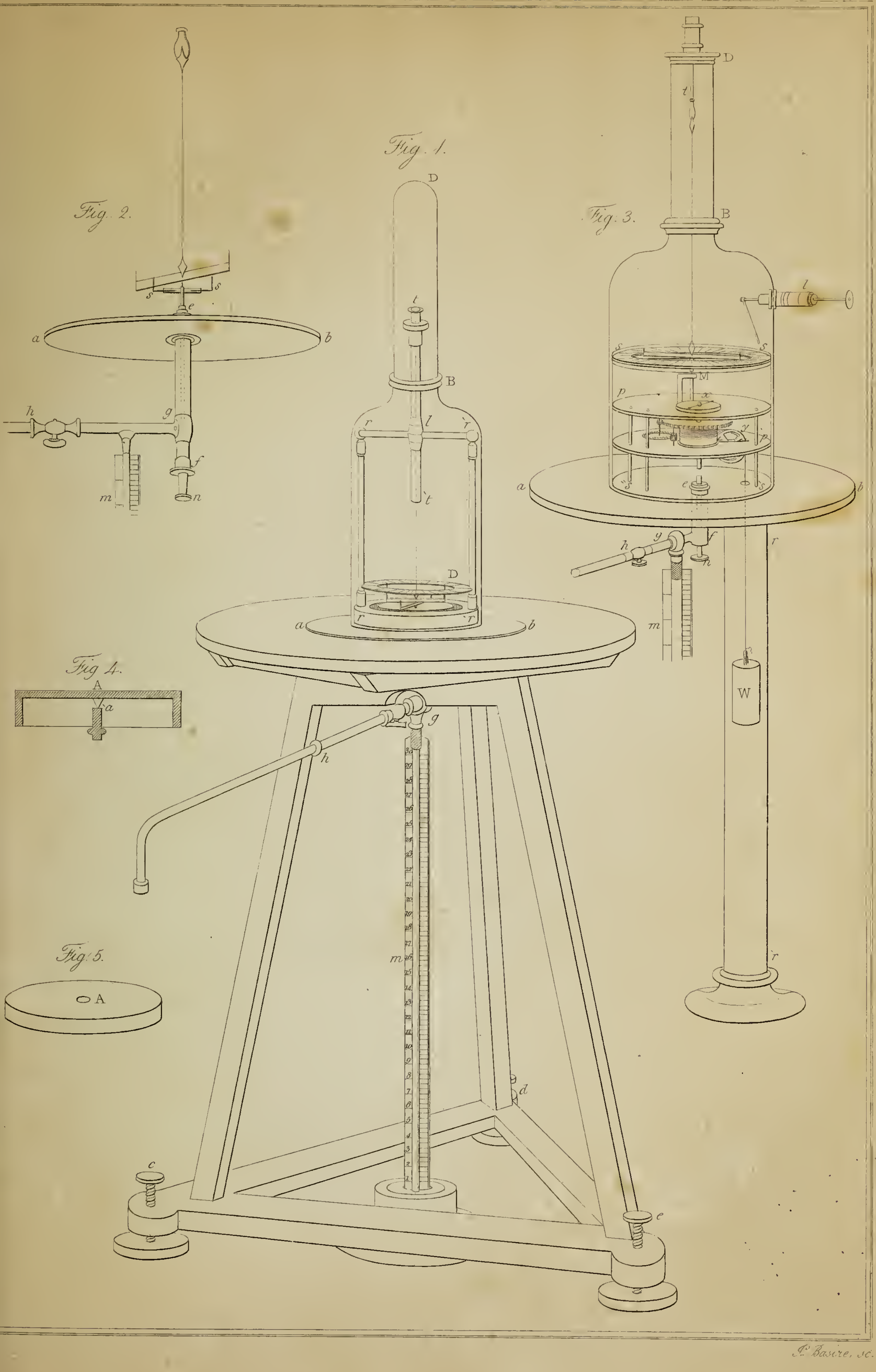
A thin circular ring of copper of about half an inch wide, ten inches in

* In 1826.

diameter, and about $\frac{1}{40}$th of an inch thick, being delicately balanced on a fine point, by means of a transverse bar of wood carrying an agate centre, was placed on a firm screen of paper strained tight on a wood frame, and covered by a glass shade; two powerful magnetic bars were caused to revolve in a horizontal plane, beneath the screen, with their poles immediately under the ring. The ring, as in all the previous experiments of the same kind, soon acquired motion in the direction of the revolving bars, which could be arrested and reversed at pleasure, by changing the direction of the rotation. It was however soon discovered, that a similar effect could be produced when the magnets were not present, merely by the action of the rotating apparatus. This last consisted of a train of wheels resting on a firm pavement of thick stone; the frame carrying the screen being supported over it on blocks of oak.

As it was of consequence to ascertain how far small vibratory impulses communicated to the screen, could cause the phenomena just observed, a continuous but gentle vibration was induced from one corner of it, by a slight tapping, which, after a few minutes, communicated motion to the ring in a constant direction; on transferring the vibrations to the opposite side of the frame, the ring was again brought to rest, and caused to revolve in an opposite direction. A similar result was obtained when a compass needle, and light needles of other substances were substituted for the ring of copper.

3. These results led me to place the copper circle in an exhausted receiver, on a strong plate of glass, and to suspend the whole on a convenient frame by means of lines passing from firm walls; the revolving bars being placed as before, immediately under it. In this instance, however, I failed at the same distance to move the ring, nor could a fine compass needle similarly placed be made to deviate more than a few degrees from its meridian, when exposed to the influence of a rapidly revolving disc. This result is quite sufficient to show how minute a cause may render delicate investigations with a rotating body unsatisfactory; more especially when connected with a heavy apparatus: even in the case of screens it does not seem quite clear that certain impulses may not be propagated through them, of sufficient magnitude to act on a very finely suspended needle or disc; when the screens are thin and porous, the chance of this is greatly increased, and in cases where a screen only is employed without a shade; similar results would doubtless ensue from vortices



J. Basire, sc.

Fig. 8.

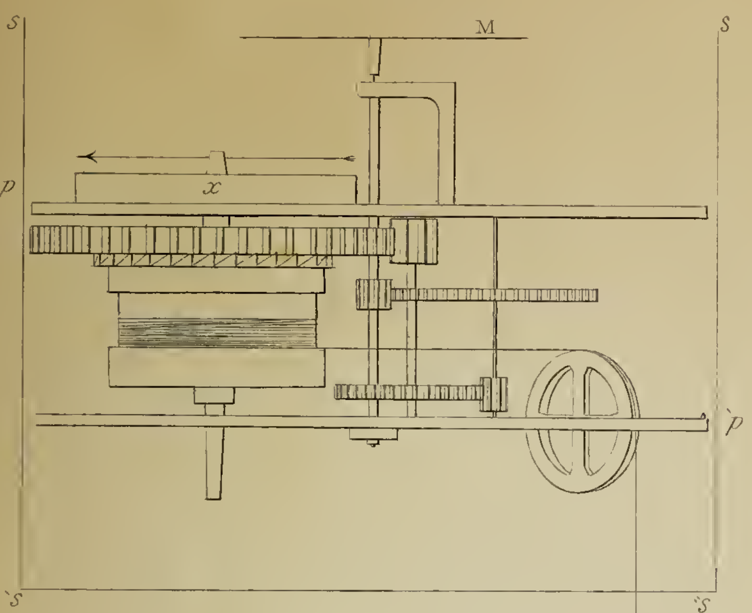


Fig 6.

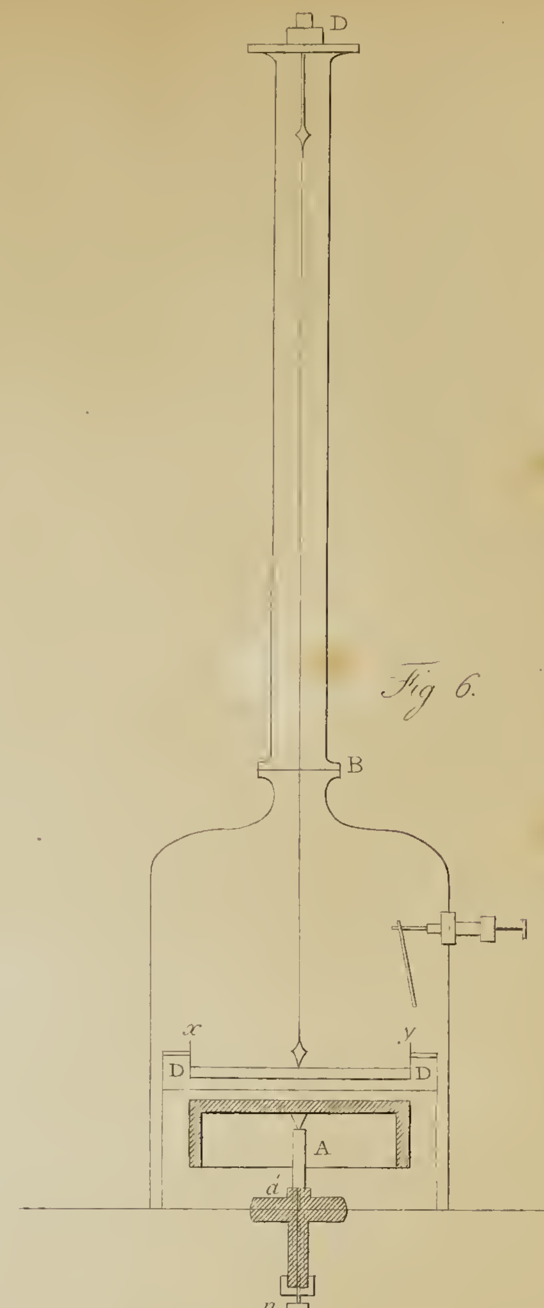


Fig. 9.

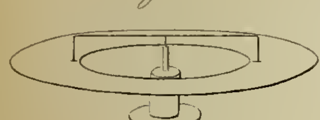


Fig. 7.

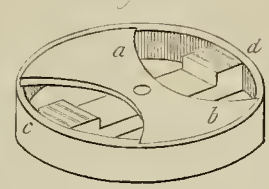


Fig. 10.

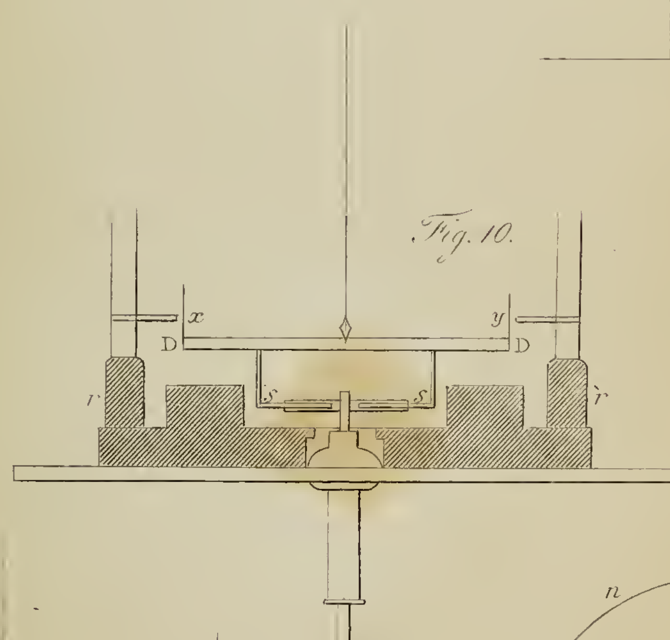


Fig. 11.

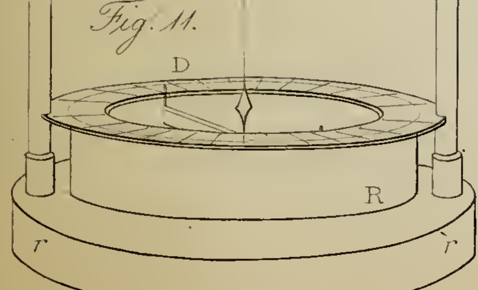
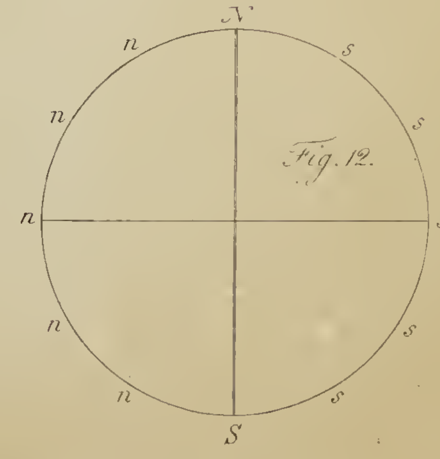


Fig. 12.



J. Basire, sc.

in the whole mass of the surrounding air. I do not however mean to infer that such sources of fallacy have not been fully calculated on, or guarded against in the many valuable papers already alluded to: I rather mention them as a reason for adopting a method of experimenting, differing from the preceding, and which might insure an unmixed result. To this end the experiments were continued in an exhausted receiver, effectually screened from vortices, and liberated from the influence of a resisting medium; and in case of employing a rotating body, with a rotation continued so smoothly and evenly, as not to cause any vibration capable of inducing motion.

4. The following is a description of the different mechanical arrangements resorted to. A circular plate of close fine-grained slate, *a b*, fig. 1, 2, 3, of about half an inch thick and a foot in diameter, being carefully ground true, in order to fit it for the purposes of an air-pump plate, was nicely finished by rubbing it over with a mixture of oil and wax.* This plate is supported on a firm stand and frame, as represented in fig. 1, furnished with levelling screws *c*, *d*, *e*. A short brass cylinder *e*, *f*, fig. 2 & 3, is fixed under the plate by means of a hole in the centre, and two brass collars ground true to its surface, and between which the plate is compressed: so that with a little grease, and a strong nut at *e*, fig. 2, 3, the joint is rendered air-tight. The cylinder *e f* connects laterally with a short pipe and stopcock *g h* leading to a good air-pump; a long barometer gauge *m*, fig. 1, 2 & 3, being attached to the pipe to indicate the state of the exhaustion. A straight rod of brass *n e*, having a milled head at *n*, passes through a compressed collar of fine cork at *f*, and is sufficiently free above in the cylinder to allow of the action of the pump; this rod is occasionally employed when furnished with a transverse bar at *e*, having two stops *s*, *s*, fig. 2 & 10, to liberate a magnetic bar in an exhausted medium, when retained at any angle of deviation from its meridian, or to effect any other required operation, as in fig. 3.

5. The revolving body A, fig. 4 & 5, consists of a circular disc of about five inches diameter, and 0.125 of an inch thick; united to a deep ring of the same

* The application of this substance to the purposes of an air pump-plate will be found extremely convenient, more especially when it is desirable to avoid the presence of a large mass of metal. It is a very economical and efficient substitute for glass or brass, and retains its form without liability to warp; and is an application as far as I can learn quite new.

substance, or otherwise to a ring of lead. This disc has smooth rounded edges, and is balanced on a fine central point A of hardened steel, in such way that the centre of gravity and centre of motion coincide as nearly as possible; the whole is sustained on a small agate cup, set in a short cylindrical piece of brass *à*, fig. 4, and screwed, when required, into the shoulder of the brass cylinder which passes through the round hole of the pump-plate as represented in fig. 6. The sliding rod *n e* above mentioned is in this case withdrawn for a short distance within the cylinder. The disc is set in motion at the rate required by means of a line wound rapidly off it, from a train of wheels; it is then covered with a screen, and with a receiver, as in fig. 6, and the exhaustion made as perfect as possible; in which case it will continue to revolve for a considerable time.

A simple or compound magnet is put into rapid rotation in a similar way, being previously mounted on a ring of lead; besides straight bars, a compound magnet was employed for this purpose, having its extremities turned up in a perpendicular direction for about an inch, as in *c d*, fig. 7, and occasionally a circular disc of magnetised steel, fig. 5. The former has its point of support fixed in a transverse bar of brass *a b*, fig. 7, which projects in a circular hole drilled through the centre of the bars.*

6. When it became necessary to measure with great precision the rapidity of rotation, the machine represented in fig. 3 & 8 was employed. It consists of a metallic disc M kept in motion by a delicate train of wheels and pinions $p\ p'$. The rapidity of the rotation being accurately measured by an index x, moving on a small circle divided into twenty parts. The weight W, fig. 3, which by its descent keeps up the rotation, falls in a tall narrow receiver $r\ r'$, supported against the under surface of the pump-plate.

A circular hole is drilled at s' through the plate for the silk cord to pass, from the pulley at y; by means of this hole also, the receiver $r\ r'$ is exposed to the action of the pump. The machine is wound up and set in motion by means of the brass rod *n e*, fig. 3, already described (4), without disturbing the state of the exhaustion; the extremity of the rod being formed into a key.

The revolving disc is screened by glass or any other substance, over which

* This method of obtaining a rapid rotation in an exhausted receiver, suggested itself on perusing the account of Mr. SERSON's horizontal speculum in the Royal Society's Transactions.

is placed a graduated circle of fine milled board, sustained in a light circular ring of wood. The screen and ring are moveable with friction in a hollow cylinder of glass $s\,s\,s'\,s''$, fig. 3, so as to be adjusted at any required altitude.

7. The receivers employed to inclose these various bodies are for the most part such as are represented in fig. 1, 3 & 6; they have an open neck at B ground to a flat surface which admits of the altitude being increased by the addition of a second narrow receiver B D: this last is also ground flat at the point of union; thus, by the aid of a little grease, an air-tight joint is easily obtained in the usual way. The upper receiver B D is either closed above, or open, with a ground flat rim in order to apply a brass plate D carrying a brass rod *t* in an air-tight collar. The rod *t* is for the convenience of raising or depressing light discs or needles suspended in the receiver below: these receivers vary from eight to forty inches in altitude, and from one to three inches in diameter; so that the filament of silk for suspension may, when required, be upwards of four feet in length.

8. The substances exposed to the influence of the revolving bodies are formed into light needles or bars, or otherwise into flat circular rings, fig. 9, and are either suspended or nicely balanced on points. For the purpose of retaining them at rest until the exhaustion is complete, and when the stop *e*, fig. 2, already explained (4) cannot be used, there is an angular lever *l*, fig. 3, which passes in an air-tight collar through the side of the receiver, carrying at its extremity a common reed *s*, which being tubular is easily fixed on it; hence it can be extended at pleasure.

9. The following are the results of some experiments with the apparatus above described.

(*a*) A circular disc of copper being put into rotation at the rate of 500 revolutions in a minute, was left unscreened under a receiver, and a fine reed suspended over it by a filament of silk; the reed was retained at rest, until the exhaustion was completed to within 0.4 of an inch of a good barometer, when it was set free. The reed rotated freely at any distance at which it could be placed from the revolving disc, and which at the greatest amounted to eight inches. The motion of the reed diminished with the rapidity of the rotation of the disc and with the torsion of the silk; but its motion could be again restored by admitting a small quantity of air into the receiver.

(*b*) In a tall receiver of about two feet high a reed similarly placed rotated at a distance of eighteen inches from the revolving disc, when the exhaustion was within six inches; but it remained at rest at a foot distance, when the exhaustion was carried to within 0.4 of an inch.

(*c*) The rotating disc being screened by a thin plate of glass or varnished paper fitted close upon a short hollow cylinder of glass or wood, the reed remained at rest, although placed within one fifth of an inch of the screen.

(*d*) A cylindrical magnetic bar revolved rapidly in an exhausted medium, when within one fifth of an inch distance from the disc; but only deviated from the meridian 40 or 50 degrees when the screen above mentioned was interposed. The disc in this experiment revolved at the rate of 400 revolutions in the minute.

(*e*) Needles of glass, wood, and metallic substances not permanently magnetic, remained at rest when the rotation exceeded 600 revolutions in a minute, whether suspended by filaments of silk, or otherwise placed on points and supported on the glass screen. When the rotation exceeded 1000 revolutions in a minute, and the exhaustion was inconsiderable, they were sometimes slightly moved.

(*f*) Rotating magnetic discs and bars induced motion in metallic discs freely suspended at one fifth of an inch distance, when their thickness was about four times as great as ordinary tinfoil, and when the rotation exceeded 500 revolutions in a minute; the rotating body being screened by any intervening substance except iron, which last, as already observed by Mr. Herschel and Mr. Babbage, completely intercepted the effect *. But light discs of wood and paper, and discs of paper covered with silver or gold leaf, remained at rest when within $\frac{3}{20}$ths of an inch distance from the rotating body.

(*g*) A disc of iron or tempered steel did not communicate motion to any

* Philosophical Transactions of the Royal Society for 1825.

Although a sheet or two of tinned iron completely intercepts the influence of a revolving magnet on metallic discs generally, it does not appear to do so when the disc acted on is also of iron. I found, however, that a screen of sheet iron of about one fourth of an inch thick materially diminished the influence even on an iron disc, so that the rotation of an iron disc by the influence of a revolving magnet is reduced nearly to that of a copper disc of the same dimensions when the iron screen is not present. This curious effect of screens may possibly lead to some further elucidation of the nature of magnetic influence.

substance not permanently magnetic; nor could motion be induced by the action of such substances on each other, however near they could be placed with an intervening screen; although the rapidity of the rotating body was caused to exceed 1000 revolutions in a minute.

(*h*) A needle of soft iron was not influenced by a copper disc revolving at the rate of 600 times in a minute; but rotated freely when surrounded by a helix transmitting an electro-magnetic current.

(*i*) A disc of tempered steel not magnetic, after simple contact with a powerful magnet acquired sufficient force to induce motion in metallic rings.

(*k*) A needle of tempered steel was not influenced by a rotating disc of soft iron at the distance of one fifth of an inch, and revolving at the rate of 600 times in a minute; but the needle revolved rapidly after simple contact with a magnet.

(*l*) Magnetic needles delicately mounted on an horizontal axis so as to admit of motion in a vertical plane, remained at rest when the axis was perpendicular to the radius of the revolving disc; but the needles appeared to be carried in the direction of its motion when the axis was turned in the same plane.

10. The above facts seem to show very clearly, that the presence of permanent magnetism, and a susceptibility of magnetic induction are essential conditions in the phenomena of rotation hitherto observed: at least this deduction is fair for all distances between the bodies at which they can be well placed with an intervening screen: without a screen it seems extremely difficult ever to arrive at a satisfactory result; since even in a very rare medium (Expts (*a*), (*b*), (*c*), (*d*)) vortices are produced by the action of a rapidly revolving body, although with perfectly rounded and smooth edges, capable of carrying round light substances; and although separated from such substances by a considerable interval. So far therefore as this method of examining the influence which bodies not permanently magnetic can exert on each other extends, it seems not susceptible of the requisite precision; and is, therefore, in a great degree inadequate to detect any very minute and delicate force, which can be supposed to arise from such an influence.

11. The law according to which metallic or other discs in a state of rotation influence a permanent magnet seems to be directly as the rapidity of the rotation, and inversely as the squares of the distances between the attracting

bodies: at least for such distances as can be conveniently resorted to with an intervening screen. In order to investigate this, I resorted to the machine already described (6) and represented in fig. 3 & 8. By a few previous trials such weights were found as might in a short time impart to the rotating disc the respective velocities of 178.5, 357 and 714 revolutions in a minute; which numbers are to each other as the numbers 1, 2, 4: and by means of the index and graduated circle x, fig. 3 & 8, the deflections of a magnet could be taken when these velocities were attained. This point was determined by means of a valuable chronometer of a peculiar description which my friend Lieut.-Col. H. Smith, F.R.S. was so good as to lend me for the purpose; it can be set going, and stopped again at pleasure, and is capable of registering an observation to the $\frac{1}{60}$th part of a second. As the graduated circle is divided into twenty parts, it is presumed that the rate of motion can in this way be ascertained with sufficient accuracy; and thus any little acceleration caused by the descending weight is not of consequence.

12. A magnetic bar being suspended as in fig. 3, was in the first place accurately adjusted at the point of contact to the plane of the body intended to be put in motion, and which consisted of a flat ring of copper of an inch wide, and about 0.05 of an inch thick. The bar was then raised from the ring through a distance equal to five turns of a micrometer-screw at D, each turn of the screw being equal to the $\frac{1}{20}$th of an inch: the screen s was then interposed, and the whole covered by a receiver, and exhausted.

(n) The machine being set in motion, the deviation of the bar amounted to 24° when the velocity was 357 revolutions in a minute: on increasing the velocity to 714 revolutions in a minute, or double the former, it amounted to 56°. The exhaustion in these experiments was carried to within 0.5 of an inch. Taking the sines of the angles of deviation as a measure of the force urging the bar, we have the respective numbers .829038 and .406737, which are very nearly in the same ratio as the respective velocities; that is to say as 2 : 1.

(o) The bar being adjusted to within a distance of the ring equal to four turns of the screw, the deviation of the needle amounted to 38° when the velocity was 357 revolutions in a minute: on raising the bar by four additional turns of the screw, the deviation decreased to 9°. Taking the sines of these angles as before, we have for the corresponding distances, the numbers .615661

and .156434, which may be considered as very nearly in the inverse ratio of the squares of the respective distances.

In the following Table are given the deviations of the bar, corresponding to other distances and velocities.

TABLE I.

(A) Velocity of rotation, 178.5			(B) Velocity of rotation, 357.			(C) Velocity of rotation, 714.		
Distance in turns of Mic^r-Screw.	Angle of Deviation.	Sine of Deviation.	Distance in turns of Mic^r-Screw.	Angle of Deviation.	Sine of Deviation.	Distance in turns of Mic^r-Screw.	Angle of Deviation.	Sine of Deviation.
4	18°	.309017	4	38°	.615661			
5	12	.207912	5	24	.406737	5	56°	.829038
			6	16	.275637			
8	4.5	.078459	8	9	.156434			
10	3	.052336	10	6	.104528	10	12	.207912

It appears by the above Table, that the influence of the ring is directly as the rapidity of the motion, and inversely as the squares of the distances: we observe a little discrepancy in some of the numbers, but the general agreement is very close and remarkable. A complete agreement cannot be expected; for supposing even the most perfect manipulation, there will always arise in experiments of this kind many causes which disturb a numerical identity*.

13. Being desirous to extend these inquiries concerning the transient magnetic state, of which various substances appear to be susceptible, I subsequently laid aside the rotating discs, as a means of detecting these minute forces for the no less refined, and perhaps still less exceptionable method of a vibrating bar; since what is termed the magnetism of rotation, seems in fact to be

* In estimating the distance between the surfaces of the attracting bodies, the revolving disc was purposely made as thin as possible, so as to admit of its being considered without any sensible thickness. With respect to the magnet, the foregoing experiments with screens clearly show, that an intervening ferruginous mass completely intercepts the attractive force upon non-ferruginous substances. If any portion of the bar beneath the surface contiguous to the revolving body be supposed to operate upon the ring, such portion must necessarily act through the intervening iron; which it cannot do (Exp. *f*) (11). This view seems to derive much confirmation from experiment, since on vibrating an extremely thin needle of sheet steel over a metallic disc, the comparative results do not vary from those obtained by means of a bar of half an inch thick.

nothing more than the presence of a temporary magnetic development induced in successive points of a body by the action of a permanent magnet, the body being supposed in motion. If the body therefore be at rest, this development will restrain the motion of a bar vibrating near it, and so diminish the amplitude of its oscillations; and thus by determining the number of vibrations performed in a given arc, we may from thence arrive at a comparative value of the force in action.

To this effect the bar already mentioned (12) was suspended, and exposed to the influence of different substances, in the following manner.

Two perpendicular rods of glass $r\,r$, $r'\,r'$, fig. 1, furnished with foot- and cap-pieces are fixed in a solid block of mahogany $r\,r'$, fig. 1; these rods by the intervention of a short wood cylinder l, and two horizontal rods, also of glass, $r\,l$, $r'\,l$, sustain a glass tube $t\,t'$ which slides with friction through the cylinder. This tube is furnished with a cap-piece at t, through which passes a brass rod: the magnetic bar is suspended from this rod by a filament of silk, and is finally raised or depressed to the required altitude by a micrometer-screw at t: the altitude of the glass tube l being previously fixed. The centre of the mahogany block is hollowed into a cylindrical cavity as represented in fig. 10, and is firmly fixed upon the pump-plate by means of the screw and shoulder projecting through the brass collars (4). Its outer part is also depressed, leaving a cylindrical projection of about one fourth of an inch deep, 4.7 inches diameter, and 1.25 inch wide.

The graduated circle of stout card-board D already mentioned (6) slides with friction between the glass rods, so as to be easily adjusted at any given point. The bar is retained at the given angle from its meridian, and again set free, when the exhaustion is sufficiently complete, by means of the lever and double stop before explained (4), and which moves in the interior circular cavity of the block, fig. 10.

This method of arresting the bar seems to be of some consequence to the experiment; for if one pole only be checked, the force operating on the other, will for an obvious reason, give the bar a swinging motion, which is very undesirable, but which is here effectually prevented; so that when set free it will appear to oscillate as if mounted on a fixed centre. The checks by which the bar is thus arrested, are so contrived as to be independent of each other,

and are moveable with friction in a small tube of brass as at *ss*, figs. 2 and 10, and may consequently be adjusted with great nicety. The tube in which the checks are fixed is sustained in a horizontal direction in the extremity of the rod, which passes into the receiver through the centre of the pump-plate.

14. The substances to be submitted to experiment, are formed into rings as at R, fig. 11; in these the bar is caused to oscillate. Each ring is about one inch in height, of any convenient thickness, and 4.75 inches interior diameter, so as to admit of its being accurately adjusted on the cylindrical shoulder of the circular base, fig. 10.

15. In applying the method of an oscillating bar to the investigation of minute and transient magnetic forces, it seems essential to keep in view the following interesting fact; viz. the influence of bodies, not susceptible of permanent polarity, on the state of oscillation is such, that the amplitude of the vibrations only is diminished, not their duration: that is to say, whether a bar vibrate in free space, or otherwise near plates, or in rings of these substances, still the number of oscillations in a given time, considered as a unit of time, does not vary, although the bar is sooner brought to rest when under the influence of such bodies, than when freed from them, whatever substance be employed, and at whatever distance the influence be exerted.

This fact seems materially to distinguish the peculiar influence of non-magnetic substances, from a case of permanent polarity; by which last, an oscillating bar is not only brought to rest in less time, but the rate of vibration is very sensibly increased. We cannot therefore, as in the latter instance, resort to the common law of pendulums, and take the square of the number of oscillations performed in a given time as a measure of the force in action, since the time of each oscillation does not sensibly vary; we must therefore adopt some other method.

16. In order to arrive at a comparative value of the influence of any substance on a vibrating bar, I have been led to employ the following formula, $\left(\frac{a}{b} - 1\right) r$; which seems to apply in a very remarkable manner to the results of experiment; in which a represents the number of oscillations in a given arc in free space, b the number in the same arc, when exposed to the influence of a substance not permanently magnetic, and r the retarding force

by which the bar tends to rest in free space; the oscillations being supposed to take place in an exhausted receiver. Thus, if in free space 420 oscillations are performed before the arc of vibration is reduced from 45° to 10°, and the number of vibrations in the same arc, are 30 and 20 respectively, when the bar vibrates under the influence of two given substances taken in succession, and whose magnetic energies we propose to compare with each other; then the energy of the one may be expressed by $\left(\frac{420}{30} - 1\right) r$, and that of the other by $\left(\frac{420}{20} - 1\right) r$; that is to say, their energies will be to each other as 13 : 20 *.

17. Previously to examining the susceptibility of substances generally to magnetic influence, the number of vibrations was first determined between 45° and 10° in air, under a glass-shade; then the number in the same arc also in air, when the bar was surrounded by a ring of wood of one fifth of an inch thick, and an inch deep; each pole being $\frac{3}{40}$ths of an inch distant from the interior of the ring: this being carefully ascertained, the same was repeated in an exhausted receiver; the results are as follow:

Table II.

Number of Vibrations from 45° to 10°.			
In air.		In exhausted receiver.	
Vibrations in space.	Vibrations in ring of wood.	Vibrations in space.	Vibrations in ring of wood.
232	210	420	308

It may be seen in this experiment that the ring of wood exerted a very sensible retarding power on the vibrations of the bar, which is more apparent when the impediment to motion arising from a resisting medium is removed. I submitted to experiment in this way, in a pneumatic vacuum, a few rings of

* A magnetic bar whilst vibrating under the influence of any substance, must be considered to be operated on by two retarding forces; one by which it would be eventually brought to rest, supposing the given substance away; and another resulting from induction. The number of vibrations in a given arc, or their differences, therefore do not seem to be a measure of the latter force, since some portion of the former will always enter into the result of the experiment.

different substances similar to the above, and found their influence on the vibrations very decided, although the differences in their respective magnetic energies was not very apparent. The substances tried, with the respective number of vibrations, and their comparative energies as deduced from the formula, are given in the ensuing Table.

TABLE III.

Substances.	Vibrations from 45° to 10°.	Comparative energies as deduced from formula.
Distilled water, temp. 20° FAHR.	330	0.27
Honduras mahogany	308	0.36
Statuary marble	306	0.37
Freestone	308	0.36
Annealed glass, $\frac{1}{16}$-inch thick	310	0.35

18. By employing the above substances in larger masses, the effect on the bar seemed to be greatly increased: although the difference in the comparative energies of wood and marble, the only two substances which I had an opportunity of trying, is still not very great. In the following Table are given the results of some experiments with a ring of statuary marble of 1.75 inch thick, and with a similar ring of Honduras mahogany, as compared with the influence of two rings of the same substances of only one fifth of an inch in thickness.

TABLE IV.

Substances.	Wood.		Marble.	
Thickness of ring in inches	0.2	1.75	0.2	1.75
Vibrations from 45° to 10°	308	280	306	265
Ratio of magnetic influence as deduced from formula	0.36	0.50	0.35	0.59

19. The influence of the substances being observed to depend in great

measure on the mass, it seemed of consequence to ascertain how far the energy might vary with the density. Several kinds of wood therefore of the above dimensions were subjected to experiment, whose specific gravities greatly varied; but there did not appear, after many trials, any perceptible difference on 280 vibrations. The woods tried were, Lignum Vitæ, Yellow Pine, African Oak, Honduras Mahogany, and Teak. All these reduced the vibrations of the bar from 420, the number performed in free space, to 280, the number performed when surrounded by the wood; the vibrations being taken from 45° to 10°, in an exhausted receiver.

20. I endeavoured to detect the magnetic energy of fluid bodies by vibrating the bar in an exhausted receiver, with its poles within the $\frac{1}{20}$th of an inch of the glass; the receiver being surrounded at that part opposite the pole of the magnet by the particular fluid, the subject of experiment. In this case the fluid was retained in its situation by means of an external check of glass ground to a glass-plate, on which the whole was sustained; the plate being supported on a ring of glass placed over and in connection with the pump-plate below; and the surfaces of contact ground, so as to fit air-tight. I failed however in arriving at any decided result; the amount of the retardation with sulphuric acid, for example, not appearing greater than was fairly attributable to the influence of the glass alone: even a saturated solution of sulphate of iron did not seem to exert in this way any sensible influence. By means of a very great number of vibrations, and a powerful bar, however, the energy of fluids, if such exist, might possibly become appreciable.

21. The inductive influence on metallic bodies being the most decisive, they are perhaps better adapted to the purposes of experiments relating to the influence of mass and density, or to the law according to which the inductive effect increases, as the distance from the magnet decreases, than any other kind of substance. Before examining, therefore, the comparative energies of various metals, I endeavoured to arrive at some further conclusions in this interesting part of the inquiry. In order to examine the comparative energies when the quantity of matter was varied, twelve concentric rings of copper were employed similar to the former ring of wood (17), except in thickness, which amounted to .025 of an inch: these rings were turned up with sufficient accuracy, and fitted very fair and close one within the other. The num-

ber of vibrations was first determined in the interior ring alone, after which a second was added, and the number of vibrations again determined, and so on to eight rings; when the increased effect became almost inappreciable.

TABLE V.

Number of rings	1	2	3	4	5	6	7	8
Vibrations	44	30	24	21	19+	17	16+	16
Ratio of force .	8.54	13	16.5	19	21+	23	24+	25

The increments in the effect on the bar appear by the above Table continually to decrease, so that after a certain number of laminæ they seem to be no longer appreciable; hence the numbers expressing the comparative energies would become at last equal, and such was found to be the case, the number of vibrations with twelve rings being no greater than with ten*.

22. The quantity of matter in the foregoing experiment being actually divided into concentric rings, I endeavoured to ascertain how far the above results depended on the sum of the whole, or of any number of them; with this view the number of vibrations was determined for each ring, when opposed to the bar in the same relative situation which it occupied in the mass, so as to obtain the separate effect due to any particular ring. These separated results were as follow.

* It is not improbable that some principle of this kind is the immediate cause of the curious result arrived at by Professor BARLOW in the course of his magnetic inquiries; who observed that a hollow sphere of iron exerted the same influence on a compass needle, as if it were a solid mass. If we consider the sphere to be made up of concentric laminæ, then, by the foregoing experiment, the number of laminæ equivalent to exhaust the inductive energy of a small compass needle, will, especially in iron, (11) (Exp. *f* note,) be very few: consequently, all the interior may be removed. This is likewise consistent with a subsequent observation of the same distinguished philosopher; a certain depth of metal being found necessary. If the experiment were accurately tried, it is probable that the depth of metal requisite, would vary with the force of the compass bar; although for small needles the differences are so inconsiderable as not to be appreciable. For very powerful magnets, however, some further quantity of metal would be required; but, perhaps, in no instance could we employ a magnet of sufficient power to influence the whole mass of a solid iron sphere of large dimensions.

Table VI.

No. of the ring	1st	2nd	3rd	4th	5th	6th	7th	8th
Vibrations....	44	76	94	124	148	166	186	210
Ratio of force.	8.54	4.52	3.45	2.38	1.83	1.53	1.25	1.0

In comparing the above numbers with those in the preceding Table, the energy of any number of rings, taken together, appears to be very nearly the sum of the same rings, taken separately, and is a curious deduction by the formula above given (16). Thus the energy of three rings combined is by the preceding Table 16.5, which is about the aggregate amount of the first, second, and third, as deduced above, and so on. It is true that some of the other numbers are not in such complete accordance as in this instance, but the approximation is nevertheless very remarkable. The intervening rings do not therefore intercept any very sensible portion of the inductive influence of the bar; a result quite consistent with what is observed in the case of rotating discs, when covered with non-ferruginous screens; and with this further condition, that whilst the inductive energy thus penetrates the intervening metal, it disturbs at the same time its magnetic distribution.

23. The foregoing deductions were verified by combining a few rings not immediately successive, so as to leave an interval between them. The results are as follow.

Table VII.

Rings combined .	1st+4th	4th+8th	1st+4th+8th
Vibrations......	−37	100	+33
Force..........	11.3	3.2	−11.7

In these, as in the foregoing instances, it may be perceived, on comparing the actual observation, as given in this Table, with the values of the respective rings in the preceding one, that the numerical approximations are very close. The energy of any ring therefore may be estimated by subtracting from the aggregate effect, the sum of the others with which it is combined.

24. It would not be difficult, from the above investigation, to arrive at some

general conclusions as to the precise ratio in which the energy increases with the separate quantities of matter, and decreases with the distance; since both are involved in the experimental results. In the following Table is given the observed effect of each separate ring, as compared with the corresponding effect deduced by calculation from the first experiment, on the supposition that the energy is directly as the quantity of matter, and inversely as the squares of the respective distances; and it will be seen that the differences are not greater than may, from the somewhat complicated nature of the experiment, be expected.

Table VIII.

Rings	No. 1.	No. 2.	No. 3.	No. 4.	No. 5.	No. 6.	No. 7.	No. 8.
Energy by calculation		4.5	3.05	2.19	1.65	1.28	1.04	0.84
Observed energy	8.54	4.52	3.45	2.38	1.83	1.25	1.25	1.0

25. There appears reason therefore to conclude, as a general principle, that the transient magnetic energy of any substance, not supposed to contain iron, varies in a direct ratio of the quantity of matter within the sphere of the action, and in an inverse ratio of the squares of the distances from the magnet; the matter being supposed to be condensed into a stratum not having sensible thickness, and taken at some mean point of distance within the surface, where the sum of the forces may be imagined to produce the same effect as if exerted from every part of the mass; and the respective distances estimated between this mean distance and the opposed surface of the magnet*.

26. It is a consequence almost necessarily resulting from the foregoing investigation, that, supposing all other circumstances alike, the energy of the same substance is directly as its density, although such did not actually appear in the foregoing experiments with various species of wood (25); but this may be readily supposed to arise from the circumstance, that the energy of wood is

* The reason why such should be the case for the magnet has been already given (12): it is equally applicable when the energy of the extremities only is supposed to be exerted: and it may be further observed, that although on either of the sides of a magnetic bar, the point of greatest attraction of that particular side seems to fall (a little) within the extremity, yet the maximum of force of every common bar will be found invariably on the surface, terminating the extreme ends.

altogether too feeble to admit of such differences becoming appreciable by the means employed. In metallic bodies, however, the fact appears to be very decided; thus, in three specimens of copper, whose dimensions were very nearly the same, but whose specific gravities varied, the resulting energies were observed to be the greatest in those which had the greatest specific gravity; as for example, when the specific gravities were to each other as 8.30 to 8.40, the resulting number of vibrations amounted in the one instance to twelve, in the other to fifteen. To obtain more than rough approximations to the precise law by this method, would be extremely difficult, since the metals, as it is evident, should be procured in a state of great purity, and the dimensions, previously to taking the specific gravities, adjusted with an accuracy which only the most refined workmanship can effect; there is however little doubt, if the experiment could be accurately tried, that the energy of the same metallic substance would, as a general result, be in the direct ratio of its density.

27. It is of importance therefore, in estimating the comparative magnetic influence of metals, to take into account their density; without which the inquiry might prove unsatisfactory. The energies of various metallic bodies, and some few alloys, were estimated by determining the number of vibrations of the bar between 45° and 10° in a ring of each; the metal being within 0.1 of an inch of its poles. The more common metals were cast in a mould, and all the rings adjusted to the same dimensions.

28. I endeavoured to estimate the energy of mercury at −50° of Fahrenheit, by freezing it into a solid ring, of the same dimensions as the preceding, in a mould of wood, accurately adjusted for the purpose; but could only obtain satisfactorily, as a general result, the place it appeared to hold in the scale of magnetic energy. The ring was fully and completely solidified, and the mould so constructed as to admit of its interior part being removed, so that the needle could oscillate, whilst the metal remained externally enveloped in the freezing mixture. The thermometer, previously to removing the interior of the mould and adjacent salts, stood at −56°. I had not an opportunity of repeating this interesting experiment, which is sufficiently practicable, and seems to promise the development of some new phenomena in magnetism. The energy of mercury in its fluid state, in a vacuum, was examined by inclosing it be-

tween two glass circular checks, so as to obtain a ring of nearly the same dimensions as the others, subsequently separating the influence due to the glass alone.

29. The place which the following metals appear to occupy in the scale of magnetic energy, and their comparative influence, as resulting from this investigation, is given in the succeeding Tables.

TABLE IX.

Metals.	Rolled Silver.	Rolled Copper.	Cast Copper.	Rolled Gold.	Cast Zinc.	Cast Tin.	Cast Lead.	Solid* Mercury.	Cast Antimony.	Fluid Mercury.	Cast Bismuth.
Comparative magnetic energy .	39	29	20	16	10	6.9	3.7	2.0	1.3	1.0	0.45

TABLE X.

Alloys.	Cast Copper and Zinc in equal parts.	Cast Copper and Bismuth in equal parts.	Cast Zinc and Bismuth in equal parts.
Comparative energy . .	12	2.3	1.4

30. Although considerable pains have been taken to make the foregoing Table as perfect as possible, yet it cannot be considered as anything more than a useful approximation; there are many conditions peculiar to this inquiry to be yet investigated, which seem for the present to preclude the possibility of obtaining results quite conclusive. The metals employed were in as great a state of purity as they could be obtained in the ordinary way of commerce; some of them, more especially the copper, gold, and silver, may be considered as very nearly without alloy.

31. If the preceding investigations are of any importance, they seem to be in great measure conclusive, as to the cause of the influence of a rapidly

* The comparative influence of solid mercury must be taken as a rough approximation; the bar in this ring in air, as nearly as could be ascertained, performed about 150 vibrations between 45° and 10°; whilst, in an analogous ring of rolled copper it performed about fourteen vibrations; in free space in air, it performed 232.

rotating metallic disc on a freely suspended magnet; whilst at the same time they are in accordance with the opinion of M. Arago, that almost every known substance can exert an influence on the compass-needle. Every species of matter therefore may be considered to be more or less susceptible of a state of transient magnetic energy, arising from induction.

32. The hypothesis advanced by Mr. Herschel and Mr. Babbage in explanation of some of these phenomena is extremely simple, and to a great extent very satisfactory; inasmuch as it agrees with all the observed operations of magnetic induction, and, as stated by them, supersedes the necessity of advancing any new hypothesis in magnetism; it is nevertheless considered by M. Arago as insufficient*. This celebrated philosopher appears to deny, that the limit of the motion produced by any force, which can be supposed to reside in the induced poles, even if their existence be admitted, can exceed a minute of a degree: whereas, in order to explain the rotation, it should exceed 90°. It may be observed however, that a ring of metal, copper for example, immediately surrounding the poles of the needle, can, as first stated by M. Arago, diminish the amplitude of its oscillations, and, as seen in the foregoing experiments, so fetter its motion, as to reduce the vibrations in a given space from 420 to 14. We must therefore necessarily conclude the force induced in each consecutive point of the ring to be very considerable. With non-metallic bodies, the force is certainly very much less; but these do not fetter the motion of the needle to anything like so great an extent, nor will these substances rotate, as far as I can find, by the influence of a magnet revolving in an exhausted receiver without sensible vibration, notwithstanding the rapidity of the rotation exceeds 700 revolutions in a minute.

33. After considering with much attention the hypothesis just alluded to, I am led to offer a few additional observations, with a view of extending the principle it involves, and which may possibly be useful in further elucidation of the perceived effects; whilst, at the same time, they will not be found in any way inconsistent with the known laws of magnetic action.

The facility with which a magnetic development is induced in bodies, may be designated by the term "susceptibility"; the time during which this effect of induction remains in them "retentive power". These terms have been

* Edin. Philos. Journal, vol. v. p. 326.

already employed by Mr. Babbage and Mr. Herschel, and they are sufficiently explanatory. In the case of magnetic energy induced in a body by any of the ordinary methods, it may be observed as a general fact, that the energy is acquired in somewhat less time than that in which it is again lost. This is particularly the case in soft iron rendered magnetic, by an electro-magnetic spiral; a simple contact between a magnet and a mass of soft iron will frequently convey to it an attractive force, which it retains for a comparatively long space of time. In regard to the susceptibility of different substances, it is found to vary considerably; and depends on some peculiar property not yet explained; it seems to be in some inverse ratio of their retentive power. Thus it is not without difficulty that hardened steel is made permanently magnetic, whilst its retentive power is considerable: soft iron, on the contrary, is observed to be comparatively feeble in its retentive power, whilst its susceptibility of magnetic change is great. It may be hence inferred, that in non-ferruginous masses, the mere susceptibility of magnetic change, is in fact also considerable, but then their retentive power is so feeble, that the subsequent attraction does not ensue to any great extent. This probably arises from some peculiar state of the particles of these substances, which allows what may be termed the new magnetic distribution, to tend more rapidly to the previous state of neutrality, immediately the tension passes a certain point. In every observed instance, however, more time seems necessary to restore perfectly, the original state of the body, than was required to disturb it, the former being the result of a progressive operation, whilst the latter is effected by a sudden and concentrated force.

34. The attraction, as usually observed, between a magnet and a mass of iron, is invariably preceded by this new magnetic distribution in the iron; and unless such new distribution can occur, it seems, as in the analogous operation of electrical action, that no attractive effect can ensue; indeed this is made evident by the repulsive efforts of two similar magnetic poles; the repulsion may be conceived to be really the immediate consequence of an inductive effort, since an attempt is made to reverse the magnetic distribution in the opposed bars, which action the already existing polarities resist: when, however, as in the case of opposing either pole of a very powerful

magnet to the similar pole of a very weak one, this new distribution can be effected, an attractive force immediately follows.

35. There are some circumstances connected with this curious result of magnetic action, which seem to apply immediately to the question under consideration, but which have not been generally observed. If two similar magnetic poles of very unequal force be opposed to each other, the greatest repulsion, taking into account the difference in distance, will be a little within the limit of their action: that is to say, the increments in the repulsive energy are comparatively less, as the magnets approach each other: evidently resulting from the change which begins to take place in the magnetism of the bars. Now if one of the bars be extremely powerful in regard to the other, the new distribution in the weaker bar is effected even before the point of contact. The precise point at which the existing polarity of the weaker bar becomes changed varies with the force of the stronger magnet. This point may be found by experiment in the following way. Let a small cylindrical magnet be suspended by a silk line from a delicate wheel, whose axis rests on friction rollers; and let it be counterpoised by a small weight at the extremity of a short cylindrical piece of wood partly immersed in a jar of water. If one pole of a powerful bar be now carefully approximated towards the like pole of the suspended magnet, by fixing the former in a brass frame carrying a micrometer-screw, the latter will be observed to recede from the bar, until the opposed poles are within a certain distance of each other; when the repulsion will cease, and a weak attractive effect ensue. The cylinder of wood as it becomes gradually immersed *, continually furnishes an equivalent to the repulsion, in the quantity of water displaced.

We do not generally perceive the attractive effect resulting from this inductive action in non-ferruginous masses; for, as already observed, their feeble retentive power admits of the magnetic neutrality being more rapidly restored, when the tension passes a certain point. So that in fact, the opposite magnetic state never becomes sufficiently intense to evince an attractive force, cognizable by the ordinary means.

* A more detailed account of this experiment may be found in the Transactions of the Royal Society of Edinburgh.

36. When, therefore, a disc of any substance is put into a state of rotation under a suspended magnetic bar, opposite poles are induced in that part of the disc immediately under the bar; these induced poles may be supposed to pass on, and being retained for a short portion of time, will, if the motion be more rapid than the time during which the impression remains, be transferred toward the opposite poles of the magnet, and exert upon them a repulsive action up to that point of distance (35) at which the poles of the magnet again reverse the transient poles impressed on the disc, and substitute opposite poles, to be again reversed as before.

37. Thus if $N s S n$ (fig. 12) be a metallic disc revolving in the direction $N s S n$ under a magnetic bar $N S$ free to move upon a central point C, and of which N is the north pole, and S the south, the effect of this rotation will be, to impress upon the semicircle $N s S$ a south polarity, and upon the semicircle $S n N$ a north polarity, in consequence of the points $s s s$ and $n n n$ having passed under the poles N and S. Now if the time required to restore the original magnetic distribution of the plate, be less than that necessary to disturb it, these impressed polarities remain for a small portion of time, and hence there will always be an attractive force in advance of the poles of the bar, and a repulsive force in arrear of them; consequently the bar becomes driven, as well as dragged in the direction of the revolving disc by the resolved portion of the oblique actions acting for the most part near the extremities. And there is little doubt, that any substance in the least degree susceptible of a transient magnetic state, might cause a magnet to rotate; provided that the rapidity of the motion be greater than the time necessary for the restoration of the original magnetic distribution of the body acted on; supposing such rapidity of rotation possible.

38. A rotating disc, therefore, circumstanced as above (37), may be considered as a circular magnet such as that already mentioned (5), one of the semicircles having a north polarity, and the opposite semicircle a south polarity; and which polarities eventually neutralise each other about one of the diameters; the only difference being, that the magnetism of the revolving disc is transient, and constantly changing its position; so that the neutral points are always near the poles of the suspended bar. The bar therefore by a well-

known law of magnetic attraction, will be constantly endeavouring to place itself at right angles to the magnetic equator of the disc, whilst the position of the latter varies in its turn with the position of the bar: it is in fact owing to this circumstance that the bar at length revolves with the disc, as may be in great measure seen by substituting for the disc a similar disc of steel made permanently magnetic; in which case the suspended bar will not revolve, it will be merely put into a state of disturbed vibration.

V. *On the Nature of Negative and of Imaginary Quantities.*
By Davies Gilbert, *Esq. P.R.S.*

Read November 18, 1830.

I AM desirous of submitting to the Royal Society some observations on the nature of what are termed Negative and Imaginary Quantities, tending as I hope to clear away the obscurity that has hitherto surrounded them.

The subject has occupied my attention for many years, and however plain and simple may be the results, they have not been obtained without much patient investigation; and, in the event of their being found correct, they will add one authority more to an observation frequently made, and confirmed by extensive experience,—That paradoxes and apparent solecisms, involving themselves with facts or with deductions known to be true, may always be found near the surface, owing their existence either to ambiguities of expression or to the unperceived adoption of some extraneous additions or limitations into the compound terms used for definition, which are subsequently taken as constituent parts of their essence.

The first misapprehension appears to consist in our considering any quantity whatever as negative per se, and without reference to another opposed to it, which has previously been established as positive.

In applying our arithmetic to whatever is continuous, some neutral point or zero must be selected, as in time, in space with reference to its three dimensions, in forces, in velocities; and the opposite directions from this point will be mutually negative in respect to each other, and must be distinguished by appropriate marks or signs. But space to the left is no more essentially negative than space to the right, nor descent than ascent, nor time past than time that is to come.

I would therefore adopt for the present investigation, and to avoid previously formed association of ideas, (a) and (b) as marks or signs for prefixing

to the same quantity taken in opposite directions, rather than the usual ones of plus and minus.

In the next place, I believe that the law of the signs has never been stated according to its full generalization.

In common language, and for ordinary purposes, multiplication is considered as an abbreviated addition; but it would be a superfluous waste of time to demonstrate before this Society that multiplication is always an affair of ratios. Length and breadth multiplied together give areas, because an unit of length by an unit of breadth has previously been established as the superficial unit. Length, width, and depth give solidity for the same reason, and from the want of such a preestablished unit, arises the utter absurdity of a question, heretofore proposed in various treatises on arithmetic, for multiplying some denomination of coin by itself, and ascertaining the product.

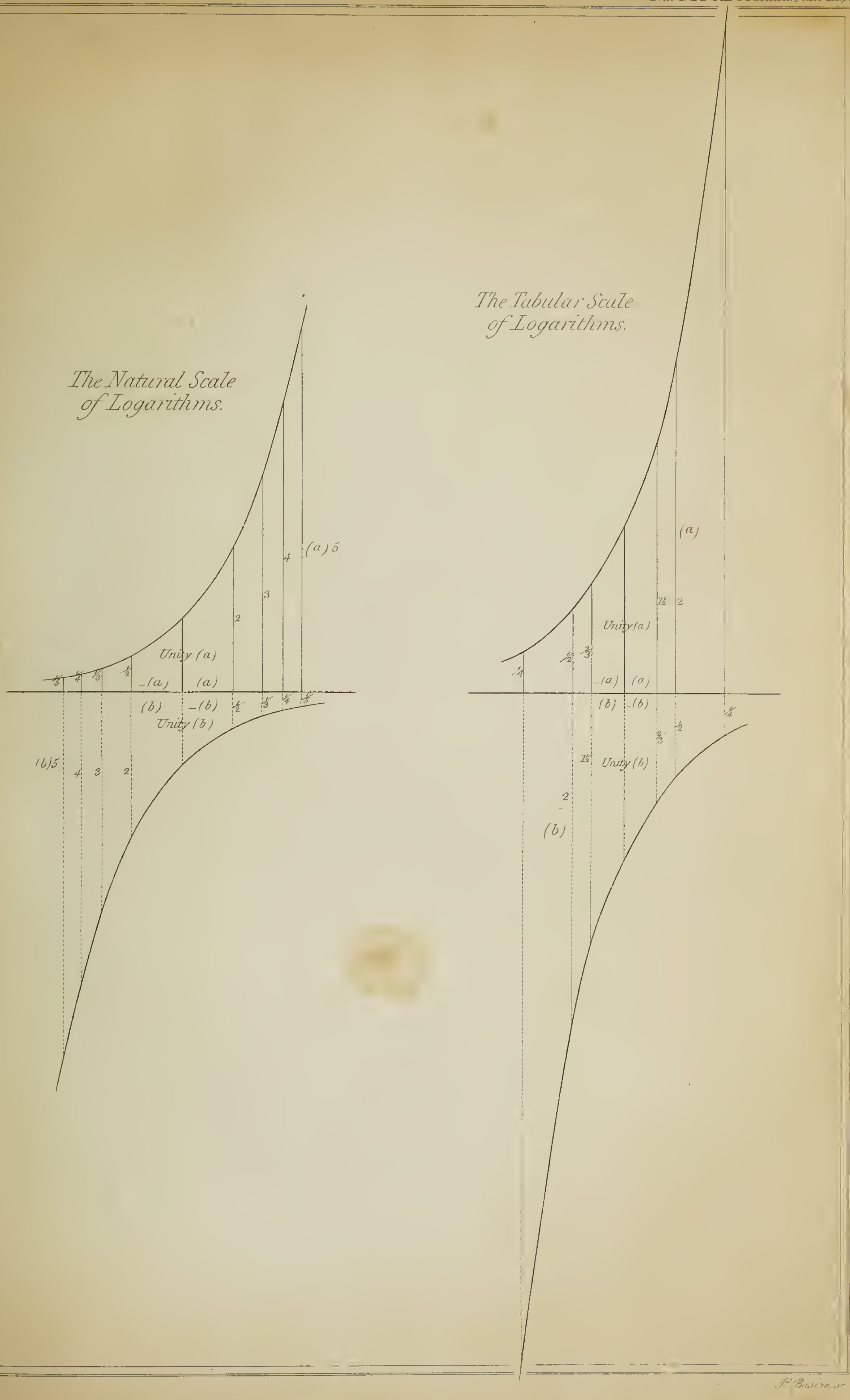
When a multiplication of two quantities is therefore to be made, unity must be understood as the antecedent; but here an extraneous limitation insinuates itself unperceived. The common antecedent taken in usual practice is not simply an unit, but unity in the scale of (a). With this limitation the law of the signs is correct, namely that like signs produce (a), and unlike signs produce (b). But let unity, the common antecedent, be taken in the scale of (b); the law will then immediately change to like signs producing (b), and unlike signs producing (a).

The general rule therefore must be, that like signs give the sign of the assumed universal antecedent, and unlike signs the contrary.

Admitting, therefore, that both scales are in themselves equally affirmative, and that either may be taken as negative to the other, it necessarily follows that by using the scale of (b), and consequently by assuming the unit of that scale as the universal antecedent, all even roots in the scale of (a) will become imaginary; and thus the apparent discrimination of the two scales is entirely removed: and in the same way, and by varying the signs according to the scale in which the universal antecedent is taken, all formulæ will become equally applicable to both.

For example: (See plate III.)

The natural numbers and their logarithms will be expressed for both scales by the correlative curves in the following figures, where all ordinates taken


The Natural Scale of Logarithms.
The Tabular Scale of Logarithms.
(a) 5
4
3
2
Unity (a)
–(a)
(a)
(b)
–(b)
Unity (b)
(b) 5
(a)
1½
2
Unity (a)
(b)
Unity (b)
(b)

continuously above and below the line of the apsides are reciprocal to each other; and their product equal to the square of either ordinate at the zero point, which, in the position of the curves corresponding to the natural system, is when the fluxions of x and y are equal.

It is obvious from inspecting this figure, that if m be a natural number and M its logarithm, both in the scale of (a), and M be taken in the scale of (b), or $-(a)$, it will become the logarithm of $\frac{1}{m}$; but M in the scale of (b) is also the logarithm of m in the same scale; and M taken as at first in the scale of (a), or $-(b)$, is the logarithm of $\frac{1}{m}$.

If the two curves are moved on each other, so that the two ordinates measuring 2.302585 in reference to the former unit be made continuous, and that length be taken as a new unit, they will then represent the tabular logarithms in both scales.

I flatter myself with having now clearly established the principle, that all properties belonging to the scales of (a) and (b) are mutually interchangeable; and that consequently imaginary quantities will be found in the even roots of either scale, as the universal antecedent is taken in the other. And this leads to the question,

What are imaginary quantities?

I answer, Creations merely of arbitrary definition, endowed with properties at the pleasure of him that defines them, the whole dispute respecting imaginary quantities turning on the point contested from the earliest times between the hostile sects of Realists and Nominalists, descending through Plato, Aristotle and the Stoics, to the Philosophers of Alexandria, and from them to the Schoolmen, who imbittered their discussions with theological controversy and persecution.

If the conceptions of the mind, all abstractions and generalizations, were considered as substantial forms, possessing existences distinct from the intelligences contemplating them; or, as some writers have expressed themselves autousia gaudentes, the μοςφαι immutable and eternal, which united to matter constitute the universe, nothing could be equally absurd with the supposition of impossible or imaginary quantities; but according to the theory now so uni-

versally prevalent as scarcely to admit of a single exception among all those who make the powers of the human mind the subject of their peculiar research; Classifications, Abstractions, Generalizations are allowed to be mere creatures of the reasoning faculty, existing nowhere but in the mind by which they are contemplated. To such unsubstantial existences any qualities may be imputed; but the only one known or useful in algebra, is the supposed even root of a real quantity taken in the scale opposite to that which has given the universal antecedent. The sign or mark indicating the extraction impossible to be performed, veils the real quantity, and renders it of no actual value until the sign is taken away by an involution, or the reverse of the supposed operation which that mark or sign represents, although by its arbitrary essence the quantity so veiled is in the mean time made applicable to all the purposes for which real quantities are used in all kinds of formulæ.

While therefore the sign of the supposed extraction of a root remains, the quantity to which it is prefixed has no more than a potential existence; but it stands ready to exist in energy whenever that sign is removed.

Consequently, without experience, it is impossible to know whether an implicit function of such an ideal quantity, will or will not be cleared by development of the symbol indicating the supposed extraction of a root, that is, whether any actual value does or does not belong to such a function.

Subject to the above conditions, namely, that the quantity veiled by the sign of a supposed extraction shall be treated in expansions and formulæ according to the laws applicable to real quantities, and that it shall exist in energy whenever an involution has reversed the supposed extraction of an even root,—

Let (A) be supposed equal to $\sqrt{-1}^{\sqrt{-1}}$ to find (A);

Then, according to the established laws of real quantities arbitrarily extended to these that are imaginary, the log. of $A = \sqrt{-1} \times$ log. of $\sqrt{-1}$; but by a well known theorem,

$$\text{the log. of } \sqrt{-1} = \left(\sqrt{-1} - \frac{1}{\sqrt{-1}}\right) - \tfrac{1}{2}\left(\sqrt{-1}^{\,2} - \frac{1}{\sqrt{-1}^{\,2}}\right)$$
$$+ \tfrac{1}{3}\left(\sqrt{-1}^{\,3} - \frac{1}{\sqrt{-1}^{\,3}}\right) - \tfrac{1}{4}\left(\sqrt{-1}^{\,4} - \frac{1}{\sqrt{-1}^{\,4}}\right) \text{ \&c.}$$

And this series $\times \sqrt{-1}$ becomes $- 2 \times \frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9}$ &c. each alternate term

vanishing. But $2 - \frac{2}{3} + \frac{2}{5} + \frac{2}{7} - \frac{2}{9}$ &c. = the quadrantial arc of a circle to radius unity.

Therefore the log. of A = — quad. arc.

$$\text{And } A = e^{-\text{quad. arc}} = 0.2078796.$$

Consequently, $\sqrt{-1}^{\sqrt{-1}}$ is an abbreviated mark or symbol, according to the above arbitrary conditions, for the radix of the natural system of logarithms raised to a negative power, indicated by the quadrantial arc of a circle to radius unity. And in the event of $\sqrt{-1}^{\sqrt{-1}}$ ever occurring in the solution of a problem, $e^{-\text{quad. arc}}$ or $2.71828^{-1.5708}$ or 0.2078796 may be substituted for it. And this is what practically happens in regard to all expressions apparently imaginary, which are found to represent real quantities, as is well known in cases of circular arcs and logarithms. These mental abstractions have moreover extended the bounds of analysis far beyond the utmost limits it could otherwise have attained; they have bestowed harmony and simplicity of form on its most recondite investigations, and eminently has their use been important in equations, by resolving them into a number of simple factors equal to the dimensions of the equation in its highest term.

It appears from these considerations, that several ingenious mathematicians have taken an incorrect view of ideal quantities, by mistaking incidental properties for those which constitute their essence; as, for example, when they are supposed to be principles of perpendicularity, because they may in some cases indicate extension at right angles to the direction here designated by (a) and (b), but with an equal degree of propriety might the actually existing square root of a quantity be considered as the principle of obliquity; because, in certain cases, it indicates the hypothenuse of a right-angled triangle.

I would here notice an error in reasoning (as it appears to me), fallen into by all authors who have endeavoured to explain the mode of arriving at a true conclusion respecting the sines and cosines of multiple arcs; which reasoning imputes actual properties to ideal quantities, instead of deriving them all from mere arbitrary convention.

Given the sine, and consequently the cosine of an arc, to find the sine and cosine of n times that arc:

Let z the original arc, v the sine, and y the cosine, x the cosine of the arc $n z$, then as $\dot{z} : -\dot{y} :: 1 : \sqrt{1-y^2} \therefore \dot{z} = \frac{-\dot{y}}{\sqrt{1-y^2}}$ therefore

$$\frac{\dot{x}}{\sqrt{1-x^2}} = n \,.\, \frac{\dot{y}}{\sqrt{1-y^2}}$$

No integration can however be effected of these quantities in their actual form; but changing the signs of the terms in both denominators,

$$\frac{\dot{x}}{\sqrt{x^2-1}} = n \,.\, \frac{\dot{y}}{\sqrt{y^2-1}} \text{ and}$$

the nat. log. of $(x + \sqrt{x^2-1}) = n \times$ the nat. log. of $(y + \sqrt{y^2-1})$

and $x + \sqrt{x^2-1} = (y + \sqrt{y^2-1})^n = y^n + n y^{n-1} \,.\, \sqrt{y^2-1}$

$$+ n \frac{n-1}{2} y^{n-2} \,.\, \overline{y^2-1} + n \,.\, \frac{n-1}{2} \,.\, \frac{n-2}{3} \,.\, y^{n-3} \,.\, \sqrt[\frac{3}{2}]{y^2-1} \text{ \&c.}$$

But since y is taken as the cosine of the original arc, and x is the cosine of the multiple arc, and consequently each less than unity, it is obvious that the second term on the left of the equation, and that every even term of the expansion on the right, can exist only in the potential form of an ideal quantity; and a conclusion has thence been drawn (but as it seems to me on no solid principle whatever), that since real and imaginary quantities occur on each side of the equation, and they are of a nature completely heterogeneous one to the other, each must be respectively equal; but this mode of reasoning clearly imputes qualities to mere symbols beyond those originally imparted to them. The double equality, on my principles, depends entirely on its being assumed; as in the solution of cubic equations.

When $a + b$ have been substituted for x in the equation $x^3 - q x + r = 0$, and it becomes changed into $a^3 + b^3 + (3 a b - q) \times \overline{a+b} + r = 0$, two unknown quantities exist with but one relation; another may therefore be assumed; and that which obviously reduces the expression to the most simple form is obtained by making $3 a b - q = 0$.

In the same manner x and y, the cosines of two arcs, having but one relation, admit of another being assumed; any relation might be taken, but the one clearly indicated is that which makes the real terms on both sides equal:

this assumption leaves the ideal quantities in their actual state without any change, and when the sign of an imaginary operation has been removed in the usual way, they also become truly equal on each side of the equation, and from this double equality, series readily present themselves expressing the sine and cosine of the multiple arc required.

On the whole, it appears to me that mystery and paradox are entirely banished from the science most powerful in eliciting truth, and where they ought least to be found,—by considering all quantity as affirmative per se, and admitting plus and minus only as correlative terms; consequently, by extending the law of the signs, so as to make the multiplication of like signs give that of the scale in which the universal antecedent has been assumed, and the multiplication of unlike signs the contrary; and, finally, by excluding all actual existence, and thereby all inherent properties, from the symbols of quantities veiled by the mark indicative of an operation incapable of being performed, but arbitrarily endowing them with the properties of real quantities in all expansions and formulæ, and with the ultimate quality of regaining their actual existence whenever the veil is removed, by an operation the inverse of that which had originally induced it.

VI. *On the probable Electric Origin of all the Phenomena of Terrestrial Magnetism; with an illustrative Experiment. By* PETER BARLOW, *Esq. F.R.S. Cor. Mem. Inst. of France, of the Imperial Academy of St. Petersburgh, &c. &c.*

Read January 27, 1831.

IN order to show the incompatibility of the observed laws of terrestrial magnetism with the supposition of the earth itself being a magnet, and at the same time their accordance with the laws which appertain to a body whose magnetism is induced by electricity, it will be necessary to take a retrospective view of the several discoveries which have been made, connected with these subjects, since the commencement of the present century, and particularly within the last ten or twelve years. Up to the period of the scientific travels of M. HUMBOLDT, it must be admitted that all the facts with which we were acquainted, relative to terrestrial magnetism, were a mere collection of observations and phenomena, uncertain in themselves, unconnected with each other, and irreducible to any specific laws; but the confidence inspired by the high character of this distinguished traveller, his accuracy of observation, and the perfection of his instruments, gave a new feature to the inquiry, and laid the foundation of our present knowledge in this science.

M. BIOT was the first to undertake the difficult task of reducing these results to some principle of calculation. Considering the earth as a magnet, he assumed an indeterminate distance to represent the distance of its two poles; and then, supposing their power to vary inversely as the square of their distance from the point on which they acted, (a law which had been already established,) he obtained a general expression for the direction of a magnetic needle. He then made his indeterminate distance vary; and comparing at every step his results with those observed, it was found that the nearer the poles were made to approach, the nearer the computed and observed results corresponded; and finally, that the errors were reduced to a minimum when the two poles were coincident, or indefinitely near to each other.

This is a highly important determination as it relates to the present inquiry; for here we have at once a demonstration that the earth is not a magnet, or at least that it does not act according to the laws of a permanent magnetic body, the distinguishing characteristic of which is, the existence of two poles, distinct and distant from each other; whereas it is shown by this investigation, that in the earth these two poles are indefinitely near to each other and to the earth's centre.

M. Biot's results, however, in consequence of the generality he had given to his first assumption, were involved in a very intricate formula; but in the mean time a similar task had been undertaken by M. Kraft, of St. Petersburgh, on different principles. This philosopher contented himself with attempting to determine empyrically some mathematical formula to connect the different dip observations with each other; and he discovered this very simple relation, viz. "That the tangent of the dip of the needle in any place is equal to double the tangent of the magnetic latitude of that place." M. Biot, in consequence of this deduction, re-examined his former investigation, and found that his formula after certain reductions, of which he was not before aware it was susceptible, became identical with that of M. Kraft; and thus the fundamental law of terrestrial magnetism was confirmed by two philosophers acting apart and independently of each other, and on principles entirely different. It followed also from M. Biot's formulæ, that the intensity of the dipping needle ought to vary inversely as the square root of 4 *minus* 3 times the square of the sine of the dip; and that of the horizontal needle inversely as the square root of 3 *plus* the square of the secant of the dip: conditions which have since been verified, at least approximatively, by observations, extending even to within one or two degrees of the pole itself. And these laws, I beg again to repeat, are entirely inconsistent with those which appertain to a permanent magnetic body, while they will be shown, in what follows, to be the fundamental laws of a body which receives its transient magnetism by induction. We see thus how it happened, that those philosophers of the last century, who endeavoured to illustrate the laws of terrestrial magnetism by what they called terrellæ, or round natural magnets, with distinct poles, failed in all their attempts on this subject to establish any mathematical principle of calculation.

It has been seen that up to the year 1809, the phenomena of terrestrial mag-

netism had resisted every prior attempt to reduce them to the power of analysis. It will now be necessary to refer to those which relate to the reciprocal action of iron bodies and a magnetic needle on each other, which were still more uncertain, and apparently more anomalous, than the former.

In 1819 I undertook this investigation, and, after an extensive series of experiments, I succeeded in reducing this reciprocal action likewise to very simple laws, derived empyrically from the results I had obtained, but which were obviously only approximative, although the approximation was very close: in the course of these experiments also a very remarkable fact was discovered; namely, that all the magnetic power of an iron sphere resides on its surface. These first experiments, with the exception of the fact last mentioned, which I had not then discovered, were presented to the Royal Society in 1819. I afterwards extended my first researches, and published the whole in a small tract, entitled, "An Essay on Magnetic Attraction."

The simplicity of these results, and particularly the circumstance of the magnetic power residing on the surface only of the sphere, led Mr. Charles Bonnycastle, (at present mathematical professor in the University of Virginia,) to undertake an investigation of the laws which an iron sphere ought to exhibit according to a certain hypothesis relative to induced magnetism; and he succeeded in eliciting most of the formulæ I had obtained empyrically from my experiments. In the second edition of my Essay, I also examined the subject analytically, making a slight alteration in the hypothesis employed by Professor Bonnycastle; and by this means have been enabled to deduce all my experimental laws without exception, and to supply the small corrective part which was obviously wanted in the several formulæ as they were first obtained. Since that time, M. Poisson has employed his powerful analysis to investigate the subject in all its generality; and I have had the satisfaction of seeing confirmed by so distinguished a mathematician all my original propositions; and an action considered till that time anomalous, reduced to laws nearly as general and certain as those which govern the planetary motions.

The application of these deductions to the present inquiry is important. It follows from these laws, that if an iron sphere, such as we have supposed in a transient state of magnetic induction, be made to act upon a magnetic needle, isolated from the terrestrial magnetism, it will produce in that needle all the

phenomena which M. HUMBOLDT observed, and which MM. BIOT and KRAFT reduced to determinate laws of action *; the two poles in this case, as in that, being indefinitely near to each other, and to the centre of the sphere. Hence then it follows,

1. That the laws of terrestrial magnetism are inconsistent with those which belong to a permanent magnetic body.

2. That they are perfectly coincident with those which appertain to a body in a transient state of magnetic induction.

These results were incontrovertible; but an insuperable obstacle seemed still to oppose itself to any rational hypothesis relative to the cause of the earth's magnetic power. Up to this period we knew of only one means of inducing magnetism, which was by the approximation of a permanent magnet to a ball or mass of simple iron, and one or two other metals: but what body could be imagined capable of inducing this power in the earth; particularly as the earth preserved its magnetic energy constantly in nearly the same direction, whereas its position with regard to any exterior body was hourly changing? Its magnetism could not therefore be induced by any foreign body; and as no

* The formula indicating the position of a magnetic needle freely suspended from the combined action of the earth and an iron sphere upon it, is

$$\tan \Delta = \frac{3 \cos \phi \,.\, \sin \phi}{\frac{d^3 M}{r^3 C} + 3 \cos^2 \phi - 1}$$

where Δ is the deflection from the axis of the sphere, ϕ the complement of the magnetic latitude, M the magnetic power of the earth, r the radius of the iron sphere, d the distance of the needle from its centre, and C a constant coefficient dependent on the magnetic power of the iron. In this expression making M vanish according to the above supposition, and substituting $\phi + \delta'$ for Δ, so that δ' becomes the complement of the dip, we have

$$\tan (\phi + \delta') = \frac{3 \cos \phi \sin \phi}{3 \cos^2 \phi - 1}$$

which after an easy reduction becomes

$$2 \tan \delta' = \tan \phi, \quad \text{or} \quad \tan \delta = 2 \tan \lambda$$

where δ is the dip, and λ the magnetic latitude. By a similar process, calling I and I' the intensity of the dipping and horizontal needle, we find

$$I = 2\,A \sqrt{\frac{1}{4 - 3 \sin^2 \delta}} \quad \text{and } I' = 2\,A \sqrt{\frac{1}{3 + \sec^2 \delta}}$$

which are precisely M. BIOT's formulæ.—See Essay on Magnetic Attractions, page 195.

power in itself could be conceived capable of producing this effect, the deductions above made, although they would have stood incontrovertible, yet the causes of them would have remained inexplicable, but for the important discovery of Mr. Oersted, which threw a new light upon magnetic investigations.

This philosopher discovered that a wire conducting an electric current was, during the interval of transmission, in a state of magnetic induction. Such a discovery, at such a time, was most fortunate; and not on this point only, but on various others, it excited the attention of all the most distinguished philosophers of Europe. The number of interesting facts which were thus elicited will always form a prominent feature in the scientific history of the nineteenth century, but the greater part of them are unconnected with the present inquiry; I shall therefore only refer to one or two, which have an important bearing upon the question under investigation. As soon as I was informed of this interesting discovery, I was anxious, by experiments as nearly similar, as circumstances would admit, to those I had adopted with the iron ball, to elicit in this case also the laws which govern the reciprocal action of the wire and the needle; and after a pretty long series of experiments I arrived at this conclusion, viz.—

"That the force of each particle in the wire on each particle of the needle varies inversely as the square of the distance, and that the nature of the force is tangential, that is, such as would place a needle, neutralized from the earth's magnetism, always at right angles to the direction of the wire, and to the direction of the line joining the needle with the centre of action of the wire." This law, with an account of the experiments from which it was derived, was read to the Royal Society, May 22nd, 1822, and was afterwards published in the second edition of my Essay above referred to.

While I was thus engaged in endeavouring to elicit the law of action between the wire and the needle, M. Ampere had entered upon a much more extensive investigation; that is, not only of the reciprocal action between the wire and the needle, but also between different wires and galvanic currents on each other. Galvanic needles, both dipping and horizontal, were constructed, which possessed all the properties of the usual magnetic needles. The law of action of galvanic currents on each other was reduced to that of attraction; and by

assuming the magnetism of the needle to be due to an infinite number of galvanic currents parallel to each other and at right angles to the axis, the action of needles on each other, of these on galvanic currents, these currents on each other, and of the earth itself on each, were all reduced to one general principle, admitting of accurate and determinate calculation.

The view which I had taken of the subject was more limited. Having obtained a law which expressed a particular class of these phenomena, I proceeded no further; but it was satisfactory to me to find, that as far as these extended, the expressions were identical, and that all the observed phenomena due to every variety of experiment were equally explicable on the one or the other hypothesis. It will not be necessary to enter further into the many beautiful effects which were obtained by the various arrangements of different galvanic conductors. I shall therefore proceed at once to describe the Experiment alluded to in the head of this article, which is intended, if not to prove, at least to show the high degree of probability, that all terrestrial magnetic phenomena are due only to electricity, and that magnetism, as a distinct quality, has no real existence.

Having, as stated in the preceding part of this paper, discovered that the magnetic power of an iron sphere resides only on its surface; having also shown that when we suppose the earth's magnetism to vanish, the fundamental laws of terrestrial magnetism are exhibited by this superficial action,—the resulting expressions being identical with those obtained by M. Biot for the earth; it occurred to me, by a very natural induction, that if I could distribute over the surface of an artificial globe a series of galvanic currents, in such a way that their tangential power should every where give a corresponding direction to the needle,—such a globe ought to exhibit, while under electrical induction, all the magnetic phenomena of the earth upon a needle freely suspended above it, the needle itself being neutralized from the earth's magnetism, so as to leave it wholly under the influence of this superficial action. This idea was put to the test of experiment as follows.

I procured a wooden globe sixteen inches in diameter, which was made hollow for the purpose of reducing its weight; and while still in the lathe, grooves were cut to represent an equator, and parallels of latitudes at every $4\frac{1}{2}°$ each way from the equator to the poles; these grooves were about one

eighth of an inch deep and broad: and lastly, a groove of the same breadth, but of double the depth, was cut like a meridian, from pole to pole, half round. These grooves were for the purpose of laying in the wire, which was effected thus. The middle of a copper wire, nearly ninety feet long and one-tenth of an inch in diameter, was applied to the equatorial groove, so as to meet in the transverse meridian; it was then turned down that groove, one end towards one pole, the other towards the other pole, as far as the first parallel; it was then made to pass round this parallel, returned again along the meridian to the next parallel, then passed round this again, and so on till the wire was thus led in continuation from pole to pole.

The length of wire still remaining at each pole was bound with varnished silk, to prevent contact, and then returned from each pole along the meridian groove to the equator: at this point, each wire being fastened down with small staples, the two wires for the remaining five feet were bound together to near their common extremity, where they opened, to form two points for connecting the poles of a powerful galvanic battery. When this connection was made, the wire became of course an electric conductor, and the whole surface of the globe was put into a state of transient magnetic induction; and consequently, agreeably to the laws of action above described, a neutralized needle freely suspended above such a globe, would arrange itself in a plane passing from pole to pole through the centre, and take different angles of inclination according to its situation between the equator and either pole.

In order to render the experiment more strongly representative of the actual state of the earth, the globe in the state above described was covered by the gores of a common globe, which were laid on so as to bring the poles of this wire arrangement into the situation of the earth's magnetic poles, according to the best observations we have for this determination. I therefore placed them according to the mean results of Sir Edward Parry and Captain Foster in latitude 72° N. and 72° S., and on the meridian corresponding with longitude 76° W., by which means the magnetic and true equators cut one another in about 14° E. and 166° W. longitude.

The globe being thus completed*, a delicate needle must be suspended above

* This globe was constructed in 1824, and exhibited by Dr. Birkbeck, March 26th of that year, at the London Institution.

it, neutralized from the effect of the earth's magnetism, according to the principle I employed in my observations on the daily variation, and described in the Philosophical Transactions for 1823; by which means it will become entirely under the superficial galvanic arrangement just described. Conceive now the globe to be placed so as to bring London into the zenith; then, the two ends of the conducting wire being connected with the poles of a powerful battery*, it will be seen immediately that the needle, which was before indifferent to any direction, will have its north end depressed about 70°, as nearly as the eye can judge, which is the actual dip in London; it will also be directed towards the magnetic poles of this globe, thereby also showing a variation of about 24° or 25° to W., as is also the case in London. If now we turn the globe about on its support, so as to bring to the zenith places equally distant with England from the magnetic pole, we shall find the dip remains the same; but the variation will continually change, becoming first zero, and then gradually increasing to the eastward as happens on the earth. If again we turn the globe so as to make the pole approach the zenith, the dip will increase, till at the pole itself the needle will become perfectly vertical. Making now this pole recede, the dip will decrease, till at the equator it vanishes, the needle becoming horizontal. Continuing the motion, and approaching the south pole, the south end of the needle will be found to dip, increasing continually from the equator to the pole, where it becomes again vertical, but reversed as regards its verticality at the north pole.

Nothing can be expected nor desired to represent more exactly on so small a scale all the phenomena of terrestrial magnetism, than does this artificial globe: besides, we know from the mathematical laws of action which have been referred to, that it is not merely an exhibition of effects, but that if we could increase our currents indefinitely, every circumstance of dip and direction would admit of actual and accurate computation.

I may therefore, I trust, be allowed to say, that I have proved the existence of a force competent to produce all the phenomena of terrestrial magnetism, without the aid of any body usually called magnetic, except perhaps it may be

* My battery consisted of twenty zinc and twenty copper plates, ten inches square: but the experiment may be shown satisfactorily by STURGEON's circular battery with about two feet of copper and one of zinc.

said the small needle employed for exhibiting the effects; and even this, but for its necessary minuteness, might be replaced by a galvanic needle on M. Ampere's construction. At the same time, I must beg again to observe, that it follows from the laws obtained by M. Biot, that no position of a single magnet, nor the arrangement of any numbers of such bodies within the globe, could by possibility exhibit the same phenomena, particularly as relates to the intensity of the needle.

I am quite aware that, after all, this does not amount to a demonstration that the magnetic phenomena of the earth are produced only by electricity; yet seeing as we do, in every operation of natural effects with which we are acquainted, that the agents employed are not more numerous than necessary, it will perhaps be admitted that I have at least shown the high probability that all terrestrial magnetic phenomena are due to some particular modification of electricity, and that magnetism, as a distinct quality, has no real existence in nature.

It is true, that as far as the discovery of Mr. Oersted goes, we have no idea how such a system of currents can have existence on the earth, because, to produce them, we have been obliged to employ a particular arrangement of metals, acids, and conductors; but, fortunately, a subsequent step, not less important than the former, was made by Professor Seebeck of Berlin, who discovered that the mere application of heat to a circuit composed of two metals, was competent to produce the same development of galvanic and magnetic effects as those above described; and there can be no doubt, that if the conducting wire of the globe I have described, were removed, and each parallel made complete in two metals, that all the phenomena it now exhibits by aid of the galvanic battery might be represented by the application of heat only.

The effect of heat is so obvious in the production of magnetic developments, that I have seen a rectangular circuit on Professor Seebeck's principle, constructed by Messrs. Watkins and Hill, which, by the momentary application of a spirit lamp, became sufficiently magnetic to deflect the needle several degrees, and even the minute change of temperature that could be produced by contact with the hand, exhibited a sensible effect: this circuit, however, was formed of two metals as proposed by Professor Seebeck; but Mr. Sturgeon of

Woolwich has been enabled to produce a similar effect with a rectangle of bismuth only.

This important discovery of M. Seebeck brings us therefore, as it were, a step nearer to our object, by referring us to the sun as the great agent of all these phenomena; indeed but one link seems wanted to connect together a chain of highly interesting phenomena, and thereby to reduce to simple and intelligent principles what has hitherto been considered amongst the most mysterious laws of nature.

P.S. I have not in the above article made any reference to the irregularity of the magnetic lines on the earth. I have spoken of the law as deduced by M. Biot, as if it answered accurately all the conditions required: it is however very well known that there are irregularities which it will not reach, and much credit is due to Professor Hansteen for the talent and industry he has applied in the collection of results, and the reduction of them to principles of calculation. These discrepancies however are by no means opposed to the foregoing view of the subject, but are, on the contrary, rather favourable to it; for if, as is implied in the preceding remark, the development of terrestrial magnetic phenomena be due to the transmission of caloric and inequality of temperature, we ought to expect the same kind of irregularities in this action as we know to exist in the temperature and climate of places situated geographically the same.

VII. *On the Equilibrium of Fluids, and the Figure of a Homogeneous Planet in a Fluid State. By* James Ivory, *A.M. F.R.S. Instit. Reg. Sc. Paris. Corresp. et Reg. Sc. Gottin. Corresp.*

Read January 13 and 20, 1831.

I. *Equilibrium of Fluids.*

1. THE nature of the ultimate particles of a fluid, and the peculiar manner of their mutual connection, are entirely unknown to us. We conceive that they obey the same mechanical laws to which all matter is subject. Experience shows that the particles of a fluid move freely among one another, yielding to the least pressure in any direction; and this is the most general property of such bodies that has yet been discovered. The perfect mobility of their particles must therefore, in the present state of our knowledge, be considered as constituting the definition of fluid bodies, and as the foundation of all our reasoning concerning them. We here confine our attention to a fluid in equilibrium, or at rest, in which state every particle is pressed equally on all sides. It is evident that the mobility of the particles among one another, and their readiness to obey any new impulse, is nowise impeded by the magnitude of their mutual pressure, since this acts at every point with the same intensity in all directions.

If we set aside the effect of gravity, and of all accelerating forces, it follows, from the definition, that the pressure will be equal in all parts of a continuous fluid at rest. In this state we must conceive that the particles are equally distant, and arranged similarly about every interior point. Their mutual distance, it is natural to think, must be connected with the magnitude of pressure; so that when they are more pressed, they will approach one another, and the volume will be diminished; and, when they are less pressed, they will recede from one another, and the volume will be enlarged. Accordingly it is found that no fluid is perfectly incompressible. But in some, such as water

and other liquids, a very great external force must be applied to produce an almost imperceptible variation of bulk; while in others, such as air and the gases, very notable changes of volume are caused by moderate compression. In the investigation of the properties of the first sort of fluids, to which our attention is here exclusively directed, we shall throw out of view the very small degree of compressibility they possess, and shall suppose them to retain the same bulk whatever changes of figure or pressure they may undergo.

In a fluid in equilibrium, the action of the accelerating forces that urge the particles must be counterbalanced by the pressure propagated through the mass: to find the relation between these opposite forces must therefore be the first object of research.

2. Assuming three planes intersecting at right angles which, by the co-ordinates drawn to them, ascertain the position of the particles of the fluid, we shall suppose two points or particles (x, y, z) and $(x + \delta x, y + \delta y, z + \delta z)$ at the infinitely small distance δs from one another; and we shall put ω for the small base of an upright cylinder or prism of the fluid placed between the two points, and having δs for its length: then the density of the fluid being invariable and represented by unit, and the quantity of matter of the cylinder or prism being denoted by dm, we shall have

$$dm = \omega \times \delta s.$$

Let all the accelerating forces which act upon the particle (x, y, z) be reduced to the directions of the coordinates; and put X, Y, Z for the sums of the reduced forces respectively parallel to x, y, z; then because $\frac{\delta x}{\delta s}, \frac{\delta y}{\delta s}, \frac{\delta z}{\delta s}$ are the cosines of the angles which the line δs makes with x, y, z, the partial forces urging the particle in the direction of δs, will be $\mathrm{X}\frac{\delta x}{\delta s}$, $\mathrm{Y}\frac{\delta y}{\delta s}$, $\mathrm{Z}\frac{\delta z}{\delta s}$, and, if we put

$$f = \mathrm{X}\frac{\delta x}{\delta s} + \mathrm{Y}\frac{\delta y}{\delta s} + \mathrm{Z}\frac{\delta z}{\delta s},$$

the whole accelerating force urging the particle (x, y, z) in the direction of δs, will be equal to f. Multiply now by the equal quantities dm and $\omega \delta s$, and the result will be

$$f\,dm = \omega(\mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z).$$

As the quantities δx, δy, δz, δs, ω may be assumed as small as we please, the force f may be considered as retaining the same value for all the particles of the cylinder or prism; and therefore $f\,dm$ is the motive force of the cylinder or prism, or the effort it makes to move in the direction of δs from the point (x, y, z) to the point $(x + \delta x, y + \delta y, z + \delta z)$.

Let p represent the hydrostatic pressure of the fluid at the point (x, y, z). This term is used to denote the pressure relatively to the surface pressed: it is the whole pressure any surface sustains divided by the extent of surface; or it is the actual pressure reduced to the unit of surface. The hydrostatic pressure is obviously variable in the different parts of a fluid, the particles of which are urged by accelerating forces; and as it can vary only when its point of action is changed, it must be a function of the coordinates of that point. The whole pressure upon the end of the cylinder or prism at the point (x, y, z) will be equal to $p \times \omega$; for we may suppose that p undergoes no change in the small extent of the surface ω: and, in like manner, the whole pressure upon the opposite end will be equal to $(p + \delta p) \times \omega$. As the pressures upon the two ends act against one another, their effect to move the cylinder or prism in the direction of δs from the point $(x + \delta x, y + \delta y, z + \delta z)$ to the point (x, y, z) will be equal to $\delta p \times \omega$; and this force, on the supposition that the particles of the fluid are at rest, must be equal to $f\,dm$, the directly opposite effect caused by the accelerating forces. We therefore have this equation for expressing the non-effect of the equal and opposite forces, viz.

$$\delta p \times \omega + f\,dm = 0:$$

and, if we substitute the value of $f\,dm$ found before, we shall get

$$\delta p + \mathrm{X}\,\delta x + \mathrm{Y}\,\delta y + \mathrm{Z}\,\delta z = 0. \qquad (1)$$

This equation must take place at every point of the mass of fluid without any relation being supposed between the variations δx, δy, δz; which condition will not be fulfilled unless p be a function of the three independent variables x, y, z. We therefore have

$$\delta p = \frac{dp}{dx}\,\delta x + \frac{dp}{dy}\,\delta y + \frac{dp}{dz}\,\delta z:$$

and, if we substitute this value of δp in the formula (1), the independence of the variations will require these three separate equations,

$$\frac{dp}{dx} = -\mathrm{X}$$

$$\frac{dp}{dy} = -\mathrm{Y}$$

$$\frac{dp}{dz} = -\mathrm{Z}.$$

From this it appears that the algebraic expressions of the forces are not entirely arbitrary; for they must be equal to the partial differential coefficients of a function of three independent variables. By differentiating we shall readily obtain the following equations which do not contain the function p, viz.

$$\frac{d\mathrm{X}}{dy} = \frac{d\mathrm{Y}}{dx}, \quad \frac{d\mathrm{X}}{dz} = \frac{d\mathrm{Z}}{dx}, \quad \frac{d\mathrm{Y}}{dz} = \frac{d\mathrm{Z}}{dy}.$$

Unless the forces possess these properties, which are the well-known conditions of integrability, the equation (1) will not hold in all parts of the mass of fluid, and the equilibrium will be impossible. But in the physical questions that actually occur, the forces of nature being either attractions or repulsions directed to fixt centres, and proportional to certain functions of the distances from those centres, they necessarily fulfil the conditions of integrability.

The whole of what has been said is succinctly expressed by the two following equations,

$$\left.\begin{array}{l} \varphi = \int (\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz), \\ p = \mathrm{C} - \varphi, \end{array}\right\} \qquad (2)$$

Here φ represents a function of three independent variables x, y, z without any arbitrary quantity; the constant C required by the integration is necessary only in the expression of p.

3. The hydrostatic pressure at every point of the mass of fluid in equilibrium, is expressed by the second of the equations (2), viz.

$$p = \mathrm{C} - \varphi.$$

But at all those parts of the outer surface of the fluid which are unconfined and entirely at liberty, there is no pressure; wherefore we have, for the equation of all such surfaces,

$$\varphi = \mathrm{C}.$$

It may be proper to remark, that although this equation is universally true, yet it is no new or independent condition of the equilibrium; it is merely an inference from the general expression of the hydrostatic pressure.

If we assume two points (x, y, z) and $(x + dx, y + dy, z + dz)$ indefinitely near one another in a part of the outer surface at liberty, we shall have, in consequence of the foregoing equation,

$$\frac{d\Phi}{dx}dx + \frac{d\Phi}{dy}dy + \frac{d\Phi}{dz}dz = 0;$$

or, which is the same thing,

$$X\,dx + Y\,dy + Z\,dz = 0;$$

and if ds represent the distance of the two points, we obtain

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = 0.$$

Now $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ are the cosines of the angles which the directions of the forces make with the line ds; wherefore the expression on the left side of the foregoing formula is the sum of the partial forces which act in the direction of ds; and as this sum is equal to zero in all positions of the line ds round the point (x, y, z), the resultant of the forces produces no effect in the plane touching the surface, and consequently its whole action is perpendicular to that plane. The nature of the case requires further, that the same resultant be directed towards the surface of the fluid.

What has been deduced from the algebraic expressions is evident in another view. For, could we suppose that the resultant of the forces is not at every point perpendicular to the surface at liberty, it might be resolved into two partial forces, one acting in the tangent plane, and the other perpendicular to that plane; and as the first force is opposed by no obstacle, it would cause the particles to move, which is contrary to the equilibrium.

If we suppose that p is constant in the general formula of the hydrostatic pressure, we shall have an equation,

$$\varphi = C - p,$$

which is exactly similar to that of the surface at liberty, and which will determine an interior surface at every point of which there is the same intensity of

pressure. By differentiating the equation of the interior surface, we obtain

$$\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz = 0\,;$$

from which we deduce, by the like reasoning as before, that such surfaces are perpendicular to the resultant of the accelerating forces urging the particles contained in them. The interior surfaces in question were named level surfaces by CLAIRAUT; and they are distinguished by the two properties of being equally pressed at all their points, and of cutting the resultant of the forces at right angles. They spread through the mass, and ultimately coincide with those parts of the outer surface which are at liberty. It may be observed, that what essentially constitutes a level surface is its equation, which must differ from the equation of the outer surface at liberty in no respect, except that the constant $\mathrm{C} - p$ takes the place of the constant C; for we shall afterwards find that, in some cases of the equilibrium of a fluid, the two properties of being equally pressed, and of cutting the resultant of the forces at right angles, belong to more sets of interior surfaces than one.

4. In what goes before, we have supposed that the density is constant, but it is easy to extend the investigation to heterogeneous fluids. Let ϱ be put for the function of the co-ordinates which expresses the variable density; then admitting that ϱ has the same value at every point of the small elementary cylinder or prism, we shall have

$$dm = \varrho\,\omega\,\delta s\,;$$

but, f being the whole accelerating force, urging every particle of dm in the direction of δs, we have

$$f = \mathrm{X}\frac{\delta x}{\delta s} + \mathrm{Y}\frac{\delta y}{\delta s} + \mathrm{Z}\frac{\delta z}{\delta s};$$

wherefore,

$$f\,dm = \varrho\,\omega\,(\mathrm{X}\,\delta x + \mathrm{Y}\,\delta y + \mathrm{Z}\,\delta z).$$

The equation expressing that the action of the accelerating forces is equal and opposite to the variation of pressure, is the same as before, viz.

$$\omega \times \delta p + f\,dm = 0\,;$$

and by substituting the value of $f\,dm$, we deduce

$$\delta p + \varrho\,(\mathrm{X}\,\delta x + \mathrm{Y}\,\delta y + \mathrm{Z}\,\delta z) = 0. \qquad (3)$$

This equation must hold at every point of the mass of fluid without any relation being supposed between the variations, wherefore p must be a function of three independent variables; and in consequence the foregoing equation implies the three separate equations following, viz.

$$\frac{dp}{dx} = -\varrho X, \quad \frac{dp}{dy} = -\varrho Y, \quad \frac{dp}{dz} = -\varrho Z.$$

It now appears that the conditions of integrability must be fulfilled, viz.

$$\frac{d.\varrho X}{dy} = \frac{d.\varrho Y}{dx}, \quad \frac{d.\varrho X}{dz} = \frac{d.\varrho Z}{dx}, \quad \frac{d.\varrho Y}{dz} = \frac{d.\varrho Z}{dy}:$$

and unless the forces possess the properties expressed by these equations, the equilibrium will be impossible.

Without pursuing the investigation in all its generality, we shall confine our attention to the case in which

$$X\,dx + Y\,dy + Z\,dz,$$

is an exact differential; a supposition that comprehends all the applications of the theory. If we represent the integral of the differential by φ, so that

$$d\varphi = X\,dx + Y\,dy + Z\,dz;$$

and convert the variations of equation (3) into differentials, we shall obtain

$$dp + \varrho\,d\varphi = 0;$$

and hence

$$p = C - \int \varrho\,d\varphi. \tag{4}$$

From this we deduce the equation of those parts of the outer surface which are at liberty, by making $p = 0$; and that of a level surface, by assigning to p some constant value. And if we differentiate the same equation (4) on the supposition that p is invariable, we shall get

$$\varrho\,d\varphi = \varrho(X\,dx + Y\,dy + Z\,dz) = 0,$$

which differential equation is common to the outer surface at liberty, and to all the interior level surfaces; and from which we deduce by the like reasoning as before, that all such surfaces are perpendicular to the resultant of the accelerating forces urging the particles contained in them.

The quantity under the sign of integration in the formula,

$$p = C - \int \varrho\,d\varphi,$$

must be an exact differential, for p must be a function of the co-ordinates; which condition will not be fulfilled unless ϱ be a function of φ. Thus both the pressure p and the density ϱ are functions of the same quantity φ, and they are both invariable where φ is constant. The density is therefore the same at all the points of any level surface. If we conceive a heterogeneous fluid in equilibrium to be divided into thin strata by level surfaces infinitely near one another, the density will be the same throughout every stratum, but it will vary from one stratum to another.

5. We have now placed before the reader the general points of the theory of the equilibrium of fluids. What has been said comprehends all that can be determined when a fluid is conceived to extend indefinitely; but in applying the theory to limited masses, it is necessary besides, that the pressures propagated through the interior parts either be supported or mutually balance one another.

In treating of the equilibrium of fluids, another mode of investigation is sometimes employed, which it would be improper to pass by without notice, as it is useful on many occasions to fix the imagination, although it leads to no new results. We allude to the narrow canals supposed to traverse the mass in various ways, of which so much use has been made by Clairaut and other authors.

Let two points (x°, y°, z°) and (x', y', z') be assumed in the interior of a mass of fluid in equilibrium, and conceive an infinitely narrow canal of any figure to pass between them; we may suppose that the whole fluid, except the portion within the canal, becomes solid without any change taking place in the position of the particles, or in their mutual action upon one another; for, as this supposition makes no alteration of the forces urging the particles contained within the canal, these particles will remain at rest after the solidification as they were at first. Suppose that the canal is divided into infinitely small parts by sections perpendicular to its sides; at any point (x, y, z) let ω be the section; δs the infinitely small part of the length of the canal; dm the quantity of matter in the length δs, that is, the product of the volume and the density, or $\varrho \times \omega \times \delta s$; and f the sum of all the partial forces that urge the particles of dm in the direction of the canal; then, the motive force of dm, or its effort to move, will be equal to $f\,dm$. Further, p being the hydrostatic

pressure at the point (x, y, z), the like pressure at the distance of δs will be $p + \delta p$; therefore the opposing pressures which act upon the two ends of the part of the canal in the length δs, will be $p \times \omega$ and $(p + \delta p) \times \omega$; and $\delta p \times \omega$ will be the effective pressure which pushes dm towards the point (x, y, z). Because every part of the canal is supposed at rest, the tendencies of dm to move in opposite directions must be equal, and we shall have this equation,

$$\delta p \times \omega + f\,dm = 0;$$

consequently,

$$\delta p + \frac{f\,dm}{\omega} = 0;$$

and by taking the sum of the similar quantities in all the parts of the canal, we obtain

$$\int \delta p + \int \frac{f\,dm}{\omega} = 0.$$

But p being a function of three independent variables, the sum of its variations, supposing the flowing quantities to follow any arbitrary law of increase or decrease, is equal to the difference of p' and p°, the final and initial values of the function; wherefore we have

$$p' - p^{\circ} + \int \frac{f\,dm}{\omega} = 0.$$

Now $f\,dm$, that is the quantity of matter multiplied by the accelerating force, is the impulse or pressure in the direction of the canal caused by all the forces urging dm; and as this pressure is exerted on the surface ω, $\frac{f\,dm}{\omega}$ is the same pressure reduced to the unit of surface. Therefore, whatever be the figure of the canal, it follows from the foregoing investigation, that the difference of the pressures at its two extremities is equal to the sum of the impulses of all the contained molecules of fluid, every impulse being reduced to the direction of the canal and to the unit of surface.

If the extremities of the canal be both in the parts of the outer surface which are at liberty, the pressures p' and p° will be both evanescent, and there will be no effort of the fluid either way, and no tendency to run out at one end. Further, if a canal be continued through the fluid till it return into itself, the

initial and final pressures being the same, the impulses of the molecules in the whole circuit will balance one another. But in this case, the reasoning we have employed will not be exact, unless p, the algebraic expression of the pressure, be such a function as admits of only one value for any three given co-ordinates; a restriction however, which, in every point of view, seems indispensable.

6. The whole theory, it will readily appear from the foregoing investigations, is built on the assumption, That the hydrostatic pressure at every point of the fluid is the same function of the co-ordinates of the point. The accelerating forces are represented by the partial differential coefficients of the pressure; and therefore they are likewise the same functions of the co-ordinates of their point of action in every part of the mass. The whole reasoning rests on these fundamental points; and if the state of a fluid were such that they are not verified, the equations for determining the required figure could not be formed, and the equilibrium would be impossible. As the hydrostatic pressure is known only by means of the given accelerating forces, it seems most suitable to employ the properties of the latter in laying down what is required for the equilibrium of a mass of fluid. It is necessary, and it is sufficient for the equilibrium of a homogeneous fluid, first, that the accelerating forces acting in the directions of the co-ordinates be, in every part of the mass, the same functions of the co-ordinates; and, secondly, that these functions possess the conditions of integrability. When these two conditions are both fulfilled, the determination of the figure of equilibrium is reduced to a question purely mathematical. For we can form the equation (1) which makes the accelerating forces balance the variation of pressure; and, by integrating this equation, we obtain the hydrostatic pressure, from which is deduced the equation of all those points at which there is no pressure, or in other words, the equation of all those parts of the outer surface which are at liberty. Nothing more is required for securing the permanence of the figure of the fluid, except that the pressures propagated through the mass be either supported or mutually balance one another.

The conditions for the equilibrium of a homogeneous fluid, as they are here laid down, do not enable us in all cases to form immediately the equation of the figure of equilibrium. If the particles attract or repel one another, the accelerating forces will, for the most part, vary as the fluid changes its form,

and they may not be at every point the same functions of the co-ordinates in all the figures, of which it is susceptible; but, notwithstanding the equilibrium may still be possible, because this indispensable condition may be fulfilled when figures of a certain class are induced on the mass. In such cases, the determination of the equilibrium necessarily requires two distinct researches; of which one is to find out what are the particular figures into which the mass must be moulded, so as to make the accelerating forces at every point the same functions of the co-ordinates. After these figures have been found, we can apply to them the equations expressing the conditions of equilibrium, and accomplish the mathematical solution of the problem. But if it shall appear that no figure whatever capable of fulfilling both the conditions laid down above can be induced on the fluid, the equilibrium will be absolutely impossible.

In the usual exposition of this theory, the equilibrium is made to depend on conditions that do not exactly coincide with those at which we have arrived. According to Clairaut and all other authors who have written on this subject, it is necessary, and it is sufficient, for the equilibrium of a homogeneous fluid, first, that the expressions of the accelerating forces possess the criterion of integrability; secondly, that the resultant of the forces in action at all the parts of the outer surface which are at liberty, be directed perpendicularly towards these surfaces. We may throw out of view what regards the criterion of integrability, about which there is no difference of opinion, and which in reality is always fulfilled by the forces that occur in physical researches. The perpendicularity of the forces to the outer surface is a property of the differential equation of that surface, and will necessarily take place whenever it is possible to form that equation. Nothing more is required for forming the equation mentioned, than that the accelerating forces at every point of it be expressed by the same functions of the co-ordinates of the point.* It follows

* The forces are perpendicular to every surface in which the pressure is constant. The outer surfaces are those at every point of which there is no pressure. In all the questions that have occurred, the forces at the outer surface of the fluid are the same functions of the co-ordinates of the point, whatever geometrical figure the fluid is supposed to assume; and on this account the equation of the outer surface can be formed without reference to any particular class of figures. But this is not sufficient; for, according to the fundamental assumption laid down by Clairaut himself, the theory of equilibrium cannot be applied, unless the forces be the same functions of the co-ordinates of their point of action in every part of the mass.

therefore, that the difference between the conditions of equilibrium hitherto universally adopted, and those laid down above, amounts to this: according to the former it is required that the expressions of the accelerating forces be the same functions of the co-ordinates at every point of the outer surface, this being all that is necessary for forming the differential equation of that surface; according to the latter, the forces will not balance the pressure, and the laws of equilibrium will not be fulfilled unless the forces be the same functions of the co-ordinates at every point whether situated in the outer surface, or in the interior part of the mass.

If a homogeneous fluid, of which the particles are urged by accelerating forces be in equilibrium, all that is required by CLAIRAUT's theory will undoubtedly be fulfilled; but the converse of this cannot be affirmed. It is no where proved generally by unexceptionable arguments, and indeed no proof can possibly be given, that the forces in the interior parts of the fluid will balance the pressure, merely because the resultant of the forces in action at the outer surface is perpendicular to that surface. All the attempts that have been made to demonstrate this point, tacitly assume that the expression of the forces is the same at the surface and in all the interior parts; which is not universally true.

In a very extensive class of problems the difference between the two ways of laying down the conditions of equilibrium disappears. This will happen when the accelerating forces are independent of the figure of the fluid, as will be the case if the particles exert no action on one another by attraction or repulsion. In such problems the forces impressed upon every particle, whatever be its situation, and whatever be the figure of the fluid, are by the hypothesis, the same given functions of the co-ordinates. The figure of equilibrium will be the same whether, following CLAIRAUT, we obtain the equation of the outer surface by means of the forces in action at that surface, or, making use of the property that the pressure vanishes at all the points where the fluid is at liberty, we deduce the same equation from the pressure that prevails generally throughout the mass.

But CLAIRAUT's theory cannot be extended to the solution of other problems than those of which we have been speaking. In no other cases is it evident without inquiry that the proposed accelerating forces urging a particle, are, in

every part of the mass, the same functions of the coordinates of the particle; and unless this be verified, the theory of equilibrium cannot be applied. In a homogeneous planet in a fluid state, there are forces which prevail in the interior parts and vanish at the surface; and, as Clairaut's theory notices no forces except those in action at the surface, it leaves out some of the causes tending to change the figure of the fluid, and therefore it cannot lead to an exact determination of the equilibrium.

II. *Application of the foregoing Theory to the Question of the Figure of the Planets.*

7. Having now explained the general theory of the equilibrium of fluids at sufficient length, I proceed to apply it to the question of the figure of the planets, in which it is required to determine the equilibrium of a fluid entirely at liberty, and unconfined by any obstacle or support. The problem is one of considerable difficulty. It is necessary to distribute the investigation under distinct heads. It would otherwise be impossible to preserve perspicuity and precision of ideas in an inquiry essentially different in different hypotheses. The equilibrium of a homogeneous fluid must occupy our attention before that of one having its density variable. For although it may at first appear that the latter problem is the more general, and includes the former, yet it will be found that the equilibrium of a fluid of variable density, depends upon that of a homogeneous fluid, and is deducible from it. And even with regard to homogeneous fluids, distinctions must be made, because what is required for the equilibrium varies with the nature of the accelerating forces. In this respect we distinguish these two general cases, of which we shall treat in two separate problems; First, when the accelerating forces depend only on the coordinates of their point of action, and are explicitly known when the coordinates are given; Secondly, when the accelerating forces depend not only upon the coordinates of the particle on which they act, but likewise upon the figure of the whole mass of fluid; as happens for the most part when the particles attract or repel one another.

Problem 1st.—To determine the equilibrium of a homogeneous mass of fluid which is entirely at liberty, when the accelerating forces are known functions of the coordinates of their point of action.

The equilibrium of a mass of fluid which is entirely at liberty, can depend only upon the action of such forces as tend to change the relative position of the particles with respect to one another. It is not affected by any motion common to all the particles, nor by any force which acts upon them all with the same intensity in the same direction; the effect of such motion, or of such force, being to displace the centre of gravity of the whole mass without altering the relative situation of the particles. In estimating the accelerating forces upon which the figure of equilibrium will depend, we must therefore begin with reducing the centre of gravity, if it be in motion or urged by any force, to a state of relative rest; which is accomplished by applying to every particle a force that would cause it to move with the same velocity as the centre of gravity, but in a contrary direction. In the investigation of this problem we may therefore suppose that the centre of gravity is at rest and undisturbed by the action of any accelerating force.

Suppose now that a mass of homogeneous fluid entirely at liberty, is in equilibrium, and conceive three planes intersecting at right angles in the centre of gravity of the mass, to which planes the particles of the fluid are to be referred by rectangular coordinates. Let x, y, z, represent the coordinates of a particle, and having resolved the accelerating forces acting upon it into other forces that have their directions parallel to the coordinates, put X, Y, Z, for the sums of the resolved parts respectively parallel to x, y, z, and tending to shorten these lines. According to the hypothesis of this problem, the forces X, Y, Z, depend only upon the coordinates of their point of action; and they are at every point the same functions of those coordinates. The equilibrium will therefore be impossible unless

$$\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz$$

be an exact differential, this being necessary in order that the hydrostatic pressure be a function of three independent variables as the fundamental assumption of the theory demands. Let φ denote the integral, and p the

hydrostatic pressure at the point $(x, y, z,)$: the equations that determine the equilibrium will be these two *,

$$\left.\begin{array}{l} \phi = \int (\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz), \\ p = \mathrm{C} - \phi. \end{array}\right\}$$

If we make $p = 0$, we shall obtain the equation of the outer surface of the fluid, viz.

$$\phi = \mathrm{C}.$$

The differential equation,

$$\frac{d\phi}{dx}\,dx + \frac{d\phi}{dy}\,dy + \frac{d\phi}{dz}\,dz = 0,$$

or which is the same,

$$\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz = 0,$$

is common to the outer surface and to all the interior level surfaces at every point of which there is the same intensity of pressure; and it shows that the resultant of the accelerating forces is perpendicular to all such surfaces †.

The figure of the fluid being determined, it remains to inquire whether the equilibrium is secured. By varying the coordinates in the formula for p, we obtain

$$\delta p + \frac{d\phi}{dx}\delta x + \frac{d\phi}{dy}\delta y + \frac{d\phi}{dz}\delta z = 0:$$

which equation proves that, if a particle be moved from its place a very little in any direction, the variation of the intensity of pressure is equal and opposite to the action of the accelerating forces. A particle has therefore no tendency to move from inequality of pressure. But we must not from this hastily conclude that there is no cause tending to change the figure of the fluid. For, as in the simple case of a fluid contained in a vessel, the equilibrium requires not only that the accelerating forces balance the inequality of pressure, but likewise that the total pressures tending outward at the boundaries of the mass, be supported by the sides of the vessel; so in the problem under consideration, there being no external support, the figure of the fluid will not be permanent

* Equation (2) § 2. † Equation (2) § 3.

unless the pressures propagated inward, which increase as any point sinks deeper below the surface, mutually compensate and destroy one another. Some further discussion is therefore necessary in order to prove that the equilibrium is completely established.

The function φ, in which we may suppose there is no constant quantity, can contain no term having the coordinates for divisors; for, were this the case, the pressure would be infinite at all those points where such coordinates are equal to zero. Let the terms of φ be arranged in homogeneous expressions of one, two, three, &c. dimensions; then

$$\begin{aligned}\varphi = {} & (A_1 x + A_2 y + A_3 z) \\ & + (B_1 x^2 + B_2 y^2 + B_3 z^2 + B_4 xy + B_5 xz + B_6 yz) \\ & + (D_1 x^3 + D_2 y^3 + D_3 z^3 + D_4 x^2 y + \&c.) \\ & + \&c.\end{aligned}$$

Differentiate this expression, and after the operations put $x = 0$, $y = 0$, $z = 0$: then

$$\frac{d\varphi}{dx} = A_1, \quad \frac{d\varphi}{dy} = A_2, \quad \frac{d\varphi}{dz} = A_3.$$

But the differentials of φ are no other than the expressions of the accelerating forces acting on a particle; consequently A_1, A_2, A_3 are the forces in action at the origin of the coordinates, that is, at the centre of gravity of the mass. Wherefore, according to what was observed, we shall have

$$\begin{aligned}& A_1 = 0, \quad A_2 = 0, \quad A_3 = 0, \\ \varphi = {} & (B_1 x^2 + B_2 y^2 + B_3 z^2 + B_4 xy + B_5 xz + B_6 yz) \\ & + (D_1 x^3 + D_2 y^3 + D_3 z^3 + D_4 x^2 y + \&c.) \\ & + \&c.\end{aligned}$$

That the expression of φ must be of this form is required by the nature of the problem: for φ must be always positive, and it must increase continually from the centre of gravity to the surface of the fluid.

Let us now put

$$\begin{aligned} x &= r \cos\theta = r\xi, \\ y &= r \sin\theta \cos\psi = r\eta, \\ z &= r \sin\theta \sin\psi = r\zeta,\end{aligned}$$

then r will be the line drawn from the centre to the point (x, y, z); and the arcs θ and ψ determine the direction of r, θ being the angle between r and the axis of the coordinates parallel to x, and ψ the angle which the plane containing r and the same axis makes with the plane of x, y. By substituting, we get

$$\begin{aligned}\phi = {} & r^2 (B_1 \xi^2 + B_2 \eta^2 + B_3 \zeta^2 + B_4 \xi \eta + B_5 \xi \zeta + B_6 \eta \zeta) \\ & + r^3 (D_1 \xi^3 + D_2 \eta^3 + D_3 \zeta^3 + D \xi^2 \eta + \&c.) \\ & + \&c.\end{aligned}$$

The symbols ξ, η, ζ, represent three rectangular coordinates of a point in the surface of a sphere having unit for its radius; and, in order to simplify, I shall write Q_2, Q_3, and generally Q_n, for homogeneous functions of ξ, η, ζ, of two, three, and n dimensions: then,

$$\phi = r^2 Q_2 + r^3 Q_3 + r^4 Q_n + \&c.$$

For the sake of distinction, let R represent a line drawn from the centre of gravity to the surface of the fluid; and r a line drawn from the same centre to any interior point at which the pressure is p, the directions in which R and r are drawn being determined by the arcs θ and ψ: the equation of the fluid's surface, and the expression of p, will be as follows,

$$\begin{aligned}C &= R^2 Q_2 + R^3 Q_3 + R^4 Q_n + \&c. \\ p &= C - (r^2 Q_2 + r^3 Q_3 + r^4 Q_n + \&c.)\end{aligned}$$

By means of these equations a radius, R or r, will be known when the arcs θ and ψ which determine its direction are assumed; and in this manner we may find all the points of the outer surface, and of any interior level surface in which p has any assigned value less than C. All these surfaces will return into themselves and inclose a space: because in whatever direction we proceed from the centre of gravity to the surface, the function ϕ passes through every gradation of magnitude between zero and the maximum.

It is now easy to complete the demonstration of the equilibrium. A stratum of the fluid between the outer surface and any interior level surface will evidently be in equilibrium, if we suppose that the level surface maintains its figure, or rather, that there are no forces urging the particles contained within that surface: for, the upper part of the stratum cuts the resultant of the forces

at right angles, and the fluid presses perpendicularly and with the same intensity at every point of the lower surface which supports the stratum. What is here affirmed is true, however near the level surface be to the centre of gravity; and as the accelerating forces urging the particles within the surface decrease without limit in approaching that centre, they may finally be regarded as evanescent when the internal body of fluid is no more than a drop occupying the centre of gravity. Wherefore, by taking the radius of the level surface small enough, the inclosed fluid may be considered free from any accelerating forces, and subject only to the external pressures; and, these being perpendicular to the surface, and acting with the same intensity, the whole mass of fluid will be in equilibrium by the known laws of hydrostatics.

It may be proper to add that the mass of fluid has no tendency to turn upon an axis. For no motion of this kind can be produced by the pressures propagated inward from the surface, the directions of which pass through the centre of gravity. Neither can the accelerating forces urging the particles, cause any such motion, these being wholly employed in counteracting the inequality of pressure.

For the sake of illustrating the problem we have solved, we shall add one example, which is besides intimately connected with the principal subject of our research.

Example.—To determine the figure of equilibrium of a homogeneous mass of fluid entirely at liberty, the particles being supposed to attract one another with a force directly proportional to the distance at the same time that they are urged by a centrifugal force caused by rotation about an axis.

At first view the proposed problem may seem one in which the accelerating forces depend upon the figure of the fluid, since it is supposed that every particle is attracted by every other. But, in the particular law of attraction assumed, the force which urges any particle is directed to the centre of gravity of the whole mass of matter, and is proportional to the distance from that point *. The hypothesis of the problem is therefore equivalent to the supposition that the particles of the fluid are attracted to a fixt centre with a force proportional to the distance; so that the accelerating forces are independent of the figure of the fluid.

* Prin. Math. Lib. i. Prop. 88.

As the centre of gravity of a mass of fluid in equilibrium must be free from the action of any force, except what is common to all the particles ; and as the attractions of the particles balance one another at that point; the centrifugal force must likewise be evanescent at the same point, and consequently the axis of rotation must pass through it. Let three planes intersecting at right angles, one being perpendicular to the axis of rotation, pass through the centre of gravity; and assuming any particle of the fluid, let r denote its distance from the same centre, and x, y, z its coordinates, z being parallel to the axis of rotation : further, let g represent the attractive force of the whole mass of fluid at the distance equal to unit from the centre of gravity ; and f the centrifugal force (that is, its proportion to g) at the distance equal to unit from the axis of rotation : then $g\,r$ will be the central attraction urging the particle, and $g\,x$, $g\,y$, $g\,z$, will be the resolved parts of the same force in the directions of the coordinates : also, $\sqrt{x^2+y^2}$ will be the distance of the particle from the axis of rotation ; $-f\sqrt{x^2+y^2}$, the whole centrifugal force estimated as tending to shorten the coordinates ; and $-f\,x$, $-f\,y$, the resolved parts of the same force, parallel to x and y: collecting, now, the partial forces which urge the particle in the respective directions of the coordinates, we shall find,

$$\mathrm{X} = (g-f)\,x, \qquad \mathrm{Y} = (g-f)\,y, \qquad \mathrm{Z} = g\,z.$$

The equations of equilibrium will, therefore, be

$$\varphi = \int (\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz) = \tfrac{1}{2}\{(g-f)(x^2+y^2) + g\,z^2\},$$
$$p = \mathrm{C} - \tfrac{1}{2}\{(g-f)(x^2+y^2) + g\,z^2\}$$

The equation of the surface of the fluid will be found by making $p = 0$, viz.

$$\mathrm{C} = \tfrac{1}{2}(g-f)(x^2+y^2) + \tfrac{1}{2}g\,z^2.$$

And, if we put $e^2 = \frac{f}{g}$, the same equation may be thus written,

$$a^2 = x^2 + y^2 + \frac{z^2}{1-e^2},$$

which belongs to an elliptical spheroid of revolution having the equatorial semidiameter equal to a, and the polar semi-axis to $a\sqrt{1-e^2}$.

8. The order of discussion that has been laid down now brings us to the

more difficult part of this research, when the accelerating forces urging the particles of the fluid, depend upon the very figure of equilibrium which is to be investigated. This must happen in fluids consisting of particles that mutually attract one another, if the attractive force acting upon a particle vary with the figure of the attracting matter. In this division of our subject, the law of attraction that prevails in nature being in reality the only one which it is of much importance to consider, will chiefly engage attention.

Problem 2nd.—To determine the equilibrium of a homogeneous fluid entirely at liberty, the particles attracting one another with a force inversely proportional to the square of the distance, at the same time that they are urged by a centrifugal force caused by rotation about an axis.

The fluid being supposed in equilibrium, the axis of rotation must pass through the centre of gravity of the mass. For, abstracting from any motion or force common to all the particles, that centre may be considered at rest and free from the action of any accelerating force; and, as the attractive forces balance one another at that point, the centrifugal force must likewise vanish at the same point.

Conceive three planes intersecting at right angles in the centre of gravity of the mass, one of them being perpendicular to the axis of rotation: let x, y, z represent the coordinates of a particle in the surface of the fluid, x being parallel to the same axis; and put V for the sum of the quotients of all the molecules of the mass divided by their respective distances from the particle: then the attractive forces urging the particle inward in the directions of x, y, z, will be respectively equal to

$$-\frac{dV}{dx}, \quad -\frac{dV}{dy}, \quad -\frac{dV}{dz}.$$

Further, if f denote the centrifugal force at the distance unit from the axis of rotation, the action of the same force at the distance $\sqrt{y^2+z^2}$ from the same axis will be $f\sqrt{y^2+z^2}$; and the resolved parts of this force urging the particle to move in the prolongations of y and z, will be fy and fz. Wherefore the total forces parallel to x, y, z, and tending to shorten these lines, are respectively,

$$-\frac{dV}{dx}, \quad -\left(\frac{dV}{dy}+fy\right), \quad -\left(\frac{dV}{dz}+fz\right):$$

and the condition that the resultant of these forces is perpendicular to the surface of the fluid is expressed by this differential equation,

$$\frac{dV}{dx}dx+\frac{dV}{dy}dy+\frac{dV}{dz}dz+f(y\,dy+z\,dz)=0;$$

and the integral, viz.

$$C=V+\frac{f}{2}(y^2+z^2),$$

is the equation of the surface of the fluid in equilibrium. This is incontestably the true equation of the surface in equilibrium, since all the forces in action at that surface have been taken into account.

Using x, y, z to represent generally the co-ordinates of any particle of the mass, and the symbol V, to denote the function of x, y, z, which is equal to the sum of the quotients of all the molecules of the mass of fluid divided by their respective distances from the particle, it will be convenient to have some means of pointing out whether V belongs to a point in the surface, or to one differently situated. For this purpose we shall put $r=\sqrt{x^2+y^2+z^2}$ for the distance from the centre of gravity, and shall write V (r) for the value of V relatively to a point within the mass; and we shall suppose that r becomes R at the upper surface, so that V (R) will denote the value of V for a point in that surface. According to this notation, the foregoing equation of the surface of the fluid in equilibrium, will be thus written,

$$C=V(R)+\frac{f}{2}(y^2+z^2). \tag{1}$$

The attraction of the whole mass and the centrifugal force, which are the only forces that urge a particle in the upper surface, likewise act upon every particle in the interior parts of the fluid. It will contribute to perspicuity if to these forces we give the name of the *principal forces*, in order to distinguish them from any other forces which an attentive examination may enable us to detect. Assuming any molecule in the interior parts, r being its distance from the centre of gravity, and x, y, z its coordinates, we have only to proceed as before, writing V (r) for V, in order to find the resolved parts of the principal

forces which urge the molecule inward in the respective directions of x, y, z, viz.

$$-\frac{d\,\mathrm{V}(r)}{d\,x}, \qquad -\left(\frac{d\,\mathrm{V}(r)}{dy}+fy\right), \qquad -\left(\frac{d\,\mathrm{V}(r)}{dz}+fz\right):$$

and if these forces be multiplied, each by the variation of its direction, the sum of the products will be the variation of the intensity of pressure, which is equal and opposite to their action, according to equation (1) of the general theory; thus, we have,

$$\delta p-\frac{d.\mathrm{V}(r)}{dx}\delta x-\frac{d.\mathrm{V}(r)}{dy}\delta y-\frac{d.\mathrm{V}(r)}{dz}\delta z-f(y\delta y+z\delta z)=0; \tag{2}$$

and, as this equation is true at every point of the mass, we further obtain

$$p=\mathrm{V}(r)+\frac{f}{2}(y^2+z^2)-\mathrm{C}, \tag{3}$$

the constant being the same as in the equation (1) of the upper surface, because the two equations must coincide when the interior molecule ascends to the surface. It must be observed that p represents the intensity of pressure caused by the principal forces alone, and not the whole pressure upon the molecule, if besides these forces there exist other causes of pressure in the interior parts.

From the nature of the function V or V (r), it has its maximum at the centre of gravity of the mass, or when $r=0$; for at that point we have the equations

$$\frac{d.\mathrm{V}(r)}{dx}=0, \qquad \frac{d.\mathrm{V}(r)}{dy}=0, \qquad \frac{d.\mathrm{V}(r)}{dz}=0,$$

because the attractive forces balance one another. While r, without any change in its direction, increases to be equal to R, V (r) continually decreases. In whatever direction the radius R be drawn to the surface, there is always a point in it, the coordinates of which will satisfy equation (3), supposing that p has any assigned value less than the maximum which takes place at the centre of gravity. All the points in which p has the same given value will form an interior surface, returning into itself and pressed with equal intensity by the action of the principal forces upon the exterior fluid. Such interior surfaces are likewise perpendicular to the resultant of the principal forces

urging the particles contained in them, as will readily be proved by differentiating equation (3), making p constant.

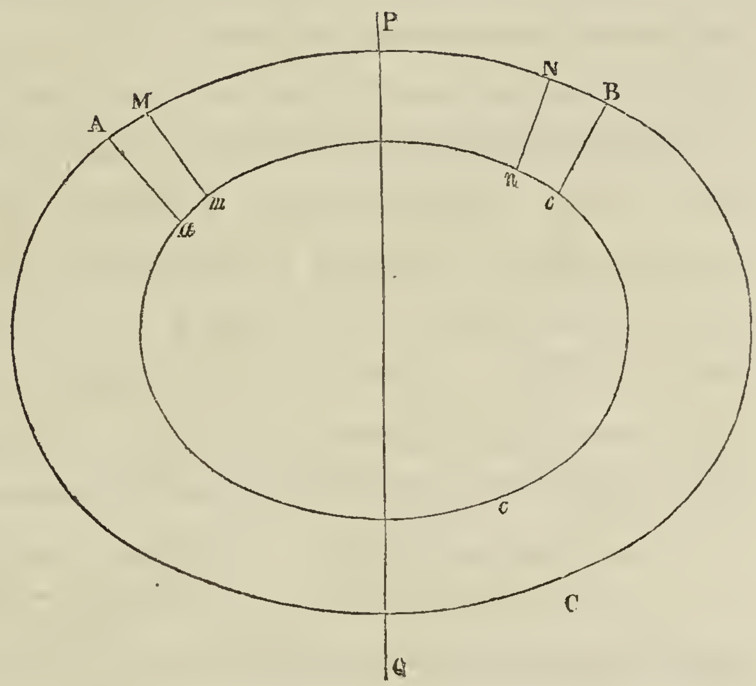
In order to place what has been said in the clearest light, let A B C represent the mass of fluid, the surface being determined by the equation,

$$C = V(R) + \frac{f}{2}(y^2 + z^2);$$

and suppose that $a\,b\,c$ is an interior surface, obtained by making p constant in the equation,

$$p = V(r) + \frac{f}{2}(y^2 + z^2) - C:$$

then, if the narrow canal A $a\,m$ M stand upon the molecule $a\,m$ of the interior surface, and extend to the upper surface of the fluid, the intensity of pressure upon $a\,m$, or the given quantity p, will be equal to the sum of all the impulses caused by the action of the principal forces upon the molecules contained in the canal, every impulse being reduced to the direction of the canal and to the unit of surface. The same thing is true of any other molecule in the same surface upon which there stands a similar canal B $b\,n$ N.



If we attend to the conditions of equilibrium required by the general theory, it will readily appear that the equilibrium of the mass A B C will be impossible, if at any point, as $a\,m$, of the interior surface $a\,b\,c$, any other pressure exist besides that represented by p, or any other forces be in action besides those expressed by the coefficients of the variations in equation (2). For, at the upper surface, there are no forces in action but the principal forces, and the equilibrium will be impossible if other forces prevail in the interior parts besides the principal forces. On the other hand, the matter contained in the stratum between the two surfaces will attract every particle, as $a\,m$, situated in the interior surface. The attraction of the stratum is an indelible force not

to be destroyed, which will produce its full effect according to the figure and quantity of the attracting matter and the situation of the attracted point. The equilibrium will therefore be absolutely impossible, unless such a figure can be induced on the mass of fluid as will set free every particle in the surface *a b c* from the attraction of the stratum. If such a figure can be found, every molecule of the mass will be urged by the principal forces only; because a surface such as *a b c*, at every point of which these forces alone will be in action, may be described through any interior molecule *a m* arbitrarily assumed. We must therefore turn our attention to investigate such figures, if there be any, as will make the irregular attraction in the interior parts disappear, so as to leave the principal forces alone in action; for, unless this can be effected, the fluid cannot maintain a permanent form.

According to the notation we have used, if r denote the distance of *a m* from G, $V(r)$ will represent the sum of the quotients of all the molecules of the whole mass divided by their respective distances from *a m*; let $V'(r)$ denote the same thing, relatively to the interior mass *a b c*, that $V(r)$ does, relatively to the whole mass A B C; then $V(r) - V'(r)$ will denote the sum of the quotients of all the molecules of the stratum divided by their respective distances from *a m*. Take a point $(x + dx,\ y + dy,\ z + dz)$ in the surface *a b c* infinitely near *a m*; and, differentiating in the surface, the expressions,

$$\frac{d.\left(V(r) - V'(r)\right)}{dx}, \quad \frac{d.\left(V(r) - V'(r)\right)}{dy}, \quad \frac{d.\left(V(r) - V'(r)\right)}{dz},$$

will be equal to the attractive forces of the stratum upon the particles of *a m*, in the respective directions of x, y, z: but, as we have shown, the equilibrium indispensably requires that these attractions be evanescent, so that we have these equations,

$$\frac{d.\left(V(r) - V'(r)\right)}{dx} = 0, \quad \frac{d.\left(V(r) - V'(r)\right)}{dy} = 0, \quad \frac{d.\left(V(r) - V'(r)\right)^{*}}{dz} = 0,$$

* The perpendicularity to the surface *a b c*, of the attraction of the stratum upon *a m*, is expressed by this equation,

$$\frac{d.\left(V(r) - V'(r)\right)}{dx}\,dx + \frac{d.\left(V(r) - V'(r)\right)}{dy}\,dy + \frac{d.\left(V(r) - V'(r)\right)}{dz}\,dz = 0;$$

and it is a consequence of the differential equations in the text. The neglect of this consideration, and the assumption that the level surfaces depend solely upon the outer surface in every case, is the great blemish of CLAIRAUT's theory.

which are no other than the partial differentials of the equation,

$$V(r) - V'(r) = \text{constant}. \tag{4}$$

This equation must hold at every point of every interior surface, such as $a\,b\,c$; and, as its differentials are separately equal to zero, it must not contain the coordinates of the surface. If such a figure can be induced on the mass of fluid as will possess the property expressed by equation (4), every particle of the mass will be urged by the principal forces alone, the equilibrium will be possible, and it will be determined in the very same manner as in the first problem.

We have now obtained a mathematical property that distinguishes the figures with which the equilibrium is possible from all others. We have also, in another place*, investigated the figures that alone possess this property; and it appears from what is there shown, that A B C can be no other but an ellipsoid, and that every interior surface, as $a\,b\,c$, is similar to the outer surface, and similarly posited about G.

Having demonstrated that the fluid in equilibrium must be an ellipsoid, it readily follows that the axis of rotation must be one of the three axes of the geometrical figure. For, as the axis of rotation passes through G, the centre of gravity, it is a diameter of the ellipsoid; and the centrifugal force being evanescent at the extremities of this diameter in the surface of the fluid, the only force in action at those points is the attraction of the mass of matter. But the whole force urging every particle in the outer surface of the mass in equilibrium, is perpendicular to that surface; wherefore, the attractive force of the ellipsoid is perpendicular to its surface at the extremities of the diameter about which the fluid revolves; and as there are no points on the surface of that geometrical figure at which the attraction of its mass is perpendicular to its surface, except the extremities of its three axes, it follows that with one or other of these, the axis of rotation of the fluid in equilibrium must coincide.

Let us now determine the relations between the axes of the ellipsoid and the centrifugal force. Of the three planes of the coordinates, one, which is perpendicular to the axis of rotation, is a principal section of the ellipsoid; and we may suppose that the other two coincide with the two remaining principal sections. We may therefore compute V (R) for a point in the surface; and by substituting this value in the equation,

* Phil. Trans. for 1824.

$$0 = \text{V (R)} + \frac{f}{2}(y^2 + z^2) - \text{C},$$

and making the result coincide with the geometrical equation of the figure, we shall obtain the expressions of the axes in terms of the centrifugal force. But it will be more simple to use the differential equation,

$$-\frac{d.\text{V (R)}}{dx}dx - \left(\frac{d.\text{V (R)}}{dy} + fy\right)dy - \left(\frac{d.\text{V (R)}}{dz} + fz\right)dz = 0,$$

which expresses the perpendicularity of the forces to the outer surface. The quantities,

$$-\frac{d.\text{V (R)}}{dx}, \; -\frac{d.\text{V (R)}}{dy}, \; -\frac{d.\text{V (R)}}{dz},$$

are the attractive forces of the ellipsoid, urging a particle of the surface in directions parallel to the axes; and these forces, by the nature of the ellipsoid, are proportional to the coordinates of the point on which they act, and may be represented by $\text{A}'x$, $\text{B}'y$, $\text{C}'z$, the coefficients A', B', C' being known quantities depending upon the ratios of the axes of the ellipsoid; wherefore, these values being substituted in the differential equation, we shall have,

$$\text{A}'x\,dx + (\text{B}' - f)y\,dy + (\text{C}' - f)z\,dz = 0;$$

and by integrating,

$$x^2 + \frac{\text{B}' - f}{\text{A}'}y^2 + \frac{\text{C}' - f}{\text{A}'}z^2 = \text{constant}.$$

Now, if h, h', h'' represent the axes of the ellipsoid, h being that about which the fluid revolves, the equation of the surface of the figure will be,

$$x^2 + \frac{h^2}{h'^2}y^2 + \frac{h^2}{h''^2}z^2 = h^2;$$

and with this equation the foregoing one must be made to coincide. On account of the arbitrary constant, we have only to equate the coefficients of y^2 and z^2, and the resulting formulas may be thus written,

$$f = \text{B}' - \frac{h^2}{h'^2}\text{A}', \qquad f = \text{C}' - \frac{h^2}{h''^2}\text{A}'.$$

But, on examining the functions that A', B', C' stand for, it will readily appear that the expressions on the right side of the two formulas will not be positive,

and consequently they cannot be equal to f, unless $\frac{h^2}{h'^2}$ and $\frac{h^2}{h''^2}$ be both less than unit: and supposing that h is the least of the three axes, the two values of f will not be equal, unless $B' = C'$, and $h' = h''$, in which case both the formulas coincide in one, viz.

$$f = B' - \frac{h^2}{h'^2} A'.$$

In conclusion, it follows that the figure of the fluid in equilibrium is an oblate elliptical spheroid of revolution, of which the equation is

$$x^2 + \frac{h^2}{h'^2}(y^2 + z^2)\, h^2,$$

the mass turning about h the less axis, and the relation between the centrifugal force and $\frac{h}{h'}$ the ratio of the axes, being determined by the equation

$$f = B' - \frac{h^2}{h'^2} A'.$$

The complete solution of the problem is now brought to the discussion of this last equation; and as this is a question purely mathematical, but slightly connected with the physical conditions of the equilibrium, which we have undertaken to investigate, we shall refer to the Mécanique Céleste of LAPLACE and to the Théorie Analytique du Système du Monde of M. de PONTECOULANT, in which works this point is amply treated.

The foregoing solution, being perfectly general, proves that the equilibrium is possible only when the elliptical spheroid is oblate at the poles. When the spheroid is oblong, and the axis of rotation h greater than the other axis h', the expression that must be equal to the centrifugal force is negative; and as that force is essentially positive, the equilibrium becomes impossible.

It will not be necessary to retrace the steps of the foregoing analytical process of reasoning, in order to show synthetically that the equilibrium will be secured if the conditions deduced be fulfilled. For, as soon as such a figure is found as will make the forces that actually urge every particle of the mass the same functions of the coordinates of their point of action, this problem comes under the hypothesis of the first one, and may be demonstrated in the very same manner.

The method of solution we have here followed may be applied to all problems concerning the equilibrium of a mass of fluid, when it is possible to form the equation of the outer surface; that is, when the forces in action at all the points of the outer surface are the same functions of the coordinates of those points, whatever geometrical figure the mass may be supposed to assume. This in reality comprehends every question that has hitherto occurred; and, as the conditions which we have laid down are necessary and sufficient for the equilibrium in every hypothesis of the forces that can be imagined, we shall not enter into any further discussion of this point.

9. The preceding analysis, by which we have investigated the figure of equilibrium of a homogeneous planet is direct and unexceptionable in point of rigour. It seems hardly possible to express simply in algebraic language, all the forces that urge the interior particles of the fluid; and this makes it necessary to have recourse to peculiar modes of reasoning for determining the figure of equilibrium. The problem, being one of great importance and difficulty, which has much engaged the attention of geometers, and which requires for its solution principles different from those that have so long passed current without suspicion of their accuracy, it may not be improper to add another investigation of it by a process of reasoning very different from the foregoing.

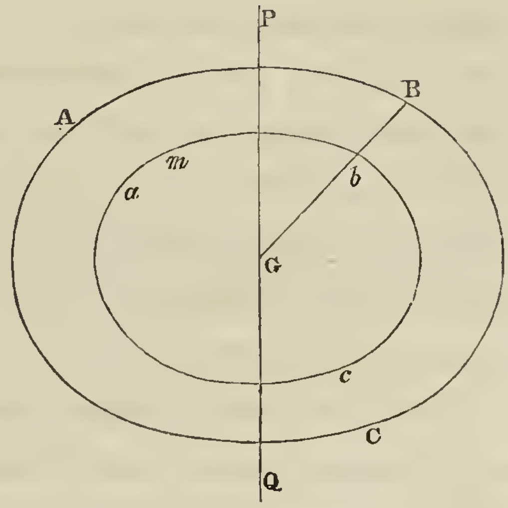
Second investigation.

We shall begin with laying down the following lemma. If a mass of homogeneous fluid, consisting of particles which attract one another inversely as the square of the distance, be in equilibrium when it revolves with a certain angular velocity about an axis; any other mass of the same fluid, the particles attracting by the same law, will be in equilibrium, if it have a similar figure, and revolve with the same rotatory motion about an axis similarly placed.

Take any two particles similarly placed in the two bodies, and having the same proportion to one another as the whole masses; it is proved in the Principia of Newton, and in the works of other authors, that the resultants of the attractive forces acting upon the particles, have similar directions, and are proportional to the linear dimensions of the two bodies. Further, the centrifugal forces urging the two bodies to recede from the axes of rotation, are proportional to the respective distances from the axes, that is, to the linear dimen-

sions of the two bodies. Wherefore the joint action of all the forces is to urge the two particles in similar directions with intensities proportional to the linear dimensions of the bodies. And as the same thing is true of all particles similarly situated in the two bodies, if there be an equilibrium in one case, there will be an equilibrium in the other; for the forces which urge the particles of one body are in no respect different from the forces which urge the particles of the other, except in being all increased or all diminished in the same given proportion.

This lemma being premised, let A B C represent a mass of homogeneous fluid in equilibrium, by the attraction of its particles in the inverse proportion of the square of the distance, and a centrifugal force caused by revolving about the axis P Q. The axis P Q will pass through G, the centre of gravity of the mass. For, abstracting from any motion or force common to all the particles, that centre may be considered at rest; and, as the attractive forces of the particles balance one another at that point, the centrifugal force must likewise vanish at the same point.



Let any radius G B, drawn from the centre of gravity to the surface of the fluid, be divided in a given proportion at *b*; and supposing it to turn round G so as to be directed successively to all the points in the outer surface of the fluid, the radius G *b*, being always the same part of G B, will describe an interior surface similar to the outer one, and similarly posited about G. And because the whole mass A B C is in equilibrium, it follows from the lemma that the interior mass *a b c*, which is similar to the whole mass, and revolves with it about the common axis P Q, will be separately in equilibrium, supposing the exterior stratum of matter were taken away or annihilated.

In the interior surface *a b c* assume any molecule *a m*: the forces that act upon *a m* are; first, the resultant of the centrifugal force and the attraction of the mass *a b c*; secondly, the attraction of the stratum of fluid between the two surfaces. Because the interior body of fluid *a b c* is separately in equili-

brium, the first of these forces, namely, the resultant of the centrifugal force and the attraction of the mass $a\,b\,c$, is perpendicular to the surface $a\,b\,c$, and destroyed by the resistance of the fluid within that surface; and from this it follows that the attraction of the stratum upon $a\,m$, must likewise be perpendicular to the same surface. For, if it acted obliquely to the surface $a\,b\,c$, it might be resolved into two partial forces, one perpendicular, and the other parallel, to the plane touching the surface; and as there is no obstacle to oppose the latter force, it would cause the molecule $a\,m$ to move, which is contrary to the equilibrium of the whole mass A B C. It appears therefore that two distinct and independent conditions are required for the equilibrium of the fluid mass: for all the particles situated in any interior surface $a\,b\,c$ similar to the outer surface, and similarly posited about the centre of gravity G, must be urged perpendicularly to the surface in which they are contained, not only by the resultant of the centrifugal force and the attraction of the interior mass, but likewise by the attraction of the exterior stratum of fluid.

Conceive three planes intersecting at right angles in the centre of gravity of the mass, one of them being perpendicular to the axis of rotation P Q: let x, y, z represent the coordinates of the molecule $a\,m$, and $r = \sqrt{x^2 + y^2 + z^2}$, its distance from G, x being parallel to P Q; and put $\mathrm{V}(r)$ for the sum of the quotients of all the molecules of the whole mass A B C, divided by their respective distances from $a\,m$: further, let $\mathrm{V}'(r)$ denote the same thing relatively to the interior mass $a\,b\,c$, that $\mathrm{V}(r)$ does relatively to the whole mass A B C: then $\mathrm{V}(r) - \mathrm{V}'(r)$ will be the sum of the quotients of all the molecules of fluid contained in the stratum between the two surfaces, divided by the respective distances of the molecules from $a\,m$. According to the known properties of this function, the partial attractions of the stratum upon $a\,m$, in the directions of x, y, z, and tending to lengthen these lines, will be respectively equal to

$$\frac{d.\left(\mathrm{V}(r) - \mathrm{V}'(r)\right)}{dx}, \qquad \frac{d.\left(\mathrm{V}(r) - \mathrm{V}'(r)\right)}{dy}, \qquad \frac{d.\left(\mathrm{V}(r) - \mathrm{V}'(r)\right)}{dz}.$$

Take any point $(x + dx,\ y + dy,\ z + dz)$ in the surface $a\,b\,c$, at the infinitely small distance ds from $a\,m$: then the resultant of the foregoing attracting forces in the direction of ds will be equal to

$$\frac{d.\big(\mathrm{V}(r)-\mathrm{V}'(r)\big)}{dx}\cdot\frac{dx}{ds}+\frac{d.\big(\mathrm{V}(r)-\mathrm{V}'(r)\big)}{dy}\cdot\frac{dy}{ds}+\frac{d.\big(\mathrm{V}(r)-\mathrm{V}'(r)\big)}{dz}\cdot\frac{dz}{ds}:$$

and this resultant must be equal to zero in whatever direction ds is drawn, if the attraction of the stratum upon $a\,m$ be perpendicular to the surface $a\,b\,c$. Wherefore we have,

$$\frac{d.\big(\mathrm{V}(r)-\mathrm{V}'(r)\big)}{dx}dx+\frac{d.\big(\mathrm{V}(r)-\mathrm{V}'(r)\big)}{dy}dy+\frac{d.\big(\mathrm{V}(r)-\mathrm{V}\ (r)\big)}{dz}dz=0: \quad (5)$$

and, by integrating,

$$\mathrm{V}\ (r)-\mathrm{V}'\ (r)=\text{Constant}, \quad (6)$$

which equation must be true at every point in the surface $a\,b\,c$.

Again, the attractive forces of the interior mass urging the molecule $a\,m$ inwards in the direction of x, y, z, are respectively equal to

$$-\frac{d.\mathrm{V}'(r)}{dx},\quad -\frac{d.\mathrm{V}'(r)}{dy},\quad -\frac{d.\mathrm{V}'(r)}{dz}.$$

Let f denote the centrifugal force at the distance unit from the axis of rotation; and, the distance of $a\,m$ from the same axis being $\sqrt{y^2+z^2}$, the centrifugal force of the particles of $a\,m$ will be $f\sqrt{y^2+z^2}$; and the resolved parts of this force acting in the prolongations of y and z, will be fy and fz. Wherefore the total accelerating forces urging $a\,m$ in the directions of x, y, z, and tending to shorten these lines, are respectively,

$$-\frac{d.\mathrm{V}'(r)}{dx},\quad -\left(\frac{d.\mathrm{V}'(r)}{dy}+fy\right),\quad \left(\frac{d.\mathrm{V}(r)}{dz}+fz\right):$$

and, the condition that the resultant of these forces is perpendicular to the surface $a\,b\,c$, is expressed by this differential equation,

$$-\frac{d.\mathrm{V}'(r)}{dx}dx-\left(\frac{d.\mathrm{V}'(r)}{dy}+fy\right)dy-\left(\frac{d.\mathrm{V}'(r)}{dz}+fz\right)dz=0 \quad (7)$$

In the equations (5) and (7) the forces expressed by the co-efficients of the differentials, act on the same particles and have opposite directions in the same lines; wherefore by subtracting the former from the latter, we have,

$$-\frac{d\,.\,\mathrm{V}(r)}{d\,x}\,d\,x-\left(\frac{d\,.\,\mathrm{V}(r)}{d\,y}+f\,y\right)d\,y-\left(\frac{d\,.\,\mathrm{V}(r)}{d\,z}+f\,z\right)d\,z=0, \tag{8}$$

in which the co-efficients of the differentials express the whole forces urging the molecule in the directions of x, y, z.

It is obvious that the equations (7) and (8) must be identical; for they are both true at every point of the same surface $a\,b\,c$. But if the co-efficients of the differentials of these two equations be identical, the like co-efficients in the equation (5) must be separately equal to zero; and this proves that the co-ordinates of the surface $a\,b\,c$ do not enter into the equation (6), which therefore contains such quantities only as remain invariably the same at all the points of that surface.

The equations (7) and (8) being identical, the latter will belong indifferently to all the similar surfaces in the interior parts, and to the outer surface which is their limit. Wherefore, if for the sake of distinction we suppose that r becomes R at the upper surface, we shall obtain the equation of that surface by integrating, viz.

$$\mathrm{C}=\mathrm{V}(\mathrm{R})+\frac{f}{2}(y^2+z^2). \tag{9}$$

The integral of (8) will likewise give the equation of any of the interior surfaces, as $a\,b\,c$, viz.

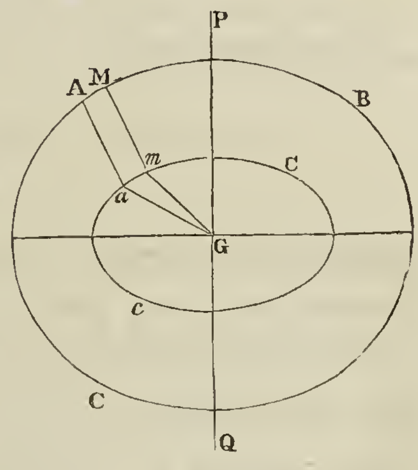
$$p=\mathrm{V}(r)+\frac{f}{2}(y^2+z^2)-\mathrm{C}, \tag{10}$$

the quantity C being absolutely constant in all circumstances, and the same as in the equation of the upper surface, and p being a new quantity which is constant when the co-ordinates are taken in the surface $a\,b\,c$, but varies when the co-ordinates belong to any point of the mass not contained in that surface. At the upper surface p vanishes; it changes its value in passing from one of the interior surfaces to another; and it is evidently the hydrostatic pressure at every point of the mass, because $-\frac{dp}{dx}$, $-\frac{dp}{dy}$, $-\frac{dp}{dz}$, are equal to the co-efficients of the differentials in equation (8) and to the accelerating forces which oppose and destroy the variation of pressure. The equations (6) and (9) and (10) at which we have arrived by this new train of reasoning are the very same with the equations (4) and (1) and (3) of the first investigation; and as the

remainder of the solution is deduced entirely from these equations, it would be superfluous to repeat here what has already been fully explained. The same procedure as in the first investigation will prove, that the figure of the fluid in equilibrium is exclusively an oblate elliptical spheroid of revolution turning about the less axis h, and that the ratio $\frac{h}{h'}$ of the two axes is derived from the centrifugal force by means of the equation

$$f = B' - \frac{h^2}{h'^2} A'.$$

10. The level surfaces of the mass in equilibrium are properly the interior surfaces similar to the outer surface, and similarly posited about the common centre. Such surfaces agree with Clairaut's definition; for they are perpendicular to the resultant of the forces urging the particles contained in them, as appears from the differential equation (8), which is common to them all. But as every particle within the mass is acted upon by several forces, it may become a question whether there are not other interior surfaces besides those similar to the outer one, which possess the properties of being equably pressed, and of being perpendicular to the resultant of the forces in action. It is this point that we are now to investigate.



Suppose that A B C represents an oblate elliptical spheroid of homogeneous fluid in equilibrium by revolving about the axis P Q; and let $a\,b\,c$ be an oblate elliptical spheroid within A B C, the centres, the less axes, and the equators of the two figures coinciding: taking any particle (x, y, z) of the interior mass $a\,b\,c$, the attractions of the whole mass A B C urging the particle in the respective directions of the co-ordinates, may, as before, be represented by

$$A' x, \qquad B' y, \qquad B' z:$$

and in like manner the attractions of the interior mass $a\,b\,c$ upon the particle, may be denoted by

$$A'' x, \qquad B'' y, \qquad B'' z:$$

and the attractions of the matter between the two surfaces upon the particle, will be

$$(A' - A'')\,x, \qquad (B' - B'')\,y, \qquad B' - B'')\,z.$$

As these forces act upon every particle of the mass $a\,b\,c$, they will cause an internal pressure; let p' denote the hydrostatic pressure at the point (x, y, z) caused by the attraction of the external matter; then, by the general theory, we shall have

$$d\,p' + (A' - A'')\,x\,d\,x + (B' - B'')\,(y\,d\,y + z\,d\,z) = 0;$$

and, by integrating,

$$p' = C' - (A' - A'')\,\frac{x^2}{2} - (B' - B'')\,.\,\frac{y^2 + z^2}{2}. \tag{11}$$

Further, the joint effect of the centrifugal force and the attraction of the whole mass A B C upon the particle (x, y, z) in the respective directions of the coordinates, is expressed by these forces,

$$A'\,x, \qquad (B' - f)\,y, \qquad (B' - f)\,z:$$

and if p be the pressure thence arising, we shall have

$$d\,p + A'\,x\,d\,x + (B' - f)\,(y\,d\,y + z\,d\,z) = 0;$$

and consequently,

$$p = C - A'\,\frac{x^2}{2} - (B' - f)\,.\,\frac{y^2 + z^2}{2} \tag{12}$$

which is equivalent to the equation (10), and expresses the whole hydrostatic pressure at every point (x, y, z) within the mass A B C.

In order to form a just notion of the pressures p and p', we shall suppose that the point (x, y, z) is in the interior surface, at $a\,m$: conduct a narrow canal from G to $a\,m$, and continue it outward to the upper surface of the fluid, at A M. Now p is the effort of all the molecules in the canal A $a\,m$ M produced by all the forces that urge them along the canal; and p' is the effort of the canal a G m caused by the attraction of the matter between the two surfaces upon the particles contained in the canal. The pressure p is always directed inward; but the direction in which p' acts will depend upon the na-

ture of the interior spheroid. If it be more oblate than the exterior spheroid A B C, A$''$ will be greater than A$'$, and the attraction $(A' - A'')\,x$ tending from the equator, the pressure of the canal a G m will be outward and opposed to that of the canal A $a\,m$ M. On this supposition, therefore, the whole action of the matter exterior to the spheroid $a\,b\,c$ will cause a pressure upon the molecule $a\,m$, equal to $p - p'$. By subtracting the equations (11) and (12) we get

$$p - p' = C - C' - A''\frac{x^2}{2} - (B'' - f)\,.\,\frac{y^2 + z^2}{2}: \qquad (13)$$

and we have now to inquire whether a spheroid can be found that will satisfy this equation, on the supposition that $p - p'$ is the same at all the points of the surface of the spheroid.

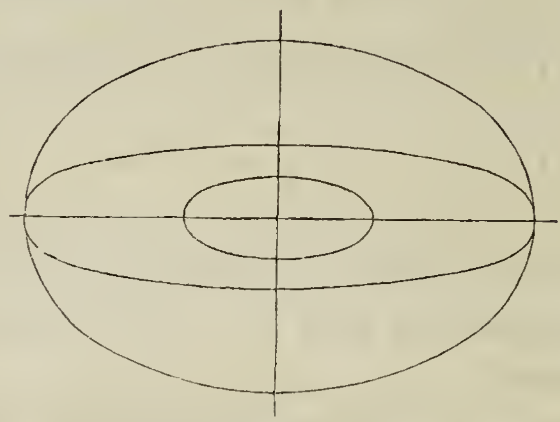
The equation (13) evidently comprehends the level surfaces, which are similar and similarly situated to the upper surface A B C: for, on the supposition that the figures are similar, we have $A' = A''$, $B' = B''$, $p' = C'$, and the equation (13) is identical to the equation (12) which, by giving different values to p, determines all the level surfaces. The equation (13) is similar in its form to the equation (12), A$''$ and B$''$ being the same functions of the excentricity of the spheroid $a\,b\,c$, that A$'$ and B$'$ are, of the excentricity of the spheroid A B C; and the centrifugal force f enters alike into both equations. It is therefore evident that the solution of the latter, supposing p constant, and the solution of the former supposing $p - p'$ constant are both contained in the equation,

$$f = B' - \frac{h^2}{h'^2}A':$$

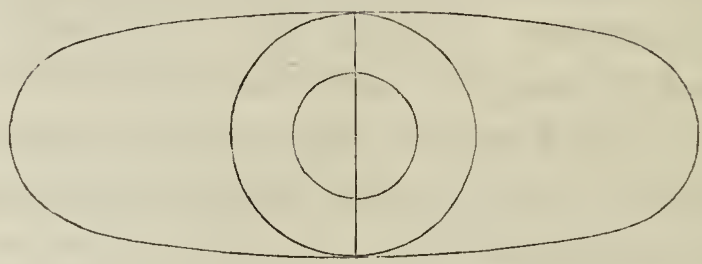
and, as from this two values of $\frac{h^2}{h'^2}$ are in general obtained, one of these results determines the spheroid A B C and its level surfaces, and the other determines the interior spheroid $a\,b\,c$, the surface of which sustains the same pressure at every point by the action of the exterior fluid, and which is therefore separately in equilibrium.

There is this difference between the level surfaces and the other surfaces of equable pressure, that the former spread through the whole mass and ultimately coincide with the upper surface, whereas the latter, on account of the dissimilarity of figure, are confined to a part of the mass. Of the two spheroids

answering to the same centrifugal force, when the exterior one is the less oblate, the greatest interior surface of equable pressure, which is not a level surface, stands upon the equator; and the rest are within this, similar and concentric to it, as in this figure



When the exterior spheroid is the more oblate of the two, the greatest interior surface is described on the less axis, and the rest are similar and concentric to it, as thus,



When the centrifugal force f has a certain relation to the attractive force, the two dissimilar spheroids A B C and $a\ b\ c$ coincide in one; and in this case there are no interior surfaces of equable pressure except the level surfaces.

It has now been demonstrated that, in every oblate spheroid in equilibrium by a rotatory motion, there are two sets of interior surfaces equably pressed by the action of the exterior fluid; and, in consequence, that there are two different figures of equilibrium, and only two answering to the same velocity of rotation. But in the hypothesis of the first problem of this paper, and according to the theory of Clairaut, which as far as regards a fluid entirely at liberty, is equivalent to that problem, there is in every case of equilibrium, only one set of interior surfaces equably pressed by the exterior fluid; and this is an incontrovertible proof that the theory of the French geometer is insufficient for determining the figure of equilibrium of a homogeneous planet in a fluid state.

Maclaurin first demonstrated synthetically the equilibrium of an oblate elliptical spheroid when it revolves about the less axis with a certain angular velocity. In examining the equation of the surface of the fluid, D'Alembert discovered that it admitted of being solved more than one way, that is, he found that there are spheroids of different oblateness which will be in equilibrium with the same velocity of rotation; and Laplace proved that there are two such spheroids and no more. Of this truth, first made known merely as a mathematical deduction from an algebraic equation, we have here attempted to give the physical explanation.

Having now fully treated of the equilibrium of a homogeneous fluid, the order of discussion laid down would lead us to investigate that of one of variable density; but the length of this paper makes it advisable to reserve this part of our subject for another occasion.

VIII. *On a simple Electro-chemical Method of Ascertaining the Presence of different Metals; applied to detect minute quantities of Metallic Poisons.* By Edmund Davy, *F.R.S. M.R.I.A. &c. Professor to the Royal Dublin Society.*

Read November 25, 1830.

1. *Introduction.*

It is now nearly a quarter of a century since the late Sir Humphry Davy, by a train of masterly researches, developed the general principles of electro-chemical action, which subsequently led him to many fine discoveries and important practical applications. Some years since, I repeated most of the interesting experiments noticed in his excellent Bakerian Lecture "On the chemical agencies of Electricity*." On the decomposition of metallic salts by the Voltaic battery, Sir Humphry is very brief. He clearly ascertained, however, that "when metallic solutions were placed in the circuit, metallic crystals or depositions were formed on the negative surface †;" and that "the metals passed towards the negative surface, like the alkalies, and collected round it †." In the course of my experiments on this subject, phenomena occurred which led me to think that some novel results might be obtained by instituting a series of experiments on metallic salts, using as a Voltaic arrangement the feeble power produced by the contact of small slips of different metals, with solutions of the common metallic salts. Operating in this manner, I could readily detect very minute quantities of different metals, coat platina with gold, silver, copper, &c., or cover gold with a surface of these metals, and tin, copper, brass, iron, &c. Several of those facts I have been in the habit of bringing forward and illustrating in my annual courses of lectures delivered both in the Royal Cork Institution, and in the Royal Dublin Society. Circumstances which it is unnecessary to mention, have hitherto prevented me from

* Philosophical Transactions of the Royal Society, 1807. † Ibid.

giving greater publicity to those facts. In the course of the present summer my attention has been directed to apply similar means to the detection of metallic poisons, a subject of acknowledged and increasing importance; and the results I have obtained, and now beg leave to submit to the Society, appear to me both novel and interesting, and afford, if I mistake not, means more simple, delicate, and effectual, than any at present known for detecting the common metallic poisons.

The fear of trespassing too much on the time of the Society, induces me to limit the present paper to one part only of the subject. At no distant period I promise myself the pleasure of communicating the remaining part, which will embrace the different electro-chemical experiments I have made on the other metals and their compounds, together with the application of the facts to the processes of gilding, silvering, tinning, &c.

In the following pages I shall notice the simple electro-chemical apparatus, (or electro-chemical method as I shall call it,) employed in my experiments; offer proofs of its efficacy to detect different metals, particularly metallic poisons; adduce instances of the extreme delicacy and facility of the method; and lastly, show, by similar evidences, that its accuracy is not impaired by the presence of organic substances whether vegetable or animal, or mixtures of both; and that the method is therefore applicable to the detection of metallic poisons in all cases. I shall be under the necessity of making some minute (and I fear tedious) details, which I trust will be excused, as they are closely connected with the elucidation of the subject.

It forms no part of my object to examine the numerous known methods of detecting metallic poisons. Experience, I may presume, has made me tolerably familiar with the details of them. The electro-chemical method here proposed appears to me to rival the very best of them in point of accuracy, whilst in facility, simplicity and delicacy, it seems superior to them all.

2. *Of the Electro-chemical Apparatus.*

The electro-chemical apparatus I used was of the simplest kind. It consisted of two different metals, generally zinc and platina, which, according to Sir H. Davy, form the most efficient combination; the zinc being positive, and the platina negative, with regard to all the other metals. The zinc was usually

in the state of foil, or of thin sheet. The platina was in some cases a small crucible, or a spatula having a spoon at the end of it, but more frequently platina foil was employed. The foil was about two inches long, and two thirds of an inch wide. The zinc foil varied from about one third to one eighth of the size of the platina foil. The size and thickness of either metal may vary indefinitely, without materially altering the results. This simple arrangement is most easily applied to the decomposition of a great number of metallic compounds. It is only necessary to mix a drop or two of acid with a little of such compounds, whether solid or fluid, and apply the zinc foil, when the platina will be soon coated with the reduced metal. Solutions of many metallic salts do not require the addition of acid. I may remark that the small slips of platina and zinc foil are very convenient for many experiments on metallic poisons; as where the object is to ascertain the presence of arsenic or mercury, in a fluid in which it may exist in considerable quantity; or to determine whether any powder contain either of those metals in combination. One slip of platina will answer for an indefinite number of such experiments; one slip of zinc, too, may be employed for many experiments. It is only necessary either to dip the end, after being used, into a little water, and wipe it, or to cut the mere point off at once. The platina spatula with a spoon at the end of it, is well adapted for concentrating, or boiling nearly to dryness, fluids which may contain metallic poisons, but in such minute quantity, as to render concentration indispensably necessary to the success of the electro-chemical method of detecting them. The small platina crucible is a necessary appendage to the apparatus, in cases, where from previous trials by the platina foil, or spoon, and zinc, the existence of a metallic poison has been proved in a fluid or solid, in order to collect it in sufficient quantity, and exhibit it in a separate state.

3. *Experimental proofs of the efficacy of the Electro-chemical method to detect different metals, and especially metallic poisons.*

Solutions of gold, silver, mercury, copper, tin, lead, &c. are not decomposed, as is well known, by platina; but if a drop of each of those metallic salts, containing excess of acid, be severally placed on a surface of platina, and a slip of zinc brought in contact with both, each salt will be decomposed, and the

respective metal deposited on the platina. These are a few instances of the apparent reduction of metallic salts on platina by the agency of Voltaic electricity. It would be extremely easy to extend the list to an indefinite number of other cases, did I not fear to anticipate details which more properly belong to the subsequent part of the subject.

The compounds of arsenic, mercury, lead and copper afford the principal metallic poisons; and a knowledge of their properties, and of the best means of detecting them, derives interest from many considerations, but is particularly valuable from the paramount importance justly attached to chemical evidence in cases of accidental or intentional poisoning. I shall proceed to offer proofs of the efficacy of the electro-chemical method to detect the compounds of those metals in the order in which they have been enumerated.

Compounds of Arsenic.— { Greyish black arsenic, protoxide of BERZELIUS, or fly-powder.
White oxide of arsenic, or arsenious acid. }

When half a grain, or less, of either of those solid compounds is placed on a slip of platina foil, mixed with a drop or two of muriatic acid *, and the zinc applied †, the arsenic will presently be reduced to the metallic state; one part will be deposited on the platina, and the other part mixed in thin filaments with the fluid. The surface of the platina will become iridescent or exhibit variegated colours (resembling heated steel), as blue with tints of red, yellow ‡, &c. Much of the arsenic is thus strongly attached to the platina, and cannot be removed by wiping or rubbing it with the finger or a cloth, nor by cold strong muriatic or sulphuric acid, nor by hot solutions of caustic alkalies; but it instantly disappears on being touched with the smallest drop of

* By "muriatic acid" in this paper, the common strong acid of commerce is meant. I have used it in preference to the pure acid, from the facility with which it can be procured.

† By the words "zinc applied," used here and in other places, is meant bringing the zinc in contact both with the platina and the substances to be acted on; or moving the zinc about on the platina, which seems the readiest way of effecting the reduction of many metallic compounds.

‡ Diluted muriatic acid occasions analogous effects, but they are more slowly produced, probably because the undiluted acid is a better solvent of the arsenical compounds, and a better conductor of electricity. Like effects occur with strong sulphuric acid, if the zinc be moved about on the platina, but they are not produced with strong nitric acid or aquafortis, and only very gradually by these acids when diluted.

strong nitric acid, or of the aquafortis of commerce. The arsenic is also readily removed by exposing the foil to the moderate heat of a spirit lamp, when it rises in the form of arsenious acid; but previous to this effect, the alliaceous, or garlick-like odour, so characteristic of this metal, is strongly produced*, and the surface of the platina remains unaltered.

The following arsenical compounds, when exposed to similar treatment, afford analogous results, viz.

Arsenious acid in solution.

Arsenites of potash, lime, &c.

Arsenic acid.

Arseniates of potash, lime, &c.

Chloride of arsenic.

Sulphurets of arsenic, obtained by passing sulphuretted hydrogen gas through solutions of arsenious and arsenic acids.

In cases where the quantity of any arsenical compound in solution is very minute, the fluid should be concentrated, or boiled nearly to dryness, previous to the addition of the muriatic acid, and subsequent trial on the platina foil by zinc.

Some arsenical compounds require a treatment somewhat different from those already enumerated, in order to exhibit the arsenic in a satisfactory manner. This is the case with the native sulphurets of arsenic, which being scarcely acted on by muriatic acid, require previous treatment with nitric acid. Thus, a little realgar or orpiment, in powder, was mixed with a drop of strong nitric acid in the platina spoon, heated, and the excess of acid expelled; a drop or two of muriatic acid being incorporated with the residual substance, the zinc was applied, and the arsenic readily reduced on the spoon, exhibiting its characteristic appearance. Arsenical pyrites, too, requires the previous addition of nitric acid. In operating on such arsenical compounds as require the use of nitric acid, as little as possible should be employed, and any excess ex-

* Dr. Christison, in his valuable work "On Poisons," proposes to discard this test altogether, chiefly, it would seem, from its being obscured entirely by the presence of a very small portion of vegetable or animal matter; but as this objection does not apply to the electro-chemical mode of detecting arsenic, the alliaceous odour is regarded, especially in these experiments, as a striking character, exhibited by no other metal, as far as my experience extends.

pelled, as a minute quantity of this acid retards the reduction of the arsenic, or re-dissolves it.

The metallic arsenites and arseniates, where two metals are present, appear also in general to require a modified treatment, according to the object in view, and seem to offer proofs of the elegance of this mode of detecting metals. To give an instance or two, in the case of the arsenite or arseniate of copper. If a little of either of these compounds be dissolved in a few drops of muriatic acid, by heat, in the platina spoon, and a few tolerably quick contacts be made with zinc foil, the arsenic only will be reduced; part of it will be deposited on the spoon, and part will remain in the fluid as a dark grey or blackish substance.

If a little of either of the above compounds be dissolved in a few drops of diluted nitric acid, (consisting of one volume strong acid, to three of water,) in the platina spoon; boiled nearly to dryness, and water added just enough to obtain a solution; if the point of a slip of zinc be now applied to the centre of the bulb of the spoon, the copper alone will presently be reduced on the platina, exhibiting an unusual metallic lustre, and forming a circle round the zinc. If the zinc be now removed, the copper will be re-dissolved by the slight excess of acid present. If the contact of the zinc be continued, in a short time the arsenic will form a circle on the platina, round the zinc, whilst beyond it the copper will make its appearance.

I found by repeated experiments, that the arsenic precipitated by zinc on platina foil from solid arsenious acid, or from arsenical solutions, and muriatic acid, could be readily obtained in the form of arsenious acid, by coiling the dried foil, and exposing it to the heat of a spirit lamp, in a tube closed at one end, or open at both, and stopped with a cork or corks, so as to allow the expanded air to escape.

With a view to gain some approximation as to the actual quantity of arsenic that could be detected by the electro-chemical method, and to ascertain if it could be procured from the platina in the metallic state, I placed on a new slip of platina weighing 22.14 grains, five drops of an aqueous solution of arsenious acid, and three drops of muriatic acid; a slip of zinc being applied, the arsenic was soon reduced, and much of it adhered to the platina, which, after being washed in pure water and dried, acquired an in-

crease of $\frac{1}{500}$th part of a grain *. The platina was put into a small retort, which was then exhausted, twice filled with pure carbonic acid gas, and heated over a jar of the same gas, until all the arsenic rose and condensed in the upper part of the bulb of the retort as an extremely delicate whitish film, exhibiting no metallic lustre, even by the aid of a magnifying glass; nor could this be expected from the minute quantity of arsenic present, and the extent of surface over which it was spread by sublimation. The platina was found perfectly clean, and of precisely the same weight as at first. Some pure water was put into the retort, and occasionally agitated in contact with the sublimate, but after thirty hours a recent solution of sulphuretted hydrogen occasioned no change in the water. The actual quantity of arsenic attached to the platina in the foregoing experiment, was ascertained to be $\frac{1}{500}$th part of a grain. But this is very far from conveying a just idea of the degree to which this mode of detection may be carried: for a single drop of the aqueous arsenious acid would have afforded ample evidence of the arsenic as it respected colour, insolubility in muriatic and sulphuric acids, alliaceous odour, and volatility; which would give the $\frac{1}{2500}$th part of a grain. Even this very minute quantity gives us by no means the extreme limits to which this truly microscopic method of detecting metals may be carried. It is, however, quite unnecessary to pursue the subject further.

In cases when the small platina crucible was used, the results were equally delicate, and much more simple and satisfactory. I put a single drop of aqueous arsenious acid, with about an equal bulk of muriatic acid, into the crucible; the zinc being applied for an instant, the arsenic was reduced on the platina. The crucible was then rinsed with pure water, dried, covered with a piece of plate glass, and heated with a spirit lamp; arsenious acid rose and condensed on the glass, whilst the surface of the crucible remained unchanged. The results were precisely similar, when a single drop of arsenic acid, of arsenite and of arseniate of potash, were exposed to similar treatment.

I put $\frac{1}{100}$th part of a grain of solid arsenious acid into a very small platina crucible, which I coated with gold on the inside, and mixed with it about half

* In my experiments I used a very delicate balance of Robinson's construction, which turns with the $\frac{1}{1000}$th part of a grain, when loaded with one hundred grains.

a drop of muriatic acid. A small slip of zinc being applied, the spot became of a dark steel-grey colour with a tint of green at its edge. Some pure water being put into the crucible, some extremely minute films of metallic arsenic appeared on the surface of it. The crucible, after being washed and dried, was heated, when the arsenic rose, leaving the gold surface unaltered.

Into the crucible used in the preceding experiment, I put $\frac{1}{10}$th of a grain of solid arsenious acid, and dissolved it in about five drops of muriatic acid. I then applied a slip of zinc for about half a minute; as soon as the contact of the zinc was made, the variegated colours from the arsenic were beautifully produced at the bottom of the crucible, an effect which was succeeded by a violent action, and all the arsenic was reduced. The crucible was now filled with pure water, and numerous steel-grey coloured filaments of metallic arsenic floated on the surface; some being collected on a slip of platina and heated, the alliaceous odour was strongly produced, and they were dissipated in a white vapour. The crucible being washed, and a few drops of diluted muriatic acid put into it, it was rinsed in pure water and dried, when its gold surface was so completely coated with arsenic of a dark steel-grey colour, that no vestige of the gold at the bottom of the crucible could be seen, even with the aid of a magnifying glass, though the surface was full of little inequalities. The arsenic on the crucible did not sensibly tarnish by exposure to the air for some days; a portion of it was then expelled by a heat of nearly 400° Fahr., when the gold became partially visible; the remainder of the arsenic continued of a steel-grey colour.

Though it seems unnecessary to bring forward more experiments in proof of the efficacy of the electro-chemical method to detect arsenic, it may be proper to recur to the evidence that arsenical compounds are by this method reduced to the metallic state. The experiments with the platina gilt crucible afford the most unequivocal proofs of metallic arsenic; whilst those with the platina foil and spoon, though seemingly more ambiguous, will, on examination, I presume, be found scarcely less satisfactory. Thus the partial oxidation of the surface exhibited in variegated colours; the strong cohesion of the arsenic to the platina, are characteristics of the metal. The alliaceous odour (in cases where no deoxygenating substance can be presumed to exist) is admitted to belong only to the metal; insolubility in strong muriatic acid, is a property of no

known oxide of arsenic, but of the metal. To which may be added the facts, that I have repeatedly coated the arsenic, deposited on platina, with copper, and also with mercury; and removed both, without affecting the arsenic. All these circumstances seem to prove the accuracy of the statements made on the subject.

Though zinc and platina were the metals used in the electro-chemical experiments on arsenic already noticed, I have also employed several other metals which generally afforded analogous results. Other metals might of course be substituted for zinc and platina, but as far as my experiments have extended I give these a decided preference. Iron and tin are much slower in their operation than zinc. Brass and copper are readily coated with metallic arsenic; but the objections to the use of the common metals as substitutes for platina, arise from the facility with which they are acted on by heat, air, acids, &c. and the difficulty with which arsenic is separated from them without injuring their surfaces. The advantage of colour which gold has over platina, when a grey metal, as arsenic, is to be reduced in contact with it, is cheaply purchased by simply gilding part of the inside of the platina crucible.

Compounds of mercury.—The compounds of mercury in general are readily reduced by being placed on a slip of platina foil, mixed with a drop of diluted * aquafortis or muriatic acid, and the zinc applied, when the mercury is soon either partly attached loosely, or amalgamated with the platina, and partly combined with the zinc. This is the case with the black and red oxides, white precipitate, the acetate, subsulphate, cyanuret, fulminate, &c. The cyanuret, however, appears to be reduced more readily, and with greater lustre, when muriatic acid, undiluted, is employed, and the acetate and subsulphate best, when diluted aquafortis is used. The compounds of mercury soluble in water, in general require not the addition of acid.

From the importance of the compounds of mercury with chlorine, our details respecting them may be more minute.

Corrosive sublimate.—If a drop of an aqueous solution of corrosive sublimate be put on a bright surface of copper, it will soon, as is well known, render the copper of a greyish black colour. If a similar experiment be made and

* By "diluted" as applied to aquafortis and muriatic acid in this paper, is meant, the strong acids of commerce, to which an equal bulk of water has been added.

the zinc applied, the mercury will presently be reduced to the metallic state, and the copper will be whitened. The results will be analogous, if similar experiments are made with solutions of corrosive sublimate in alcohol and ether, provided a little water be employed when the zinc is applied, or if platina be substituted for the copper.

The following is a beautiful and extremely delicate mode of reducing corrosive sublimate, and of obtaining the mercury in a sensible form; and it may of course be applied to other compounds of mercury. Put a drop or two of an aqueous solution of corrosive sublimate into a small platina crucible, add about an equal bulk of muriatic acid, and apply the zinc for a short time, when the mercury will be reduced; part of it on the platina, which will appear of a brighter white colour, and part of it will amalgamate with the zinc, whitening it and making it brittle at the point of contact. Wash the crucible with pure water, dry, and cover it with a piece of plate glass, and heat it with a spirit lamp: the mercury will rise and be condensed as an extremely fine white powder without metallic lustre, and it may be easily collected from the glass by the finger, so as to be exhibited in minute globules, visible by a high magnifying power, even when a single drop only of the solution has been operated on.

I repeated the preceding experiment in the platina gilt crucible. The instant the zinc was applied the surface of the gold was whitened. Afterwards the mercury was collected by sublimation on glass.

The solution of corrosive sublimate may of course be readily reduced on platina when the zinc is applied without the addition of any acid, as in the ingenious mode proposed some time since by Mr. Sylvester, and afterwards simplified by Dr. Paris. I am disposed to prefer the use of platina to that of gold, not only on account of the great difference in their commercial value, but because I found the surface of gold tarnished after the mercury had been sublimed from it, which was not the case with the platina. I did not, indeed, ascertain if the gold I employed contained any alloy;—it was supposed to be pure.

A very simple mode of detecting corrosive sublimate, whether solid or dissolved in water, alcohol or ether, which I have not seen any where noticed, is, to put on a bright surface of copper, a bit of the solid compound, or a drop of

its solution, and to add a drop of muriatic acid to it, when the mercury will be immediately reduced, and the copper whitened. Similar effects also take place when a number of other compounds of mercury are similarly treated, as the oxides, cyanuret, the yellow precipitate from corrosive sublimate by the fixed alkalies, the white precipitate by ammonia, &c. Acetate of mercury does not yield to such treatment, unless heat be applied, when the copper is readily whitened; but with diluted aquafortis, the acetate is easily reduced on copper.

Calomel is slowly decomposed in a platina crucible, when mixed with diluted muriatic acid, and the zinc applied; the reduced mercury is deposited on the platina and combined with the zinc, or in case gold is present, it is whitened.

The following is a much readier and better mode of treating calomel. Mix a little of this compound, with a few drops of diluted aquafortis, and boil for an instant or two in a small platina crucible, add a little water, and apply the zinc; the mercury will be readily reduced. Break the contact of the zinc with the platina, and the bright mercurial surface of the latter will at once become tarnished. Restore the contact, and the lustre of the surface will reappear; and these changes will occur as often as the contact is broken and renewed. Add some pure water whilst the zinc is in contact, and the surface will remain bright. The mercury may be collected by sublimation, as in the case of corrosive sublimate. Operating in this manner, the mercury from a very small quantity of calomel may be obtained. Calomel may also be readily detected, by placing it on a clean surface of copper, mixing a little muriatic or diluted muriatic acid with it, and applying a gentle heat; when the mercury will be instantly reduced, and the copper whitened.

Compounds of lead.—The soluble compounds of lead, in general, like those of mercury, are readily reduced to the metallic state when placed on a surface of platina, and the zinc applied; as the nitrate, acetate, &c. The insoluble compounds, as the oxides, patent yellow, the carbonate, sulphate, tartrate, &c. require previous mixture with diluted aquafortis or muriatic acid. The lead thus reduced is usually of a dark grey colour, and exhibits little lustre, unless pressed, as by the blade of a knife, when its metallic lustre becomes quite apparent. It feels quite soft as the zinc is drawn over it. Its cohesion to the platina is generally so feeble that most of it may be easily removed by a cloth

or the finger. It is also presently dissolved by the aquafortis and muriatic acid of commerce, and by strong nitric acid.

I coated one side of a platina spatula with gold, mixed a drop of acetate of lead with one of diluted aquafortis, and applied the zinc; when the lead being reduced, the surface was washed, and by slight friction most of the lead was removed; but there remained sufficient to show the colour and lustre of the metal. It dissolved by diluted aqua regia, leaving the surface of gold apparently unaltered.

Compounds of copper.—The soluble salts of copper, as the perchloride, permuriate, sulphate, nitrate, &c. when dissolved in water, require only to be placed on platina, and the zinc applied, when the copper will be presently reduced and cover the platina. The compounds insoluble in water, as the protochloride, proto-muriate, the carbonate, oxides, &c. when previously mixed with a little muriatic acid, or diluted aquafortis, are also readily reduced. The last acid seems best adapted in the case of different pigments of copper, as Olympian and Scheele's greens, common verdigris, refiner's verditer, &c.

The copper reduced on platina, by zinc, usually exhibits the colour and lustre of the metal. Its surface is in some cases blackish, and requires gentle rubbing to show the metallic appearance. It has, in general, greater lustre, in cases when diluted aquafortis is used, and the acid only in slight excess when the zinc is applied. It is soluble in muriatic acid, but more readily in strong or diluted aquafortis.

4. *Experimental evidences that the accuracy of the Electro-chemical method of detecting metals is not impaired by the presence of mixed vegetable and animal substances, when applied to the detection of metallic poisons; particularly to the common compounds of Arsenic, Mercury, Lead, and Copper, in such mixtures.*

I have ascertained that the electro-chemical method is competent to the detection of very minute quantities of the different metals, when their compounds are mixed with vegetable and animal substances. But the object of this section is to notice briefly a number of miscellaneous experiments on the principal well known poisonous compounds of arsenic, mercury, lead, and copper, when mingled with organic fluids and solids, &c.

Arsenious acid.—I mixed a small quantity of solid arsenious acid in powder, with the following substances, separately, viz. wheaten flour and its paste; bread; cake with currants and caraway seeds; starch in powder, and in solution; rice in powder, and boiled with water to a pulp; potatoes boiled, and roasted; sugar in powder, and in syrup; vinegar, and raspberry vinegar; port and sherry wines; gruel, thick and thin, with sugar and milk, also with sugar and wine; milk and cream; white and yolk of egg, both fluid and coagulated; gelatine (isinglass) in solution; bile discharged from the stomach, mixed with saliva, and watery fluid.

In a number of those instances, it was only necessary to put a little of the mixture into the platina spoon, incorporate a few drops of muriatic acid with it, and apply a slip of zinc, when the arsenic was readily precipitated on the platina. In cases when the arsenious acid existed only in very minute quantity, or when the water present was considerable, the fluid was concentrated by boiling, or evaporated nearly to dryness, previous to the application of the zinc, when the arsenic was in like manner deposited.

Arsenious acid was also mixed with butter, lard, and oils; some of the respective mixtures being boiled a short time in the platina spoon with muriatic acid, or in solution of fixed alkali, then muriatic acid added in excess, and the zinc applied, when the arsenic readily appeared.

A few drops of an aqueous solution of arsenious acid were mixed with some sheep's blood; muriatic acid was added; the whole formed a deep wine yellow solution. A drop or two being put on the platina foil, and the zinc applied, coagulation took place, and the arsenic precipitated. Ox-bile being treated in a similar manner afforded analogous results.

In one instance, I mixed five grains of solid arsenious acid in about half a pint of tea sweetened with sugar and milk. It was kept hot for some minutes, and occasionally stirred. A little of the tea being put into the platina spoon, and muriatic acid added, the application of the zinc produced no immediate effect. But on boiling the spoonful of tea nearly to dryness, then adding two or three drops of muriatic acid to the residual brown substance, and agitating, a yellow solution was formed. A slip of zinc being now applied, a white coagulum, which soon changed to brown, appeared, and the arsenic was copiously precipitated on the spoon.

In another experiment, a small quantity of arsenious acid, in powder, was mixed with hot tea. Some hours after it had cooled, it was filtered, and on being treated as in the preceding experiment, the arsenic appeared in a similar manner. Coffee, on being exposed to like treatment, afforded analogous results.

I mixed five grains of pulverized arsenious acid in a small basin of warm and rather thick pea-soup, having fragments of the fibrous part of beef diffused through it. The platina spoonful of it was boiled nearly to dryness, several drops of muriatic acid were added, and by agitation most of the solid matter was dissolved, forming a thickish fluid. The zinc being applied, a white coagulum changing to brown appeared, and the arsenic soon covered the surface of the spoon. Similar results were obtained when giblet soup was treated in like manner.

Corrosive sublimate.—A few drops of a solution of corrosive sublimate in water, were well mixed with a solution of gelatine (isinglass): a drop or two of the mixed fluids being placed on the platina spatula, and the zinc applied, the mercury presently precipitated; and this effect was more readily produced when a little diluted muriatic acid was previously added to the mixture. When a solution of nutgalls was added to the mixed gelatine and corrosive sublimate, also when corrosive sublimate was added to yolk of egg, the results, by similar treatment, were precisely analogous.

A small quantity of solution of corrosive sublimate was put into fluid albumen (white of egg). To some of the precipitate in a platina crucible, a little diluted muriatic acid was added, and the zinc applied; the albumen coagulated, and in about a minute some mercury was reduced in the crucible, and by continuing the experiment for a few minutes the quantity increased considerably. A similar experiment being repeated in a platina gilt crucible, in a short time the gold partially assumed a dark blueish grey colour. The mercury from both experiments was collected by sublimation on glass, as detailed before.

When solution of corrosive sublimate was mixed with flour into a soft paste, put into the platina crucible, and diluted muriatic acid added, in the course of a few minutes after the zinc was applied, similar results as in the foregoing instances, were obtained.

Corrosive sublimate in powder was mixed with butter, and diluted muriatic acid was incorporated with the mixture; a little of the same being placed on

the platina spatula, and the zinc applied to the fluid, the mercury was presently reduced. The experiment was successfully repeated with diluted aquafortis, instead of muriatic acid.

Solution of corrosive sublimate was respectively mixed with saliva, sheep's blood, and ox bile; the zinc being applied to each, on platina, the mercury was soon reduced.

A few drops of solution of corrosive sublimate were mixed with some human bile, and watery fluid discharged from the stomach, and rather more than half its bulk of muriatic acid was added; a small teaspoonful of the mixed fluids was put into the platina crucible, and the zinc applied for about two minutes, when the surface of the crucible had acquired a dark grey colour; on being heated, metallic mercury was obtained. The experiment was repeated, with similar results, in a platina gilt crucible.

A solution of corrosive sublimate, in small quantity, was mixed with milk; muriatic acid, equal to about one half the milk, was added. Some of the mixed fluids was put into the platina crucible, and the zinc being applied a short time, a grey surface of mercury appeared on the platina. A similar experiment was made in the platina gilt crucible, which soon exhibited a partially whitened surface. From both experiments mercury was sublimed.

A few drops of solution of corrosive sublimate were mixed in a cup of tea sweetened with sugar and cream. About a teaspoonful was put into the platina crucible, with nearly an equal bulk of muriatic acid. A slip of zinc being applied for about a minute, a greyish substance was found at the bottom of the crucible, which, when washed, dried, and heated, afforded metallic mercury. These experiments were repeated with success, both in the platina crucible and in the platina gilt crucible; and precisely similar results were obtained with coffee, under like treatment.

It is proper to remark, that the experiments noticed in this section, were, in general, carried on only for a few minutes, so that the decompositions were mostly partial, and probably in no instance complete. This statement is confirmed by facts, for I repeatedly found that after a given effect had taken place in the platina crucible in a few minutes, a still further effect was soon produced, by transferring the same materials to the platina gilt crucible.

When a solution of corrosive sublimate (in small quantity) was added to some sherry or port wine in a platina crucible, and the zinc applied, a beautiful effect gradually took place. In the course of about a quarter of an hour, calomel was formed, and deposited at the bottom of the crucible in successive circles, which extended beyond each other, and differed slightly in their shades of colour, being alternately lighter and darker. The zinc became brittle at its point of contact, from amalgamation. These experiments were repeated in the platina gilt crucible with similar results, part of the gold was whitened, and calomel and mercury were afterwards sublimed from both crucibles.

With raspberry vinegar and solution of corrosive sublimate, the results appeared to be similar to those produced with the wines.

Sugar of lead, acetate of lead.—A solution of sugar of lead, in small quantity, was separately mixed with flour into a thin paste, with saliva, port and sherry wines, raspberry vinegar, and yolk of egg. A little of each mixture was put on the platina spatula, and the zinc applied, when the lead was soon reduced on the platina.

A little of the solid compound of albumen (white of egg) and sugar of lead was placed on the spatula, and mixed with a drop or two of diluted aquafortis; the zinc being applied, the lead was reduced to a dark grey colour; by gently rubbing it with the finger the lustre of the metal became apparent. The results were analogous when sugar of lead was mixed with sheep's blood and ox bile, and exposed to similar treatment.

A mixture of gelatine (isinglass) and sugar of lead was put on the spatula, and the zinc applied; the lead was slowly reduced: the effect was more rapid when a drop of diluted aquafortis or muriatic acid was previously added. The results were precisely similar when a solution of nutgalls was added to the mixed solutions of gelatine and sugar of lead. The yellow soft solid readily yielded metallic lead on platina by means of zinc.

A small quantity of sugar of lead was put into some tea sweetened with sugar and cream; some of the yellow matter produced being put on the spatula, and the zinc applied, the lead was slowly deposited. The results were similar when sugar of lead was mixed with milk and with coffee. The addition of a little diluted aquafortis appears, in a number of cases, more readily to

reduce the lead, but as a slight excess of acid will redissolve the lead (partially or wholly), a longer contact of the zinc is necessary to neutralize the acid, and render the lead permanent.

Sulphate of copper, blue vitriol.—A small quantity of sulphate of copper in solution was added to the following substances, viz. flour forming a soft paste, milk, tea and coffee sweetened with sugar and cream, gelatine (isinglass) in solution, and with nutgalls, albumen (white of egg) and yolk of egg, saliva, sheep's blood, and ox-bile. On a little of each mixture being placed on the platina spatula, and the zinc applied, the copper was presently reduced on the platina. In some cases, the surface was of a blackish colour, but its proper colour and lustre became apparent on rubbing it. The addition of a little diluted aquafortis or muriatic acid to the mixtures, usually facilitated the reduction of the copper.

A little sulphate of copper in powder was mixed with butter and lard, placed on the spatula, and a drop of diluted aquafortis incorporated with them. On the zinc being applied, the copper was readily reduced.

5. *Some general Remarks, and Conclusion.*

The experiments detailed in this paper seem to prove, that the common compounds of arsenic, mercury, lead, and copper, may be readily reduced to the metallic state, on platina, by the electro-chemical method described; and that this method is also competent to the detection of those metals, in cases when their compounds are mixed in very small quantity with vegetable and animal substances. It may, I think, be further deduced, that the method is applicable in cases when those compounds exist in the most complicated mixtures of organic substances. For, though I have had no opportunity of applying it to the contents and tissues of the stomach in instances of poisoning, yet its efficacy in such cases can scarcely be doubted, as no animal or vegetable substances can, I apprehend, resist the action of the mineral acids, which are almost indispensably necessary to the success of the method. In some instances, where the common tests will not act at all, and in others, where they only act fallaciously, the electro-chemical method will, in general, be found to act with certainty.

The general results appear to be strictly electro-chemical, or to arise from

the union of a chemical and an electrical action, and seem to be comprehended in the principle developed by Sir HUMPHRY DAVY relating to the changes and transitions by electricity, viz. that "the metals are attracted by negatively electrified metallic surfaces, and repelled by positively electrified metallic surfaces, and these attractive and repulsive forces are sufficiently energetic to destroy the usual operation of elective affinity *."

If the decompositions which I have noticed were effected by the mere contact of the zinc with acid solutions of the metals, there would, indeed, be reductions, but no depositions on the platina. The electrical action arising from the contact of the two metals, in cases when small slips of zinc and platina were employed, was quite sufficient to destroy or suspend the usual operation of chemical affinity. Thus, in numerous instances of the decompositions of metallic compounds, the platina remained covered with the reduced metal during the contact of the zinc; but on removing the zinc, the coating was readily redissolved by the slight excess of acid present. The transition of the reduced metal to the surface of the platina, is the effect that gives to the method a beauty, simplicity, and delicacy, exceeding, in my opinion, all other modes of detecting the metals already referred to, together with a number of others which remain to be noticed.

As this paper has already exceeded the limits at first intended, I shall defer noticing a number of useful applications of the facts, until I shall have the honour of submitting to the Society the remaining part of the subject.

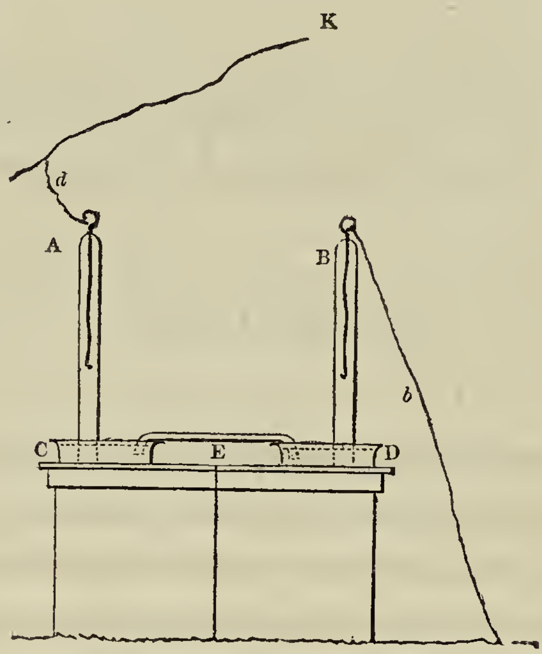
* Phil. Trans. 1807.

IX. *On the Chemical Action of Atmospheric Electricity.* By ALEXANDER BARRY, *Esq. F.L.S. Communicated by* JOHN GEORGE CHILDREN, *Esq. Sec. R.S., &c. &c.*

Read Feb. 24, 1831.

THE intimate connection existing between the chemical constitution of various substances and their condition in regard to electricity, has now been long known. So true is this, that the very order of their affinities has also, for a considerable period, been acknowledged as greatly within the control of this agent. With this object in view, the electrical influence, as exhibited in the ordinary way, as well as by the Voltaic battery, has successively confirmed its results in the experiments of NICHOLSON, CAVENDISH, Sir H. DAVY, CHILDREN, and the French chemists. This being the case, it is not my purpose to advert to any new source of electrical accumulation, but to describe what appears to me as the link connecting the researches of Dr. FRANKLIN with the electro-chemical theory of Sir H. DAVY. With this view, in August 1824, I elevated the kite in an atmosphere favourable to the exhibition of its phenomena. It was raised from an apparatus firmly fixed in the earth, and was insulated by a glass pillar. The usual shocks were felt on touching the string, which simple fact I am induced to mention from the circumstance of no electrometer having been employed. The portion of string let out with a double gilt thread passed through it, was about five hundred yards. I then made the connection shown in the accompanying sketch, where the straight glass tubes A B, having platina wires passed from above half way down their axes, and standing in their respective glass cups C D, were filled with a solution of sulphate of soda coloured with syrup of violets, connected also with each other by the bent glass tube E, likewise filled with the above solution in the usual manner. A portion of gilt thread *d* was then brought from the tube at A and united to the kite-string K, whilst a similar thread *b* was carried from B to the earth. Bub-

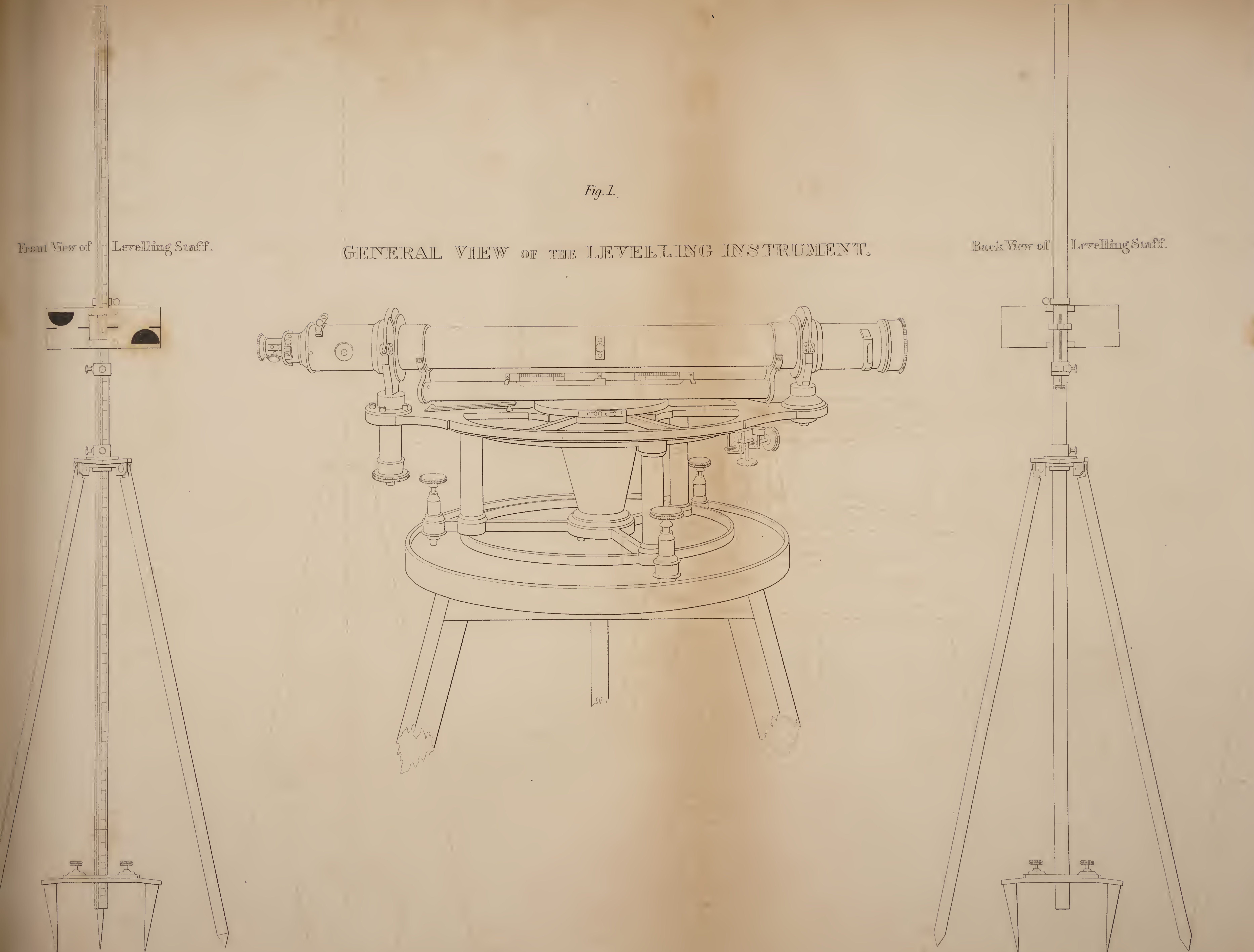
bles of hydrogen in A and of oxygen in B, soon appeared. In about ten minutes, the blue liquid in A became green from the separation of the soda,



whilst the sulphuric acid, by passing to the pole in the tube B, changed its contents, as usual, red. The experiment was then discontinued. In having the honour to communicate the result to the Royal Society, I must remark, that it was my intention to have pursued this simple application of electricity, in the expectation of arriving at more extended and important conclusions, which, however, numerous circumstances have hitherto prevented me from accomplishing.

Fig. 1.

GENERAL VIEW OF THE LEVELLING INSTRUMENT.



Gardner sc.

X. *An Account of Operations carried on for ascertaining the Difference of Level between the River Thames at London Bridge and the Sea; and also for determining the Height above the Level of the Sea, &c. of intermediate Points passed over between Sheerness and London Bridge. By* John Augustus Lloyd, *Esq. F.R.S. F.R.G.S. & F.S.A.*

Read March 3, 1831.

In February 1830, at the suggestion of the Royal Society, I had the honour to receive directions from the Lords Commissioners of the Admiralty to make such observations as I might consider necessary, to ascertain the difference, if any, between the level of the waters at certain points on the river Thames, and the mean level of the sea near Sheerness, as well as the height of different intermediate points above the sea, such as Gravesend, Greenwich Observatory, &c.

Having found, while employed in the Isthmus of Darien, how inadequate the present levelling instruments were to obtain very accurate results, and being desirous of conducting the interesting observations, I now had orders to make, with the most scrupulous exactness, I thought it necessary, in the first instance, to bestow some attention to the improvement of the instruments required to be used, endeavouring to combine superior steadiness and motion in azimuth, more delicacy in the level itself, more permanency in its position, and greater power in the telescope.

After several trials, and by the assistance of Mr. Cary, (to whom I am indebted for many valuable suggestions,) I determined upon having an instrument made exactly after the accompanying Plate.

Fig. 1. is a perspective view of the instrument. It is supported by three foot-screws, similar to those used in the best altitude and azimuth circles, with thirty threads to an inch, and serving to place the instrument horizontal.

The stand is formed by two plates of brass, which are firmly connected together by three pillars. To the limb that carries the telescope, in very sub-

stantial Ys, is fixed the inner conical centre, three inches in diameter and eight in length, ground to move perfectly smooth in the hollow conical centre fixed to the stand.

This limb projects over the upper part of the frame about one inch, and is bevelled at the edge, in order to slide in a groove attached in two pieces to a clamp, screwed underneath the upper part of the frame, and which, by means of a tangent screw, gives a slow motion in azimuth to the limb. This limb, having two solid projections, allows the Ys being at some distance over the periphery of the circle, by which means the supports to the telescope are nearer its extremities. One of these Ys has a short vertical motion, with a pushing screw at the side, which fixes it at any height required. The telescope itself is of thirty inches focal length, and magnifies about twenty-two times, having two thick gun-metal collars, by which the telescope rests on and is turned in the Ys. These were quickly worn by the continued friction from the supporting part of the Ys, and would have caused an error in using the instrument; I therefore had small thick steel plates, highly polished and hardened, dovetailed into each supporting part of the Ys. The friction on the collars was now transferred to the pieces of steel; and although they likewise continually wear, it does not affect the correctness of the instrument in any other way, than in altering the adjustment of one of the Ys. (See page 20.)

At the eye end of the telescope is an adjustment with a rack and pinion, for distinct vision, and another by the same means to regulate the distance of the eye from the wires: this I found to be indispensable, as the eye occasionally becomes fatigued, and requires a different focus to view the wires distinctly.

The wires themselves are adjusted as usual, by a sliding piece for azimuth, and two pushing screws for vertical motion.

In the centre of the telescope there are two orifices opposite one another; the one to receive a small lamp, and the other admitting a spindle and speculum at the end, ground to an angle of 45°, which, by reflection, illuminates the wires for night observation. The speculum being made to turn, the quantity of reflected light may be regulated at pleasure.

As the glass would not admit of distinct vision for objects at a less distance than one hundred feet, and it being necessary to use the instrument, at times,

not more than three feet distance from the station-staff, several lenses were made of different focal distances, as sixty, thirty, fifteen, eight feet, and thirty inches; which, being applied to the object end of the telescope, converts it, in fact, to a microscope.

To the lower part of the telescope, within the collars, are affixed the cocks, into which fits the tube protecting the level: one of these cocks gives a vertical, and the other a horizontal motion to the tube, in order to place the level parallel to the axis of the telescope.

When the instrument was first made, these motions were effected by means of endless screws; but I found it so difficult (almost impossible) to keep a delicate level in adjustment by this mode, that I substituted the old fashion of capstan-headed pushing screws.

To the upper part of the telescope are attached (outside either points of support) cocks* or braces, carrying a swinging level, having, as well as the cocks, separate adjustments.

This additional level was intended as a check to the lower level, and to detect any occasional variation in the figure of the tube itself.

The glass bubbles themselves were placed in the tubes at first with paper, and wedged at either end with small pieces of wood; but the wedges are liable to distort the bubble itself, and after some time get loose in the tube; and the level alters in its position, and is never to be depended on. I found it better to push the level into its proper place in the tube, not tightly, and with paper underneath, taking care that the paper touches the middle part, and then filling up the two ends with plaster of Paris.

To the upper part of the telescope, near the eye-piece, is fixed a small level, adjusted to the horizontal wire of the telescope, and by the assistance of which the same surface of the large level is used at each observation. There is also another small level at the other end of the telescope, to adjust the vertical wire by, but of less use than the former.

On the upper limb, near one of the Ys, is fixed a small thermometer, with the bubble inclining downwards, at an angle of about 10°, the use of which will be explained hereafter.

The large stand of this instrument is made with a solid top of African oak.

* These cocks are not shown in the Plate.

The legs, which are very strong, are iron-shod, and braced at the bottom, about six inches from the shoeing, with three thick iron rods, rendering the whole steady, and affording the men a purchase to press down with their feet, and rough level the stand ready for the reception of the instrument.

Adjustments.

The above instrument requires several adjustments, which I shall endeavour to describe in the order they are made.

First, To make the lower level parallel to the axis of the telescope :

Place the telescope directly over one of the foot-screws, and clamp it; then bring the bubble of the level, by means of the foot-screw, to the same division on either side the graduated scale affixed to and over the level, taking care that the bubble of the little level at the eye end of the tube (and at right angles to the large level) is in the centre; then reverse the telescope in its collars, observing if the bubble reaches to the same division, and correct one half of that number by the pushing screws on the level itself, and the other half by the foot-screw: this must be repeated until the bubble remains in the same spot.

Second, To place the large level in the same vertical as the axis of the telescope:

Move the telescope in its collars until the level is brought considerably to one side, and observe if the bubble still remains at the same division; if not, move the side pushing screws on the level, until the bubble has returned to its proper place; move the telescope again as much to the other side, and observe if the bubble comes to the same division; if not, it must be re-adjusted, until it is as near as the accuracy of the grinding of the level will allow.

These two adjustments are naturally dependent on one another, therefore they must be both examined, until no alteration in the bubble can be perceived.

Note.—As the collars may wear a little hollow in time, care must be taken that one particular shoulder of the two collars rests against the Y when reversed, in order to use the same point of support.

Third, To place the axis of the telescope parallel to the plane of the instrument:

Loosen now the clamp which confines the telescope over the foot-screw, and adjust the bubble exactly to the centre; bring the telescope and limb half round on its conical centre, and observe the bubble ; half the number of divisions, in error on the scale, must be corrected by the foot-screw, and the other half by the vertical motion to the Y, securing it, when it is sufficiently adjusted, by the little side screw.

Fourth, To place the vertical and horizontal wires at right angles, and to connect them with the little levels on the telescope:

Place one of the station-staves about five feet from the ground in a horizontal position (by means of a small hand level and two nails); from the centre thereof, suspend a white plumb-line; adjust the vertical wire to the horizontal one, by loosening the two screws, which admit of its moving diagonally, and making them coincide with the surface of the staff and the plumb-line; then by loosening the screws of the small level at the eye end, move it until the bubble rests in the centre, reverse the position of the wires, making the vertical horizontal, and adjust in the same manner the other little level.

Fifth, To make the hanging level parallel to the plane of the instrument:

Adjust by the foot-screw, until the bubble of the lower level is in its position; then observe the variation of the riding level, and alter it one half the error by the vertical screw on the cock, and the other half by the pushing screws on the riding level itself.

Or, By the foot-screw, bring the bubble of the hanging level in the centre; then reverse it on the cocks; one half the difference is to be altered by means of the adjustment on the level itself, and one half by the foot-screw: now place the lower level perfectly horizontal, and by the vertical screw on the cock, bring the bubble of the hanging level to correspond.

The usual adjustment for collimation is here purposely omitted. The eye tube of the telescope altering continually in its position, renders it most difficult to make this adjustment correctly; for although it may be found in perfect adjustment for a distant object, when directed to a near one there will be a considerable error, which would affect the results in levelling, if the station-staves were not equidistant from the instrument.

To avoid this inconvenience therefore, as well as to avert the difficulty of placing the extra lenses for short distances, so as not to alter the line of colli-

mation, I permitted the wires to be some distance from the axis of the telescope; and, in levelling, a mean of four or six observations were taken, with the telescope turned half round in its collars at each observation.

Note.—I take this opportunity of mentioning a substitute I have occasionally used for wire or cobweb with success, viz. asbestus; very fine fibres of which can be obtained by being thrown into hot water, when it easily divides. These fibres are tough enough to be placed with ease on the diaphragm, and have the advantage of being opaque.

Being now in possession of an instrument equal to perform the most delicate observations, my next object was to make some improvement in the station-staves, so that they might point out as minute a quantity as the instrument could detect a difference of. This was something difficult, without encroaching on the portability and quick application of the staff to its use.—The following is a description of the staves I used, a Plate of which is given.

The staff itself is a rod of six feet six inches in length, of solid seasoned mahogany on the face of which is let-in a slip of brass, riveted at intervals from end to end; by the side of this is also a slip of holly, fixed in the same manner. The divisions are laid down in feet and tenths, on the holly; and in feet, tenths and hundredths, on the brass. The lower part of the staff is fitted into a square tube of brass about eight inches long, four inches of which are occupied by the staff, and the remainder filled up with lead.

The vane is a plane of seasoned holly, with two semicircles of stained ivory let into the face of it (see Plate). It is fixed by screws to a brass box with tightening springs on the staff itself: there is another small slide on the staff, having two clamping screws, and a long tangent screw, which is attached to the box of the vane by a female screw; the last gives a slow vertical motion to the vane.

On the top of this vane, and at right angles to one another, are two small spirit-levels, mounted in brass; affixed to the vane, in a square hole in the centre, and levelled on one side, is a small brass vernier, the edge of which slides on the divisions of the staff reading to the 1000dth, and, by practice, to the 10,000dth of a foot.

The station-staff thus described is in itself complete; but for accurate observation it requires to be immoveable on the picket: a three-legged support

is therefore added, having a box and ring with double compass gimbles, and a horizontal motion; into this the staff slides.

There is also a small brass tripod with iron legs, having a hole in the centre plate, over which another small plate slides, fixing to the tripod by two clamping screws; this is used to confine the bottom of the station-staff over the picket.

Adjustment.

The only adjustment required to this instrument, is to enable it to be placed vertically over the picket.

Slide the staff into the piece carrying the gimbles; suspend it with the gimbles as nearly as possible at right angles, and as distant as conveniently may be from the ground. When the staff does not oscillate, observe if the two bubbles on the vane are correct; if not, by means of the small screws at either end raise or depress them, until the bubble remains in the centre.

Having now the means of placing the staff immoveably and vertically over a spot, it is to be accomplished as follows: Fix the support that carries the staff, as nearly as possible over the required spot; pushing the legs into the ground, place the small tripod as shown in the Plate, unclamp the milled heads, and pass the staff through the gimbles of the support into the plate of the tripod. As this plate will move in any direction horizontally, the staff is to be adjusted, until perfectly upright, by means of the levels on the vane; then clamp the plate firmly, and push the staff down to the head of the picket, and turn it until it is directly in front of the level.

In March 1830, having every thing prepared, I departed for Sheerness, determining to commence my observations at that point.

As part of the main object of my commission was to ascertain the height of different places above the level of the sea, it was necessary to endeavour to determine that point with accuracy. From the observations made from time to time, at the caisson at Sheerness Dock Yard, of high and low water, I could only obtain this point within a certain degree of accuracy. I therefore determined to take advantage of the permission granted me by the Admiralty, to commence the erection of a tide-gauge at the Dock Yard. Accordingly, after having a model made on a principle that I hoped would be the most simple

and yet the most accurate, I examined the whole of the front of the Dock Yard, and selected the corner of the boat basin adjoining the ordnance basin, as the only spot indeed at all eligible for the erection of a tide-gauge. (The spot is shown on the plan of the Yard.)

By the friendly and prompt assistance of His Majesty's Commissioner at the Dock Yard (J. Lewis, Esq.), I was enabled to have my tide-gauge quickly and substantially erected, under the excellent superintendence of Mr. Mitchell, the master millwright at the Yard.

Description of the Tide-Gauge.

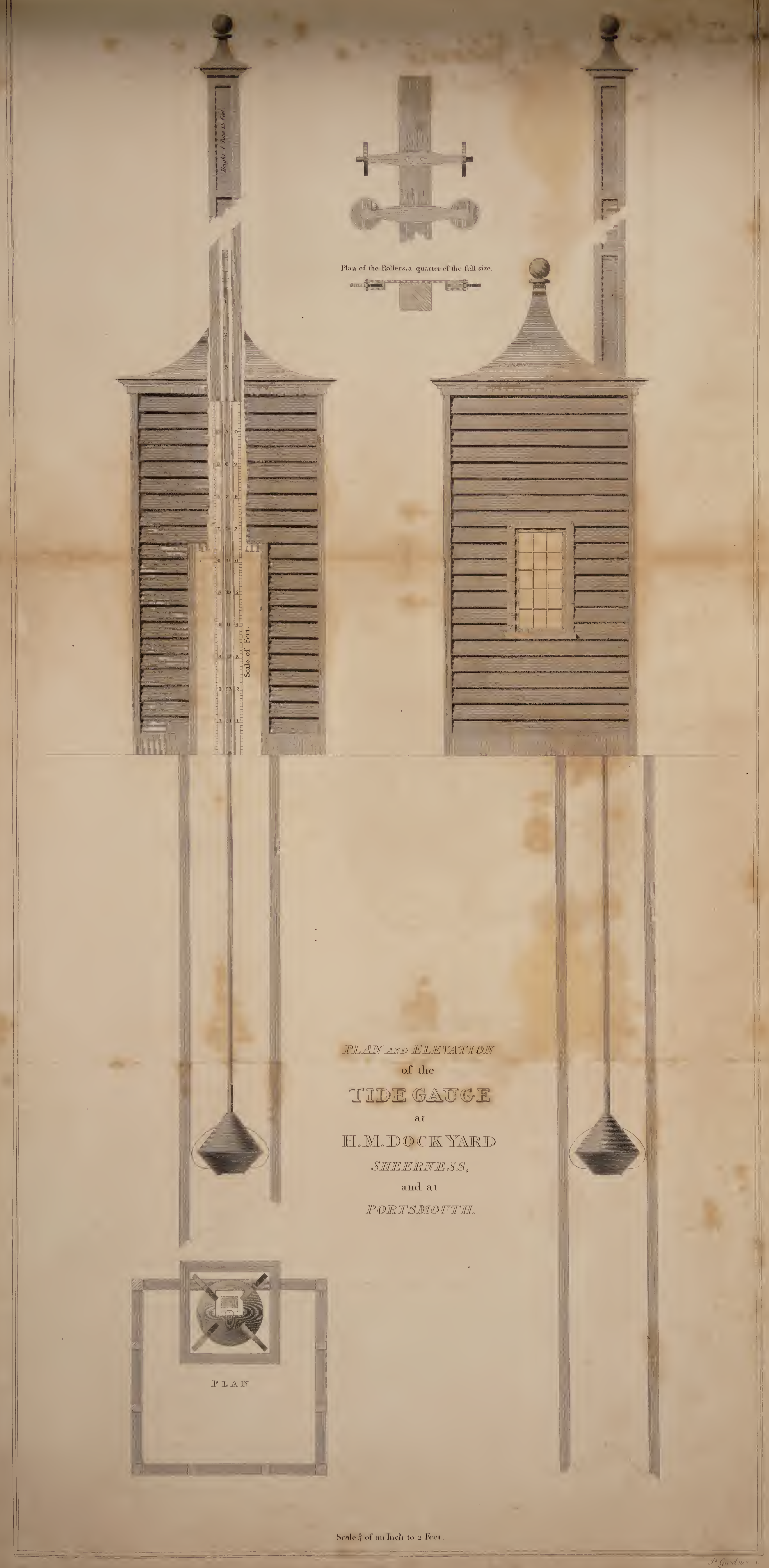
The model is on a scale of three quarters of an inch to a foot.* The flat board on which the house stands, represents the Wharf at Sheerness; it is made long at the back, and balanced so as to be placed on a table, to show the trunk in the actual position of the gauge. The top or cap to the covering of the staff is left unfixed, in order to be taken off, and allow the house to be lifted over it, and show the gauge alone.

The slide rods in the model are of iron wire, and out of proportion; but no smaller was at that time to be found in the Dock Yard; in the original they are $\frac{7}{8}$ copper bolts.

The slides or pointers have springs to tighten them to the rods, but they are too minute to act in the model. The lower end of the tube in the model is nearly filled up with wood, in order to secure it to the trunk, leaving only a part for the gauge-rod to pass; but in the original the tube is left open below, there being sufficient strength in the timber to allow of its being bolted to the trunk and platform. There are three small friction-wheels at the upper end of the rod, to steady it.

There is a distance of fifteen feet between the top catch or point on the rod which brings the slide down to indicate low water, and the point on the middle of the rod which takes the other slide up to indicate high water. Therefore allowing an eighteen-feet tide, the top catch will bring the slide down to four on the index, and the middle catch on the rod will raise the slide to seven on the other index, the difference of which, three added to fifteen (the length between the points), gives eighteen. Therefore it will be observed that the dif-

* The model is deposited at the Royal Society's Apartments in Somerset House.


Height of Tube 15 Feet
Plan of the Rollers, a quarter of the full size.
Scale of Feet.
PLAN AND ELEVATION
of the
TIDE GAUGE
at
H.M. DOCKYARD
SHEERNESS,
and at
PORTSMOUTH.
PLAN
Scale ¾ of an Inch to 2 Feet.

ference between the slides is always to be added or subtracted, to or from the length between the points on the rod. The divisions on either index are continued from 0 feet to 11 feet, to allow for any extraordinary tides. The little brass standard in front of the tide-gauge, and about six or eight feet from it, has been connected with it; therefore the height of any particular tide above the standard is easily found, knowing the exact distance from the middle catch to the line of immersion (which is registered). For example: Let the distance from the centre catch or point, to the line of immersion be 12.444, and the face of the brass standard on a level with 0.747 of the indices; and that high water carries the pointer to 6.546 on the index, then $12.444-(6.546-0.747) = 5.151$, the height of that particular tide below the standard. Again, assuming 4.328, as pointed on the index by the upper catch for low water, then $15+12.444-(4.328-0.747) = 23.863$, the height of standard above that of low water. The difference between the two, gives 18.712 for the rise and fall of the tide.

Many interesting observations may also be made by the same instrument on the irregular rise and fall of the tides, as there is an excellent clock within the house, and I believe an index connected with a weathercock on the outside.

The trunk of the gauge is substantially fixed between large piles, driven in for the purpose, and the whole partitioned off to low-water mark, to render the gauge secure from boats or vessels*.

My next object was to select a standard mark, from whence to commence my levelling, and to form a zero point.

I made examinations of the Dock Yard in different parts, particularly that part facing the Medway. I at first fixed on the northern part of the Dock Yard wall adjoining the garrison; but upon due inquiry I found that part was not considered so good a foundation, being built on piles driven in at some distance from one another, and to no great depth. Wishing, however, to shorten the distance between my zero point and the long level I should have to take across the Medway to the Isle of Grain, and desirous of having the

* This tide-gauge is on the same principle as that mentioned by Mr. Lubbock in the Companion to the British Almanac.

standard in the vicinity of the tide-gauge, so that future observations might be easily connected with the standard, I selected a large block of granite in the southern pier of the entrance to the boat basin, the position of which is shown in the plan of the Yard. I caused a block of gun-metal (cast for the purpose), two inches and a half square and eight inches long, to be sunk in the centre of the granite, about an inch below the surface, thereby allowing a brass box and cover to be placed over the standard, to protect it from injury.

In order that there should be a sufficient number of checks to the stability of this standard mark, I caused three more to be placed in the Yard; viz. one near the southern extremity on the wall of the Dock Yard, one at the eastern side of the great basin, and one in a large block of stone resting on the brick-work of the navy well 330 feet deep.

But however satisfied I might be that these standards were sufficiently firm, I thought it advisable to seek some spot more unquestionable in its foundation than Sheerness Dock Yard. I found a place that possessed this advantage; it was a slight eminence about two miles and a half to the southward of the Dock Yard, and surrounded by a moat.

On this eminence formerly stood the old castle of Queenborough, within a short distance of the present town of Queenborough. The castle, which was in the form of a pentagon, was some years ago pulled down, but a very small part of the foundation was left. On this foundation, which is rubble and chalk, some feet under the surface a very large block of granite was placed for me, by order of the commanding officer of engineers, and into which was let one of the brass standards. The place is now covered over, but marked by a small mound of earth near it, and reference can be easily made to it if required.

Having now standard marks enough to ensure, by comparison, the knowledge of any alteration (if any should occur) in the zero point, I commenced levelling.

My first business was to ascertain the difference of the several standards in the Dock Yard above the level of the sea.

From a series of observations made at the caisson at the entrance to the great basin, in the years 1827, 1828, 1829, the mean of the tides was as follows:

Mean high water spring tide for	1827,	26.475	Low water	8.700	Mean	17.587
Do. . . . do. . . .	1828,	26.50	. do. .	8.61	do.	17.55
Do. . . . do. . . .	1829,	26.09	. do. .	8.93	do.	17.51
Means . .		26.355	. do. .	8.74	do.	17.549
High water neap tides . . .	1827,	22.56	. do. .	11.12	do.	16.84
Do. . . . do. . . .	1828,	22.69	. do. .	11.44	do.	17.06
Do. . . . do. . . .	1829,	22.72	. do. .	11.45	do.	17.08
Means . .		22.656	. do. .	11.336	do.	16.993

Mean for the three years.

Spring tide high water,	26.355	Spring low .	8.74	Mean level	17.549
Neap tide do. . .	22.656	Neap low .	11.336	. do. .	16.993
Means . .	24.50		10.03		17.27

As there are many blanks in the three years' observations that these were taken from, I have also selected the most perfect year (1827), and taken the mean of all the tides for each month in the year.

The following is the

Summary.

January,	Mean high water	24.78	Mean low	10.20	Mean level	17.49
February . .	do. . .	24.08	. do. .	10.00	. do. .	17.04
March . . .	do. . .	24.70	. do. .	9.60	. do. .	17.15
April	do. . .	24.25	. do. .	9.90	. do. .	17.07
May	do. . .	24.35	. do. .	10.35	. do. .	17.35
June	do. . .	24.31	. do. .	10.29	. do. .	17.20
July	do. . .	24.39	. do. .	10.16	. do. .	17.27
August . . .	do. . .	24.39	. do. .	10.26	. do. .	17.32
September . .	do. . .	24.29	. do. .	10.00	. do. .	17.14
October . .	do. . .	24.79	. do. .	9.66	. do. .	17.22
November . .	do. . .	24.65	. do. .	9.90	. do. .	17.27
December . .	do. . .	24.53	. do. .	9.62	. do. .	17.27
Means . .		24.46		9.995		17.23

These results agree so very nearly, that they may be safely taken as correct. I have given them at length, to afford the data for finding the true level of the sea. It will be seen that the mean level taken from spring tides differs 0.556 from the mean level deduced from neap tides. I shall, however, assume the mean level as 17.27, differing only 0.04 from the same of the whole year's observations of 1827.

At the north-western end of the caisson I caused a small cross (+) to be cut in the granite, which corresponded exactly with the XXXI feet of the indices cut in the masonry at the side of the caisson. This cross is situated as follows, with regard to the standard marks in the yard. (See page 3 of Levelling Book.)

The northern standard, which is my zero point, is 0.5789 below the cross. The southern standard at the other end of the Dock Yard wall is 0.0259 above the cross; therefore 0.0259 + 0.5789 = 0.6048 gives the height of the south standard above the northern one, although they are set equally beneath the surface of the granite.

The standard at the eastern edge of the great basin is 0.6735 above the northern, and 0.6735 − 0.6048 = 0.0687 above the southern standard, and 0.6735 − 0.5789 = 0.0946 above the +.

The standard at the navy well is 6.0656 below eastern standard.
5.9968 below south standard.
5.3920 below north standard.
and *5.9710 below the cross.

Now, the mean level of the sea being 17.27 feet, the + or XXXI feet of the indices is 13.73 above the mean level of the sea, and 4.645 above the mean spring tide high-water mark. And

North standard or zero is .	13.73 − 0.5789 = 13.1511	above the level of the sea, and	4.0661	above spring tide high-water mark.
South do .	13.73 + 0.0259 = 13.7559	. . . do.	4.6709	. . do.
East do .	13.73 + 0.0946 = 13.8246	. . . do.	4.7396	. . do.
Navy well .	13.73 − 5.9710 = 7.7590	. . . do.	4.3260	below do.
Tide-gauge standard .	13.73 − 0.0946 = 13.6354	. . . do.	4.5504	above do.

* In page 4 of Levellings there will be observed a difference of 0.0001 in the two sets of levels, the first set taken in March, the last in June.

Having now completed this part of my observations, I levelled from the Dock Yard to Queenborough:—the result of the operation will be seen in pages 1 and 2 of the Levelling Book. And it will be there observed, that between the levels and proofs there is a difference (in a distance of nearly three miles) of 0.0269 about $\frac{3}{8}$ths of an inch. In the commencement of my commission I had a new instrument, and many little difficulties to overcome. In part of the Queenborough observations, I made use of another level which I had from Troughton, on a different principle to the one by Carey; and in page 3 of the rough Levelling Book, an observation is there made on the difficulty I had in getting the level to adjust, from the fault I have mentioned in my remarks, of the bubble having been placed in its tube with only a little paper, and becoming loose.

I used also my large instrument with a level adapted to it, made by Riechenbach, which was lent me by Captain Sabine. It was of a large diameter, and ground most accurately, but not hermetically sealed, having two plates of glass ground in to the ends. It was exceedingly delicate, but at any considerable motion, the bubble would break into many globules, requiring a long time to collect again. The second day I used it in the marshes the day was hot, and I had not at that time the means of shading it from the sun; the bubble, while observing, suddenly contracted, and almost immediately disappeared. I threw some water on the instrument, and the bubble gradually appeared again, but much lengthened. The next day I placed the level in the sun with a thermometer; when at 68° the end was forced out, and I found it contained ether, which appears to be quite unfit for a level in a high temperature. For future observations I replaced it by one of Cary's make.

The mean of the observations give as follows:—Queenborough standard is higher than southern standard 9.9981, and 10.6029 higher than northern standard; and 14.6690 above spring tide high-water mark, and 23.7540 above the mean level of the sea.

Having driven a large picket into the embankment at the Isle of Grain, at a point the shortest distance I could obtain from the north standard (viz. about 5000 feet), across the Medway, I now commenced the observations necessary to ascertain the difference of level between the zero or north standard, and the picket at the Isle of Grain. One instrument was placed near

the standard mark, and the other on the Isle of Grain, about fifty feet from the picket.

Between the high-water mark at the Isle of Grain and the standard at Sheerness, there is a long ledge or bank of mud, extending nearly half a mile, and quite dry at low water. At these times I found it very difficult to make observations, and together with the wind occasionally interfering, and the interruption from the sun's rays, which caused a great vibration in the atmosphere, I spent day after day uselessly waiting at the instrument. I was, however, more successful afterwards, by taking advantage of the first hour or two of day-light, which was more favourable for observation, as the wind had generally lulled, and the air was in a quiescent state.

After the instruments were in perfect adjustment, an observation was made at each side of the river alternately; first by myself at Sheerness, and then by my assistant at Grain. A mean of a considerable number was taken, the difference of which, after the curvature was subtracted from each, would give the refraction, provided the instruments were in good adjustment, and the observations correctly taken. (See page 5 of Levellings.)

Not being satisfied with the observations made at my first visit to Sheerness, I returned at a subsequent period, and made a great number most satisfactorily, from which the best were selected, and a mean taken.

As there was not a fit place for a standard mark near the margin of the river Medway, I caused a block of granite, about 900lbs. weight, to be placed on the foundation of the old church wall (the church was once much larger) of St. James, near the porch-door, into which block was sunk one of the brass standards. (See page 6 and 8.)

The face of this standard is 28.7454 feet above the north standard mark; 32.8115 above spring tide high-water mark, and 41.8965 above the mean level of the sea.

From St. James I almost immediately came into the marshes, levelling in a direct line towards Yantlet Creek. In consulting the Ordnance map of Kent, there appear but two or three ditches in the marshes; but in a distance of less than three miles I passed thirty-three dykes and ditches, from fifteen to twenty-five feet broad, and four or five deep, over which there is no mode of passing, but a bridge perhaps a mile distant.

It was desirable therefore to find some means of conveying my instruments, men and apparatus over, without losing so much time by going round. The men first used leaping-poles; and a case for the instrument was constructed water-tight, so as to swim over the ditches. But from the sides of the ditches being steep, the instrument was constantly subject to heavy blows in being taken out of the water; I therefore adopted another mode. On our arrival at a broad ditch, two leaping-poles, about eighteen feet in length, with cross pieces at the bottom to prevent their sliding too far in the mud, were connected together at the top by passing the eye of a rope just over the ends of them, where it was confined by two thumb-cleets; to this rope was attached, at any height that the depth of the ditch might require, a hook having four tails, and likewise hooks to them; these fixed into the four arms of the box, through which slid the poles for carrying it. The ends of the two leaping-poles were merely placed in the centre of the ditch, forming a pair of sheers, the apex of which was inclined on one side or the other by guys. The instrument was in this manner taken from one bank and landed on the other with the greatest gentleness; it was quickly unhooked, and men and apparatus passed over in the same manner. In this mode 263 ditches were passed between Sheerness and Greenwich.

On my arrival at the marshes, I found, from the nature of the soil, the greatest difficulty in adjusting the instruments; the movement of any of the men at a considerable distance caused a motion in the bubble, and the least alteration in my position whilst standing at the instrument shifted the bubble. In order to avoid moving, which heretofore was necessary to examine the level, I caused a square mirror, about eight inches by five, to be mounted on a mahogany frame, which permitted it to stand on the upper limb with the face placed at an angle of about 60° from the telescope, by which I could, without moving my head from the eye end of the telescope, read off the length of the bubble. This precaution was not however sufficient in some parts of the marshes, where the ground was of hardly more consistence than a bog, but invariably most unsubstantial where the most dry.

At these places, large pickets, from four to six feet in length, were driven in, having a groove on the iron head of each. On three of these, the iron-shod points of the stand rested; and after the observations were made, the pickets were drawn.

The pickets used for the stations were also of a larger class than those used in other parts; but the ground was occasionally so spongy, that it was with much difficulty that a picket could be driven, and frequently the iron heads of several would break off, before we could succeed in getting one down.

It will be seen by the Observation-book, that from a mean of pickets (page 5 to 11), the Queenborough and Sheerness marshes are in some parts 6.3652 below the northern standard; but these marshes are unlike the rest passed over, being particularly rugged and undulating.

In the commencement of the marshes between St. James and Yantlet Creek, it appears by a mean from picket 11 to 17, that the surface of the marshes is 5.8524 below the standard, or 1.7863 below mean spring tide high-water mark; and opposite Allhallows, by a mean from picket 35 to 44, the marshes are 3.7247 below the north standard, rising there 5.8524−3.7247=2.1277.

About half a mile past the Decoy in St. Mary's marshes, nine miles distant from Sheerness, the marshes again fall: the mean from picket 75 to 86 gives 5.9916 below the north standard at Sheerness, and 5.9916−4.0661=1.9255 below mean spring tide high-water mark.

Some distance past Cliff Canal, and between that and the Gravesend Canal, in Higham marshes, by a mean from 120 to 124, the surface is 6.6356 below the north standard, which −4.0661 gives 2.5695 below spring tide high-water mark, having fallen in a distance of seven miles and a half 2.9109, and in a distance of five miles 0.6440; the marshes between Northfleet and Greenhithe, by a mean from picket 208 to 211, are 7.4889 below the north standard, and −4.0661=3.4228 below spring tide high-water mark.

On the eastern side of Dartford Creek, the marshes, by a mean from picket 247 to 252, are 8.8676 below the north standard, therefore 8.8676−4.0661 =4.8015 below spring tide high-water mark.

On the western side of the marshes, as far as the mean from picket 256 to 259 will show, the marshes are 9.7207 below the north standard, and 5.6546 below spring tide high-water mark, and lower by 0.8531 than the marshes on the eastern side of Dartford Creek.

The marshes to the eastward, and in the immediate vicinity of the arsenal at Woolwich, are (from a mean of pickets 305 to 312) 10.1404 below north standard, and 6.0743 below spring tide high-water mark, and only 3.0107 above the mean level of the sea.

The only remaining pickets that were directly in the marshes, are from 347 to 352, the mean of which gives 9.6321 below north standard, and 5.5660 below spring tide high-water mark.

The following therefore is a statement of the depression of the marshes, from Sheerness towards the source of the Thames.

The Yantlet Creek marshes are 0.5128 higher than the Sheerness marshes.

The Allhallows marshes, in a distance of three miles, are 2.1277 higher than the Yantlet Creek, and 2.6405 higher than the Sheerness marshes.

The St. Mary's marshes are 0.1392 lower than the Yantlet Creek marshes, and 2.2669 below the Allhallows marshes, having fallen that quantity in a distance of three miles.

The Higham marshes are lower than those of Yantlet Creek by 0.7832, than those of Allhallows by 2.9109, and than those of St. Mary's by 0.6440, having fallen the last quantity in five miles.

The marshes between Northfleet and Greenhithe are lower than Yantlet Creek 1.6365, than Allhallows 3.7642, than St. Mary's 1.4973, than Higham 0.8533, being a fall of this last quantity in 6½ miles.

On the eastern side of Dartford Creek, the marshes are 3.0152 below those of Yantlet Creek, 5.1429 of Allhallows, 2.8760 of St. Mary's, 2.2320 of Higham, and 1.3787 below the marshes between Northfleet and Gravesend; being a fall of the last quantity in 4½ miles.

The marshes near Woolwich Arsenal to the eastward of the practising ground are 4.2880 below those of Yantlet Creek, 6.4157 of Allhallows, 4.1488 of St. Mary's, 3.5048 of Higham, 2.6515 of the marshes between Northfleet and Gravesend, and 1.2728 below the eastern Dartford Creek marshes, being a fall of 1.2728 in six miles.

The marshes at Greenwich, as far as the few observations I had the opportunity of making, are 0.5083 higher than those of Woolwich, therefore less that sum than the comparison of the Woolwich marshes.

At picket 131 (page 14), I intersected the Thames and Medway Canal, three miles from its mouth at the Thames.

The above picket was driven into the water's edge, another was at the same minute driven to the water's edge in the basin close to Gravesend. I then levelled along the banks, imagining that from a mean of simultaneous observa-

tions made at the two pickets at the water's edge, I should obtain the best proof to my levellings; the results (page 6 of Proof Levels) will show how little confidence is to be placed on water, as a true level, under such circumstances.

At the Lock Gate of the canal, close to the Thames (see page 15), I made some observations of the tides, and found, June 7th, that high-water mark at Gravesend was 1.1018 higher than at Sheerness, and June 8th, was 0.8367 higher; but these two observations are not to be depended on as giving a mean difference between the two places, as the height of the tides at Gravesend are much affected by any winds.

On the new pier at Gravesend, I caused one of the brass standards to be sunk in the granite on the eastern side, the face of which is 0.1828 below the north standard mark, and 3.8833 above mean spring tide high-water mark at Sheerness.

On my arrival at Greenwich Hospital, I commenced a set of branch levels, from thence to the Royal Observatory at Greenwich, in order to determine the height of that place above the level of the sea. From the abruptness of the ascent, the operation was very tedious; and I here found the advantage of the extra lens to the telescope, as there was seldom a distance of more than twelve or twenty feet between the pickets.

I levelled up to a small brass standard already placed for me by the direction of the Astronomer Royal in the block of stone immediately under the eye-end of the transit instrument pointing southward.

This standard (page 31 of Levels) is

140.6806 above the north standard at Sheerness.

153.8317 above the mean level of the sea.

144.7467 above the mean spring tide high-water mark.

162.3611 above the mean spring tide low-water mark.

These observations being completed, it occurred to the Astronomer Royal, after minutely examining my instrument, that it might be used as a proof in ascertaining the correctness of the horizontal point of the two mural circles.

By his directions I placed my level upon the high window-frame in front of and about eight feet from the object-end of the mural circle, pointed towards the north, and at such a height that (from a known principle in optics that all rays are parallel *,) I could intercept some of the rays from the object-glass.

* If any object be placed at the focus of a lens (viz. the wires), the emergent rays are parallel.

Having adjusted my level most exactly, I directed my telescope into the tube of the great telescope of the mural circle; and adjusting as for infinite distance, by placing a disc of white paper about an inch from the eye-end of the great telescope, I observed all the wires most distinctly. I then adjusted my horizontal wire for collimation.

The mean horizontal point was then taken, and the circle adjusted to that by the micrometer; and after again observing my instrument to be in perfect adjustment, I sought for the horizontal wire of the circle, and I was astonished and delighted to see so perfect a coincidence of the horizontal wires of the two instruments, that, until I slightly depressed the eye-end of my telescope, I could not see the horizontal wire of the circle separate from my own. The circle was then altered, and the wires were again made to coincide; the quantity was then read off, and found to agree within a very few hundredths of a second to the horizontal point.

The Astronomer Royal was present at these experiments, and expressed himself much pleased at the proof given of the coincidence of the two instruments.

From the stairs of Greenwich Hospital I crossed the river in order to level up to the different places where tide-registers had been kept.

After crossing the Isle of Dogs, I arrived at a spot on the south side of the lock of the City Canal at Limehouse Reach (see No. 381, page 26), which was 1.9008 above the north standard at Sheerness; and this spot was 3.8446 above the $\begin{array}{c}\text{TRINITY}\\ \text{HW}\\ \text{1800}\\ \hline \wedge\end{array}$ marked on the face of the masonry. Therefore $3.8446 - 1.9008 = 1.9438$ is the height of Trinity mark at the canal below northern standard at Sheerness, and 4.0661 (height of north standard above mean spring tide high-water mark) $- 1.9438 = 2.1223$, the height of Trinity mark above mean spring tide high-water mark at Sheerness.

This spot (picket No. 381) is also 0.5202 above a particularly high tide 21/ft. 11 in., 1827, marked on the masonry; but upon referring to the tide-register at Sheerness, of the 20th and 22nd of November 1827, no particular rise in the tide is to be remarked. It must therefore have been caused by land floods, which are the occasion of most of the extraordinary tides near London.

I next levelled to a standard placed at the West India Docks on the S.S.E. side, close to the entrance. This standard is 1.3018 above the northern standard, and 2.3367 above the index mark XXIII. Therefore 2.3367 − 1.3018 = 1.0349, the index mark below the north standard at Sheerness, and 4.0661 − 1.0349 = 3.0312 is the height of XXIII above mean spring tide high-water mark at Sheerness.

Not having succeeded in procuring a copy of the observations on the tides at the West India Docks, I cannot make any other comparison.

At the Regent's Canal Dock, in a very large stone near the office of the canal works (see page 26), I placed another standard mark. This standard is 2.4418 above the northern standard at Sheerness, and 2.2308 above the index mark XXI; therefore the index is 0.2110 above the north standard at Sheerness.

From two years' observations on the tides (1828—1829), the following are the results.

Mean Spring Tide High-Water Mark for				
	1828.		1829.	
	Spring Tide, High Water.	High Water, Neap.	Spring Tide, High Water.	High Water, Neap.
	Ft. In.	Ft. In.	Ft. In.	Ft. In.
January........	19.8^{3}	15.	19.1	14.8
February	18.11	14.3	19.1^{5}	13.11
March	19.9	14.3	19.5	14.8
April	19.9^{6}	14.3	19.6	13.7
May	18.9	15.5	19.	14.9
June	18.10	15.7	18.9^{5}	15.2
July	19.2^{6}	15.9	18.9^{6}	15.7
August	19.3	15.4^{5}	19.3	14.7^{6}
September......	19.9	14.6	19.7	14.7
October........	19.7	14.5		
November	19.1^{5}	14.3	19.3	14.3
December	19.2^{5}	13.9	18.7	14.9
Means	19.325	14.733	19.141	14.592

Spring 1829 ...... 19.141
—— 1828 ...... 19.325

Mean Spring Tide for two years 19.233

Neap 1829 ...... 14.592
—— 1828 ...... 14.733

Mean Neap Tide for two years 14.662

Mean Spring Tide 19.233
Mean Neap Tide 14.662

Mean High Water 16.947

The mean high-water mark, taking every high water through the months, is as follows:

	1828.	1829.
January	17.342	16.904
February	16.904	16.804
March	17.070	17.208
April	16.925	17.270
May	17.352	17.175
June	17.258	17.091
July	17.595	17.350
August	17.466	17.100
September	17.345	17.591
October	17.000	,,
November	17.001	17.412
December	16.691	16.787
Means	17.1624	17.1538

$$\underbrace{17.1624 \quad 17.1538}_{17.1581}$$

The difference therefore between the mean of spring and neap tide and the means of the months, as above, is (17.1581—16.947) 0.2111.

The index mark XXI at the canal is 0.2110 + 4.0661 = 4.2771 above spring tide high-water mark at Sheerness; and the spring tide high-water mark at the Regent's Canal being 19.233, XXI — 19.233 = 1.767, and 4.2771 — 1.767 = 2.5101, the height of mean spring tide high-water mark at the Regent's Canal above the same at Sheerness.

The index mark XXI is also 6.1321 above mean high-water mark at Sheerness.

And the mean high-water mark at Regent's Canal being

16.947, XXI — 16.947 = 4.053 and 6.1321 — 4.053 = 2.0791

or 17.1581, XXI — 17.1581 = 3.8419 and 6.1321 — 3.8419 = 2.2902 the height of mean high-water mark at the canal above the same at Sheerness.

And the mark XXI is 7.9761 above mean neap tide high-water mark at Sheerness. The mean neap tide mark at the canal being 14.662, XXI — 14.662 = 6.338 and 7.9761 — 6.338 = 1.6381, the height of mean neap tide high water at the canal above the same at Sheerness. Therefore the water of spring tides at the canal above spring tides at Sheerness is higher by 0.8720 than it is at neap tides at both places.

Having no observations of low-water mark at the canal, I have not the means of ascertaining the difference at the two places.

The next place where a comparison of the tides was made was at the London Docks, at No. 418 of the Levels (see page 27), being on an iron on the south-west pier of the main entrance close to the first lock. This iron was 3.7570 above the northern standard, and 5.7682 above the index mark by the side of the gates xxiii, which mark answers to the 18-feet Trinity $\underset{\wedge}{\underline{\overset{\text{HW}}{1800}}}$

Therefore $3.7570 + 4.0661 - 5.7682 = 2.0549$ the height of xxiii above spring tide high-water mark at Sheerness. And $2.0549 + 1.855$ (the difference between spring tide and mean tide high-water mark at Sheerness) $= 3.9099$, the height of xxiii above mean high-water mark at Sheerness.

And $2.0549 + 3.699$ (difference between spring and neap tides at Sheerness), $= 5.7539$, the height of xxiii above neap tide at Sheerness.

And $2.0549 + 9.085$ (the difference between mean level and spring tide at Sheerness), $= 11.1399$, the height of xxiii above the mean level of the sea at Sheerness.

And $2.0549 + 17.615$ (difference between spring high and low water at Sheerness), $= 19.6699$, the height of xxiii above spring tide low-water mark at Sheerness.

By the kindness of Mr. Lubbock I have received the following results of twenty-six years' observations on the tides at the London Docks:

	Mean High Water.	Spring High Water.	Neap High Water.
	ft. in.	ft. in.	ft. in.
January	21.275	22.729	19.750
February	21.275	23.002	19.145
March	21.291	23.541	19.125
April	21.395	22.854	19.375
May	21.475	22.708	20.125
June	21.395	22.50	20.250
July	21.291	22.604	19.687
August	21.250	22.708	19.562
September	21.291	23.979	19.292
October	21.291	22.937	19.229
November	21.395	22.687	19.833
December	21.312	22.458	19.870
Mean	21.333	22.812	19.604

Therefore xxiii — 21.333 — 1.667.

And 3.9099 — 1.667 = 2.2429, the height of mean high-water mark at the London Docks above the same at Sheerness.

And xxiii — 22.812 = 0.188, therefore 2.0549 — 0.188 = 2.0361, the height of spring tide high-water mark at London Docks, above spring tide high-water mark at Sheerness.

And xxiii — 19.604 = 3.396, and 5.7539 — 3.396 = 2.3579, the height of neap tide high water at London Docks above the same at Sheerness.

No observations have been made on low-water mark; but from the Trinity mark it appears the spring tide low-water mark is considered to be 17.833 below Trinity mark, or rather below the high-water mark.

Therefore 22.812 — 17.833 = 4.979, and xxiii — 4.979 = 18.021, the height of xxiii above spring tide low-water mark.

And 19.6699 — 18.021 = 1.6679, the height of spring tide low water at London Docks, above spring tide low water at Sheerness.

Taking 22.812 and 4.979, the mean level of the sea is 13.896.

Therefore xxiii — 13.896 = 9.104, the height of xxiii above the mean level.

Then 11.1399 — 9.104 = 2.0359 gives the mean level at London Docks above the mean level of the sea.

The following is a Summary of the different heights.

Spring tide H. W. at London Docks, above the same at Sheerness				2.0361	
					0.2068
Mean H. W. mark	ditto	ditto	ditto	2.2429	
					0.1150
Neap tide Ditto	ditto	ditto	ditto	2.3579	
					0.6900
Spring tide low water	ditto	ditto	ditto	1.6679	
					0.3680
Mean level of the tides	ditto	ditto	ditto	2.0359	

Or taking more correctly the ½ difference between spring high and low water at Sheerness, the mean spring level is 10.8289 below xxxiii, therefore 10.8289 — 9.104 = } . . 1.7249

Note.—It seems by the above summary that as the water decreases in height, so the height of the water's surface at London Docks, above the same at Sheerness also decreases, with the exception of spring tide at London Docks and the neap tide; these are means, not of the highest tides, but of tides at a particular time of the moon's southing.

The next spot levelled to where any tidal observations were made, was St. Catherine's Docks, where a brass standard was placed close to the south-west side of the Dock-gates at the entrance.

This standard is 4.3143 above the north standard at Sheerness, and 6.2563 above the index mark XXVIII, upon a level with which is a —+— denoting Trinity $\underset{\wedge}{\overset{\text{HW}}{\underline{1800}}}$

Therefore $4.3143 + 4.0661 - 6.2563 = 2.1241$, the height of this index or Trinity mark above mean spring tide high-water mark at Sheerness.

After passing along the Tower Wharf, and placing a standard mark in one of the large blocks of granite lately put down near the Traitor's Arch (see Levellings, page 28), I arrived (not without much vexatious interruption and annoyance in passing along Thames-street and Billingsgate) at the starlings of Old London Bridge, where I had some difficulty in making observations, owing to the tremor caused by the vehicles above. I levelled up to the Trinity mark on the western side of the bridge, and found as follows (see page 29):

No. 443 was 4.1047 below the north standard mark at Sheerness, and the Trinity $\underset{\wedge}{\overset{\text{HW}}{\underline{1800}}}$ was 1.9426 above No. 443.

Therefore $4.1047 - 1.9426 = 2.1621$, the height of Trinity mark below north standard mark at Sheerness.

And $4.0661 - 2.1621 = 1.9040$ gives the height of Trinity mark above the mean spring tide high-water mark at Sheerness.

Having now made all the observations that time and means afforded me, I concluded my levellings at a standard mark sunk in the large plinth of the landing-place (near the wall) of the stairs on the north-east side of the New London Bridge.

This standard was (page 29) 2.3967 below the north standard mark at Sheerness.

I have subjoined a list of the different Trinity marks I have observed, and their respective heights, and also of the brass standards placed by me, as well as various substantial points passed over in the levellings.

	Below North Stand. Sheerness.
Trinity high-water mark near the dock-gates of the City Canal .	1.9438
Ditto, answering to XXIII of the indices marked on the south-west side of the entrance to London Docks	2.0112
Ditto, at the west side of the lock-gate at the entrance to St. Catherine's Docks, answering to the index mark XXVIII . .	1.9420
Ditto, on the west side of London Bridge	2.1621

Finding so great a difference between the marks at the City Canal, London Docks, &c. and that at London Bridge, I levelled again in October last from St. Catherine's Docks to London Bridge, but found the same results. (See page 33.)

List of Standard Marks and other Points of Reference between Sheerness and London Bridge.

		Feet.
North standard at Sheerness		0.0000
South ditto ditto	higher	0.6048
Eastern ditto ditto	ditto	0.6735
Navy Well ditto ditto	lower	5.3921
Little ditto near tide-gauge at Sheerness	higher	0.4843
Queenborough standard	ditto	10.6029
St. James ditto	ditto	28.7454
The boundary post of Hoo, on a little eminence in the marshes, &c. (See page 10.)	below	2.6317
The 3 mile stone at the bank of canal. (See page 14)	above	2.4204
2 mile ditto	ditto	2.0038
1 mile ditto	ditto	1.1448
An iron clamp at second gate of Gravesend Canal. (See page 15)	ditto	0.5564
Brass standard mark on the pier at Gravesend . .	below	0.1828
The boundary stone of Swanscomb. (Page 18) . .	ditto	4.7578
Erith Church. (Page 20)	ditto	1.5669
Standard in Woolwich Arsenal	above	1.2019
A in a corner-stone near the officers' guard-room	ditto	1.5576
The top of the $4\frac{1}{2}$ mile post in the river (□) . . .	ditto	1.7182
□ on a stone at the west end of the dock-yard . .	ditto	1.9975

		Feet.
Brass standard in the dock-yard on the eastern point of mast slip	above	1.6545
A ⊥ in small north-east gateway of Greenwich College, from the main road	ditto	9.9975
Small brass standard underneath the transit at Greenwich Observatory	ditto	140.6897
Little brass standard on the plinth of the statue of George II. in Greenwich Hospital	ditto	5.1157
□ on one of the iron plates near the south side of the lock of the City Canal	ditto	1.9008
Brass standard at the West India Docks	ditto	1.3018
Brass standard at Regent's Canal	ditto	2.4418
A □ on the top of a granite post close outside the entrance from Ratcliffe Highway to the London Docks	ditto	16.0915
On the top of a gun in the Docks near the bridge of the eastern basin	ditto	13.1646
Brass standard St. Catherine's Docks	ditto	4.3143
Brass standard near the Traitor's Arch in the Tower	ditto	1.2854
Brass standard at landing-place of New London Bridge	below	2.3967

As it may be expected that I should state the manner in which I made the preceding observations, in order that a judgement may be formed of the confidence to be placed in them, I shall give a concise description of the field operations, and of the calculations necessary to complete the observations, and place them as they are in the book*.

After examining the ground, the particular line for carrying on the levellings was selected and marked out, when pickets were driven in at proper distances, according to the range of the instrument over the ground.

In the first part of the work the levelling was carried on as follows, the instrument being perfectly adjusted, and the station staves placed on their respective pickets. I made four observations, the telescope being in a different

* The field-book and all the other papers connected with the preceding operations are preserved at the Royal Society's Apartments.

position at each observation, and making a complete revolution in its collars: my assistant read off each, and noted it down in a book. I then went to the staff and examined its position on the picket, and read off the last observation myself; my assistant then read from his book the last observation; the two, of course, when correct, would correspond.

The telescope of the instrument was then reversed in its collars (which is a good check to the adjustment at each observation), and by the motion in azimuth directed to the next station staff, when the same mode was used in observing: the spot was marked, and the instrument moved to the next station, and the station staff turned half round in its collars, and gimbles ready for the next level. At the end of the day's work an additional picket was driven in about twelve or fifteen feet from that just used, and compared with it; this was to ensure the detection of any alteration in the pickets during the intervening time, either from mischief or accident, being compared before the commencement of the day's work. After a few days the ground was gone over again, generally from the opposite end, and two observations taken at each picket, which were sometimes more and sometimes less in number than in the former levellings; these were the proof-levels. The distances were then measured, and the necessary angles taken to lay down the work. In this manner proof levels were taken up to picket No. 111; but finding in this method that I was liable to great inconvenience and loss of time from many circumstances, and amongst others by the pickets being mischievously drawn or moved; and if a trivial mistake occurred in the levellings (of several days back perhaps), it was only detected at the summary of the levels. I determined to endeavour to adopt some method of proof, not liable to the inconveniences of the former.

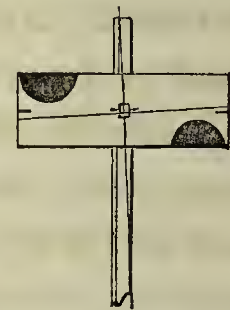
I accordingly made use of the following mode, which I have found, after repeated trial, to be a most correct proof.

After having finished the four observations at one picket, I threw the instrument out of adjustment by the foot-screws; and after adjusting it again most correctly, I took a pair of observations; but instead of reading off the staff from the picket, it was now read from above, the staff being imaginarily continued to the length of ten feet. The mean of these observations, therefore, gave the complement to 10,000 of the true observation: by this mode much time was saved, and an error, if any, was immediately detected of a tenth of a

foot, or a foot (which are the errors most likely to occur), without the trouble of bringing the instrument and adjusting it two distinct times at the same spot. The mean of the four and the mean of the two observations ought of course to make 10,000 *. In the detail of the proof-levels, the correction for curvature is additive instead of subtractive; and in fact the whole operation is reversed, + standing for −.

In the course of my levellings, having instruments not generally used, I made some few notes, which I take leave here to transcribe.

In adjusting the station-staff, it is difficult to know when the zero on the vane is made to coincide with the horizontal wire of the telescope. I have found the most convenient and correct mode to be, to observe with the wire forming a diagonal to the lines on the vane, by which, when the staff is near, the two small black lines at each end of the vane could be seen; and when the vane was adjusted to the proper height, one of these lines was above and the other below the horizontal wire at equal distances, thus:



—At a greater distance, the two little white right-angled triangles, formed by the edges of the vane and bounded by the black semicircle, are very distinctly seen, the one above and the other below the (now-placed) horizontal wire, and can be compared in size with great nicety.

But in observing with the wire diagonally, great care must be taken that, by the vertical wire (the error of which, if any, will by practice be accurately known), the axis of the telescope shall bisect the centre of the staff.

In an instrument with a very sensitive level, there is usually some difficulty in adjusting the level. It arises from no fault in the level itself, but from a

* In the proof-levels it will be found, that generally, upon adding them to the mean of the four observations, there will be a quantity of from .0010 to .0060 to make up the 10,000. I could not discover the cause for some time.

The wires being at right angles to each other, of course have been at different distances from the eye-glass; but the difference not being much more than the thickness of a hair, I did not alter the eye-tube: however, upon examination, I found that at the usual distance I took levels, when I altered from extreme distinct vision of the horizontal wire to extreme distinct vision of the vertical wire (which was the one I used for proofs), it made a difference of from .0020 to .0050, the distinct vision of the vertical wire being that quantity lower than when observed with the same adjustment as for the horizontal wire.

difference in weight of those parts of the telescope outside the collar. I found this difficulty in my instrument: to remedy it, I measured off from one of the resting points half the distance between the two Ys on the collars, and suspended the telescope by a fine wire from this point, which was the proper centre of the telescope. I found the eye-end with the tube close in to be more than four ounces heavier than the object-end. To remedy this, I caused a thick ring of lead of the above weight to be placed inside the tube near the object-end, by which the telescope was balanced; and I found it, when adjusted, to reverse without differing a quarter of a degree.

Distinct vision is certainly desirable, but not so absolutely requisite as that there shall be no parallax of the wires. The best way to avoid this, is, after adjusting for distinct vision, to move the eye as far as the hole in the eye-piece will admit of, and observe if the wires have any motion over the object or vane: if so, it must be remedied by sliding the eye-tube in or out, until the objects appear motionless.

Mirage.

I have found that the tremulous motion or jumping in the air, termed as above, appears not to be caused so much by evaporation as probably by some oscillation in the particles of light: for I have remarked, when the sun shines brightly and is occasionally obscured by clouds, that while the sun is out, the tremor is so great as to prevent the possibility of making a correct observation; yet the moment the sun is obscured, the intermediate space between the instrument and object (provided the sun is obscured so as to cast a shadow the whole distance) will be immediately perfectly tranquil; and again, at the instant of the sun's appearance, the same tremor will be observed.

I have found this motion to be exactly equal above and below any object; for upon placing the wire of the telescope one half the distance between the extreme oscillations, whenever the sun is obscured, the wire will be found to bisect the object.

Description of the Observation-Book.

Each page of the book will be found to contain sixteen columns: the first and ninth contain the numbers of each staff or picket; the second and eleventh the mean from the rough book of the four observations at each staff; the third

and eleventh the distance in feet from staff to instrument and instrument to staff; the fourth and twelfth the correction for the curvature of the earth; the fifth and thirteenth the mean observations minus the curvature; the sixth and eighth are the length of the bubble and the thermometer attached to the instrument. This last is useful as a check to the bubble, which, when the instrument is moved suddenly, shakes into several small globules, that sometimes do not immediately unite again: this is detected by the length of the bubble, which ought to correspond to a certain degree of the thermometer. Column fourteen is the difference between columns five and thirteen; column fifteen is that difference + or − ; and column sixteen is the amount of that difference added or subtracted, according to the sign, from the former quantity. These quantities are a continuation of heights above or below the first picket, or the northern standard in the dock-yard at Sheerness.

In order to prove the correctness of the different columns, they are summed up at the bottom, when the gross sum for curvature, being deducted from the gross sum of columns two and eleven, show the correctness of columns five and thirteen; and the difference between the sums of columns five and thirteen + or −, added to or subtracted from the little figures above the top line in column fifteen, ought to give the last true level in column sixteen at the bottom of the page.

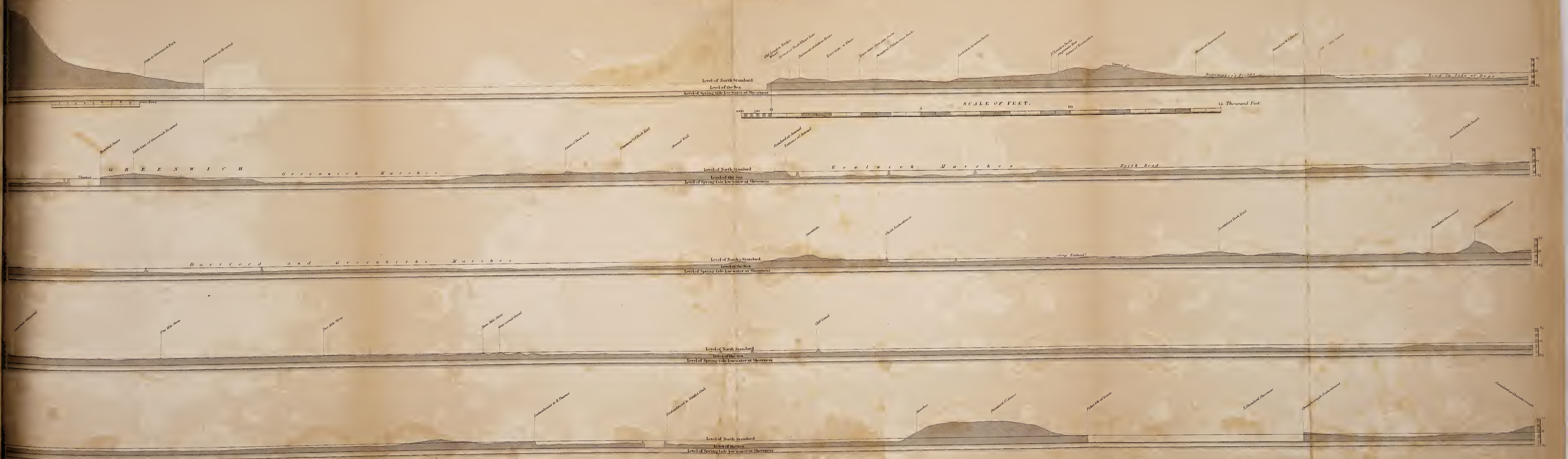
The whole is further proved by taking the sums of columns five and thirteen of each page, and the difference of the whole amount of each gives a proof of the correctness of the whole work, by giving the difference at once between picket 1 and 445.

The same method is pursued in the proof-levels up to No. 112: after that, all the corrections become reversed, the curvature being additive (as the complement to 10,000 is read off on the staff). The difference between the proof-levels continue from standard at Sheerness to New London Bridge, and will be found to be 0.0110, and varying at different distances from Sheerness, but never more than 0.0300.

As there is not, in any work that I have seen, a correct table of curvatures of the earth for small distances, I have added one for every five feet from 60 to 1000.


PLAN of the LINE of LEVELLINGS, from SHEERNESS to LONDON.
PLAN of HIS MAJESTY'S DOCK YARD, at SHEERNESS.
LONDON
Regent's Canal
W. I. DOCKS
Isle of Dogs
GREENWICH
Park
Observatory
Charlton
WOOLWICH
Plumsted
ERITH
Purfleet
Grays Thurrock
Crayford
Dartford
Stone
GREENHITHE
NORTHFLEET
GRAVESEND
Milton
Gravesend Canal
Chalk
Higham
Canal
Cliff
H. Halstow
Decoy
St. Mary's
Allhallows
Stoke
Yantlet Creek
St. James
Isle of Grain
RIVER THAMES
RIVER MEDWAY
Shoebury Ness
Nore Light
SHEERNESS
Blue Town
Mile Town
Queenborough
SHEPPY ISLE
SCALE OF MILES.
1 2 3 4 5 6 7 8 9 10 Miles
THE LINES
Beach
New Road
Chapel
Commissioners House
Officers Houses
High Street
BLUE TOWN
Pay Office
Navy Well Standard
Military Hospital
Ordnance Well
DOCK YARD
THE GARRISON
Saw Pits
West Street
To Queenborough
Smithery
Store Ho.
Dry Docks
Tide Gauge and Standard
Mast and Boat House
Mast and Boat Pond
Admiral's Ho.
Small Basin
GREAT BASIN
Working Mast House
Boat Basin
North Standard
South Standard
THE HARBOUR

SECTION OF THE LAND PASSED OVER IN THE LEVELLINGS FROM SHEERNESS TO LONDON BRIDGE.



A Table of the Curvature of the Earth, the mean diameter being 41,807,803 feet.

Dist.	Curvature.	Dist.	Curvature.	Dist.	Curvature.	Dist.	Curvature.	Dist.	Curvature.	Dist.	Curvature.
Feet.		Feet.		Feet.		Feet.		Feet.		Feet.	
60	000086	220	001157	380	003453	540	006975	700	011721	860	017691
65	000101	225	001216	385	003544	545	007104	705	011888	865	017897
70	000117	230	001265	390	003637	550	007235	710	012057	870	018104
75	000134	235	001321	395	003730	555	007368	715	012228	875	018313
80	000153	240	001378	400	003827	560	007501	720	012399	880	018523
85	000173	245	001435	405	003923	565	007636	725	012572	885	018734
90	000194	250	001495	410	004021	570	007771	730	012746	890	018946
95	000216	255	001555	415	004119	575	007908	735	012921	895	019159
100	000239	260	001617	420	004219	580	008046	740	013098	900	019379
105	000264	265	001680	425	004321	585	008186	745	013275	905	019591
110	000290	270	001744	430	004422	590	008326	750	013454	910	019807
115	000316	275	001809	435	004526	595	008468	755	013634	915	020025
120	000344	280	001875	440	004621	600	008611	760	013816	920	020245
125	000374	285	001943	445	004735	605	008755	765	013998	925	020466
130	000404	290	002016	450	004843	610	008901	770	014181	930	020687
135	000436	295	002082	455	004952	615	009047	775	014366	935	020911
140	000469	300	002152	460	005060	620	009195	780	014552	940	021135
145	000503	305	002224	465	005171	625	009344	785	014740	945	021360
150	000538	310	002298	470	005284	630	009494	790	014928	950	021586
155	000575	315	002373	475	005397	635	009645	795	015118	955	021824
160	000612	320	002449	480	005511	640	009798	800	015308	960	022043
165	000651	325	002526	485	005626	645	009951	805	015500	965	022274
170	000691	330	002605	490	005743	650	010105	810	015693	970	022505
175	000732	335	002684	495	005861	655	010262	815	015888	975	022738
180	000775	340	002764	500	005978	660	010419	820	016083	980	022972
185	000818	345	002847	505	006100	665	010577	825	016280	985	023207
190	000863	350	002929	510	006221	670	010737	830	016478	990	023443
195	000909	355	003014	515	006324	675	010898	835	016677	995	023681
200	000957	360	003100	520	006467	680	011061	840	016877	1000	023919
205	001005	365	003187	525	006593	685	011223	845	017079		
210	001054	370	003275	530	006719	690	011383	850	017281		
215	001105	375	003363	535	006846	695	011554	855	017485		

NOTE.—I received the following rule for curvature and refraction together, from DAVIES GILBERT, Esq., and it will be found useful in ascertaining heights approximately.

Rule.—Assume the diameter of the earth as 10,000 instead of 7918 miles, and which will give the refraction about one-tenth the intercepted arc.

XI. *On the Variable Intensity of Terrestrial Magnetism, and the Influence of the Aurora Borealis upon it. By* Robert Were Fox. *Communicated by* Davies Gilbert, *Esq. M.P. V.P.R.S.*

Read March 17, 1831.

IN the annexed Table are given the results of a series of observations on the vibrations of the magnetic needle, which I undertook last summer, for the purpose of ascertaining whether its intensity is or is not affected by the changes in the earth's distance from the sun, or by its declination with respect to the plane of his equator; for, if we refer the nodes of the planetary orbits to this plane, there appears to be so considerable a degree of coincidence in most of them, as would seem to imply the existence of a more definite law than we are accustomed to attach to the abstract principle of gravitation*.

I am not at present prepared to say much respecting this part of my investigation; but I have obtained results, which appear to be interesting, relative to the variable force of the magnetic attraction, and the action of the aurora borealis on the direction and intensity of the needle.

I have used two needles, one possessing the north, and the other the south polarity in excess. To effect this, I have employed, in each case, a slip of seasoned oak, split with the grain, and suspended near the centre by unspun silk several inches long. This slip of wood serves as a support for a magnetic bar or needle, which is firmly riveted to it near one of its poles—say, for example, the north pole; whilst the south pole is at liberty to yield longitudinally to any

* I had been making numerous experiments with the same object in view some time before I commenced the series now given; but the results were unsatisfactory, from my not having employed a *stationary* magnetic apparatus, which should always be done in cases in which great accuracy is required. If the needle has been touched with the hand for one or two seconds only, its action is disturbed for some little time; indeed, it requires not a little practice to appreciate all the precautions that are necessary in vibrating the magnetic needle.

contracting or expanding force*. The magnetic bar is eight inches long, half an inch wide, and one tenth thick: its south pole extends rather more than two inches beyond the centre of suspension of the wooden support, so that there is nearly an equal portion of the southern half of the needle on each side of the centre, the north end being extended as a lever to control, in great measure, the magnetic movements of the needle†. A counterpoise of lead is placed at the other extremity of the support, sufficient to allow the north pole to dip at an angle of about forty degrees from the horizontal direction. The foregoing description also applies to the needle in which the south polarity predominates, except that the poles are reversed, the counterpoise being at the depressed end of the support. The former I shall distinguish as No. 1, and the latter as No. 2. Each is inclosed in a box of slate; as, if of metal, the action of the needles might have been disturbed: and the boxes are mounted steadily on bricks, in a room appropriated to them. A slip of glass at one end of each box enables me to observe the vibrations, which are from east to west, like the horizontal needle.

I have observed that the magnetic intensity is subject to frequent slight variations; but these I have been mostly unable to refer to any obvious cause, except when accompanied by the appearance of the aurora borealis, which evidently affected the needles on many occasions. Their vibrations, I think, generally became less rapid with a moist atmosphere, and more so when it was very dry; but I do not speak with full confidence on this point, as I have only recently made any regular hygrometrical observations. I might, perhaps, also mention changes of the wind, and snow storms, as being sometimes attended with fluctuations in the intensity of the needles. If the times of their vibration at different periods are compared, differences in their relative intensity will be noticed; that of No. 2. having on the average been diminishing, during the last three months nearly, in a more rapid ratio than No. 1. This must, I think, be considered an interesting fact, if confirmed by future observations. It does

* If the rivets which attach the magnetic bar to the support are placed at a proper distance from the acting pole, the equipoise of the two extremities will not be affected by the contraction or expansion of the bar.

† The neutralization of the pole would have been more fully effected had a smaller proportion of it extended beyond the centre of suspension; $1\frac{1}{4}$ to $1\frac{1}{2}$ an inch would probably have been sufficient.

not seem easy to account for the discrepancies which not unfrequently occurred in the indications afforded by the two needles at the same time, and I hesitate to adopt any distinct conclusions from this circumstance without further experiments, which I intend to make with horizontal needles on the same principle.

I have endeavoured by various experiments to ascertain whether there is any decided and permanent difference in the directive force of the opposite poles;—for example, I have alternately neutralized in an equal degree the poles of a needle by means of a sliding axis, and suspended it horizontally by unspun silk, but the two ends vibrated in nearly equal times. Other needles on the same principle gave corresponding results, or at least the differences were so inconsiderable, and were so nearly compensated on an average of many observations at different times, that I think it may be safely assumed that there is an equality in the yearly mean of the magnetic intensity of the two poles separately considered. But on the hypothesis of a central magnetic force, ought not the north pole in this latitude to be acted on with much greater energy than the south? for if my experiments may be depended upon, the alternative can scarcely be adopted, of supposing that one pole of a needle is necessarily repelled as strongly as the other is attracted, since it appears that their relative intensity is not always the same. It therefore seems most reasonable to refer the phenomena of the earth's magnetism to the agency of electrical currents existing under its surface, as well as above it: indeed, I think it is impossible to doubt that the changes in the intensity and direction of the needle, which are often so transitory, must be due to meteorological causes *.

The aurora borealis which has frequently appeared more or less distinctly this winter, generally affected both needles during some part, and only a part, of the time of its being visible. On the seventh of last month, the aurora was seen from this place † as soon as it became dusk, and was still visible some time after midnight. It extended from N.N.E. to N.W. or W.N.W., and at intervals

* Is it not probable that the small anomalies which have been sometimes observed in the oscillations of the pendulum may be owing to the same causes which produce the much more considerable irregularities in the magnetic needle?

† Falmouth.

sent up streams of red and white light which occasionally nearly reached the zenith. These were most striking early in the evening, and more especially about eleven, or a little later, when the coruscations were beautiful. At 7 P.M.* I found the needles at 0, but soon after their north ends moved towards the east, and at 8 to 8½ their easterly variation was 1° 15′. They began to return westward at 8¾, and soon after 10 P.M. were again stationary in the magnetic meridian. At 11 to 11½ P.M., I found their intensity had diminished. Many instances are given in the table of the appearance of the aurora in the horizon about the magnetic north, and extending more or less considerably on each side of it. It was generally of a pale white, and sometimes I could perceive faint streams shooting upwards from the horizon a very few degrees, but more often I could not. It will be seen that considerable variations of the needle usually occurred in the course of the same evenings; and sometimes these variations took place on evenings when I did not remark any luminous appearance in the northern horizon.

All the variations at night were towards the east †, whether the aurora was actually visible or not; and hence may we not conclude, taking it for granted that it is an electrical phenomenon, and that it usually moves from about the north towards the south, that it must be of the nature of positive electricity? And, by a parity of reasoning, may we not assume the existence of an opposite state of electrical action by day, in order to account for the diurnal westerly variation, which is most considerable in the summer in these latitudes, when the aurora prevails about the south pole? This idea seems to be strengthened by a fact I have noticed, that the magnetic intensity is usually less considerable in summer before the middle of the day than it is afterwards; so that the minimum intensity commonly occurs some hours before the maximum temperature.

It is evident that the elevation of the aurora must often be exceedingly great, probably much more than a thousand miles, as it seems to be generally seen from places very distant from each other at the same time, and in nearly the same

* I did not observe the needles before 7 P.M.

† I have since observed a slight westerly variation in the needle at night, but this seems to be of rare occurrence.

direction: thus, for example, the observations made on the beautiful aurora of the 7th ultimo from the vicinity of London, and of this place, seem very nearly to accord in almost every respect.

In conclusion, I will venture to express a hope that the subject of terrestrial magnetism may obtain the attention which it appears to deserve, and that experiments may be made at the same time in different countries, in order to develop its more obscure properties. It will then perhaps be found that its relations to other natural phenomena are as extensive as they are interesting and important. At any rate, it seems probable that some light might be thrown on the hypothesis of electrical currents under and above the surface of the earth, and their relative influence on the magnetic needle, if observations on its intensity were to be made on small islands, as remote as possible from any large tracts of land, and the results compared with others obtained by the same apparatus, on extensive continents at stations as nearly as may be in the same magnetic parallels: or, instead of employing the same apparatus, several magnetic needles might be forwarded to different places, after having been carefully compared with a standard needle; and with these, simultaneous observations might be made, not only in the same parallels, but likewise in different parallels of latitude and longitude in both hemispheres.

A Table showing the Times of Vibration of two Magnetic Needles, described in the annexed paper, in the neighbourhood of Falmouth.

Needle No. 1.					Needle No. 2.										
Time of Observation.			Arcs 80° to begin, ended as under.	Number of seconds in making 80 vibrations.	Time of Obs.		Arcs 80° to begin, ended as under.	Number of seconds in making 80 vibrations.	Baro-meter.	Ther-mometer in room with needles.	Ther-mometer out, ex-posed to the north.	Lunar phases.	Extraordinary variations of the needles by night.	Winds.	General Remarks.
1830.	A.M.	P.M.			A.M.	P.M.									
Aug. 2	......	10	......	1183†.00											
3	10	......	39.00	1179†.00											
4	......	10	39.00	1184†.00	......	......	......	....	....	......	......	○	....	......	Clear.
5	10	......	39.00	1186†.00											
6	......	10	39.00	1181†.00	......	......	......	....	....	......	......	......	....	......	Fine weather.
9	......	10	39.00	1183†.00											
Mean for the month				1182.66											
[a]Oct. 7	......	10	38.30	1180†.00	......	10.30	43.00	1137.00	30.30	60.00	......	☾ 8th	....	......	[a] Suspended nearly two months in consequence of absence from home.
9	......	3.30	38.30	1180†.00	......	4	43.00	1132.00	30.40	......	......	......	....	......	Fine clear weather.
13	......	10	39.00	1177†.00	......	10.30	43.00	1131.00							
15	......	10	39.00	1180†.00	......	11	43.00	1133.00							
Mean for the month				1179.25	......	......	......	1133.25	....	......	......	● 16th			
[b]Nov. 8	......	8	39.00	1177†.00	......	8.30	42.00	1127.50	....	......	......	......	....	......	[b] Again suspended from absence.
12	......	8	39.00	1178.00	......	8.15	43.00	1130.00							
13	10.30	......	39.00	1180.00	11	......	44.00	1134.50	....	......	......	......	....	S	A great storm with rain.
15	......	7.30	39.00	1179.00	......	8.30	43.00	1134.00	29.20	58.00	......	●	....	SW	A great storm without rain.
16	10	......	40.00	1182.00	10.30	......	44.00	1136.00	28.50	58.00	......	......	....	”	Calm throughout the day, but a great storm at night.
19	......	7.30	39.00	1176.00	......	9.45	44.00	1133.00	30.00	54.00	......	......	....	”	Fine.
22	......	8	39.00	1176.00	......	9	44.00	1131.00	29.90	55.00	......	......	....	”	Nearly calm and clear weather.
23	10	......	39.30	1176.00	10.30	......	44.00	1130.00	30.00	54.00	......	☽	....	”	Nearly calm and clear weather.
24	......	2.15	40.00	1177.00	......	2.30	44.00	1126.50	30.20	55.00	......	......	....	SE	A strong dry wind.
25	11.30	......	40.00	1177.00	10	......	44.30	1125.00	30.10	55.00	......	......	....	”	A strong dry wind, very clear.
26	11.30	......	40.30	1175.00	noon		44.30	1123.00	29.80	52.00	......	......	....	”	Clouded, and wind stronger—some rain.
27	......	11	40.00	1176.00	......	11.30	44.30	1124.00	28.90	52.00	......	......	....	ESE	Rain in morning, evening clear.
29	10.30	......	39.30	1180.00	11	......	44.00	1128.00	29.40	56.00	......	......	....	......	Strong wind and dry weather.
Mean for the month				1177.61	......	......	......	1129.38	29.60	54.90		○ 30th			

3	......	10	40.00	1177.00	......	10.30	44.30	1126.00	29.30	54.00	.....	......	....	NE	Clouded, and light wind.
4	10.30	......	39.30	1176.00	10.50	......	45.30	1126.00	29.50	54.00	......	......	....	SE	Clouded, and light wind.
6	......	9.30	40.00	1176.00	......	10	45.00	1128.00	28.10	54.00	......	......	....	„	A great storm to-day, which abated in the evening.
7	9.30	......	41.00	1175.50	10	......	46.00	1128.00	28.40	55.00	......	☾	....	NE	Rain.
8	......	8.30	41.00	1174.00	......	7.30	45.30	1127.00	28.70	54.00	......	......	....	SE	Rain.
9	......	10	41.30	1173.00	......	10.20	45.30	1125.00	28.60	52.00	......	......	....	NE	Clouded.
10	9	......	41.30	1173.00	......	......	46.00	1125.50	28.80	51.00	.....	......	....	NW	Showers.
11	......	7.30	41.00	1175.75	......	7	46.00	1126.50	29.20	49.50	......	......	....	W	Strong squalls with hail showers.
„	......	11.15	42.00	*1177.30	......	11.40	45.30	*1129.00	*29.*20	*49.00	......	......	....	„	Aurora visible this evening; the variation not observed.
12	......	......	......	....	......	......	......	....	....	......	......	......	6 P.M. 45′ E.	NNE	Aurora again visible in N.N.E. to N.W. from 6 to 11 P.M.
13	......	4	41.00	1172.00	......	4.30	46.00	1122.50	30.30	47.50	......	......	....	SW	Hard frost in the morning, but now gone.
14	......	11	41.30	1172.00	......	......	46.00	1123.50	30.50	48.00	......	......	....	SSW	Sunshine.
15	......	10	40.00	1172.00	......	......	45.00	1124.00	30.45	49.00	......	●	....	S	
16	9	......	41.00	1174.00	9.30	......	44.30	1126.00	30.25	50.00	......	......	....	N	Clouded, and small rain.
„	......	9.30	40.45	*1171.00	......	10	44.30	*1124.00	„	„	......	......	9 P.M. 20′ E. $10\frac{1}{2}$ P.M. 15′ E.	ESE	Strong wind, and clear.
17	9	......	41.15	1172.00	9.30	......	44.30	1124.50	29.30	49.00	......	......	....	NNE	Nearly calm.
18	......	10.30	41.15	1169.50	......	11.30	45.00	1122.50	30.30	46.00	......	......	....	W	
20	9	......	41.30	1173.50	9.30	......	45.15	1126.00	29.50	47.50	......	......	....	N	Showers.
21	......	10	41.15	1174.00	......	10.45	45.15	1126.50	29.30	49.00	......	......	10 P.M. 20′ E.	NNW	Clouded.
22	......	7	42.00	1173.50	......	7.30	45.30	1126.00	29.40	49.00	.....	☽	....	WNW	Strong gusts of wind, and rain.
23	9	......	41.30	1171.50	9.30	......	46.00	1125.00	29.45	46.50	......	......	....	NW	Hail showers.
„	......	11	40.30	*1168.50	......	11.30	45.30	*1123.00	*29.50	46.00	30.00	......	$9\frac{1}{2}$ P.M. 30′ E.	N	Snow showers.
24	......	11	40.30	1167.00	......	11.30	45.30	1128.00	29.50	41.00	27.00	......	....	„	Mostly clouded; deep snow on the ground.
25	......	2	40.15	1167.00	......	3.15	45.30	1128.50	29.30	41.00	25.00	......	....	„	Clear weather, and deep snow on the ground.
„	......	9.30	40.00	*1173.50	......	10	46.00	*1134.00	*29.20	*41.00	*33.50	......	$9\frac{1}{2}$ P.M. 35′ E. 10 P.M. 50′ E. 12 P.M. 30′ E.	„	Snow and sleet falling.
27	9.15	......	40.30	1171.50	10	......	46.30	1131.00	28.70	41.50	36.00	......	....	W	Clouded.
„	......	11	40.00	*1172.00	......	11.30	46.00	*1132.00	*28.70	*42.00	*39.00	... ..	9 P.M. 15′ E.	SW	Clouded, and rain occasionally.
28	9	......	40.30	1171.00	10	......	45.30	1132.00	28.90	42.50	39.00	......	....	W	Clouded, and showers of hail and rain.
29	9	......	40.15	1172.00	9.30	......	44.45	1132.00	29.50	43.00	39.00	○	....	SW	Rather hazy.
„	......	10.45	40.40	*1174.00	......	10	45.00	*1133.00	*29.35	*44.00	*44.00	......	9 P.M. 25′ E. $9\frac{1}{2}$ P.M. 50′ E. $10\frac{1}{2}$ P.M. 15′ E.	S to E	Just commencing to blow strongly—showers.
30	9.30	......	40.30	1174.50	10	......	44.30	1134.50	28.85	44.00	47.00	......	....	E	Strong wind and much rain.
31	9	......	41.30	1176.00	10.30	......	45.00	1134.00	29.00	46.50	45.00	......	....	W	Clouded and damp.
„	......	11.30	41.00	*1173.00	11	......	45.30	*1131.00	*29.50	*45.50	*38.50	......	....	W	Clear, and fine night.
Mean for the month				1173.30	......	......	......	1126.72	29.39	48.61	36.60				

TABLE of the Vibrations, &c. (Continued.)

Needle No. 1.					Needle No. 2.				Baro-meter.	Ther-mometer in room with needles.	Ther-mometer out exposed to the north.	Lunar phases.	Extraordinary variations of the needles by night.	Winds.	General Remarks.
Time of Obs.			Arcs 80° to begin, ended as under.	Number of seconds in making 80 vibrations.	Time of Obs.		Arcs 80° to begin, ended as under.	Number of seconds in making 80 vibrations.							
1831.	A.M.	P.M.			A.M.	P.M.									
Jan. 1	......	9	41.00	1174.00	9.30	......	44.30	1133.50	29.40	48.00	50.00	......	....	S	A strong wind and some rain.
3	......	10	41.45	1175.00	......	11.30	45.00	1134.50	29.50	51.50	50.00	......	6¼ P.M. 15′ E.	E	Much wind—clouded.
4	9	......	41.00	1175.00	9.30	......	45.00	1133.25	29.50	50.00	49.00	......	....	"	Clouded.
5	......	4	41.30	1175.50	......	3.30	45.00	1134.25	29.65	51.50	51.00	☾			
6	......	10	40.30	1173.50	......	10.30	45.40	1133.00	30.50	49.00	41.00	......	....	NE	Clear weather.
7	9	......	40.30	1171.75	9.15	......	45.00	1131.50	30.60	48.00	41.00	......	....	E	Clear weather.
"	......	11.30	41.00	*1177.30	......	11	45.00	*1135.00	*30.60	*48.00	*39.00	......	7 P.M. 0. 8 to 8¼ P.M. 1.15′ E. 8¾ P.M. 1.10′ E. 9 P.M. 40′ E. 9.20 P.M. 20′ E. 10 P.M. 5′ E.	"	Clear weather—Aurora visible at 5½ P.M. in N. & N.W., and afterwards extended to N.E. & W.N.W.; it was seen after midnight.
8	9	......	40.30	1173.50	9.30	......	44.30	1132.25	30.65	46.00	42.00	......	9½ P.M. 15′ E. 11 P.M. 0′ E.	"	Hoar frost in the morning, and afterwards clear—a faint white light in N. & N.W. most of the evening, till after midnight.
10	......	11.30	41.30	1173.00	......	12	45.00	1133.00	29.90	46.50	46.50	......	10 P.M. 10′ E. 10½ P.M. 20′ E. 11 P.M. 10′ E.	NE	Luminous in N.N.W. this evening.
11	......	......	40.30	1172.50	9.30	......	45.00	1133.00	29.92	46.00	39.00	......	....	"	Clouded.
"	......	10	41.00	*1172.00	......	11	45.30	*1126.75	*29.94	*45.00	*38.00	......	No variation observed.	"	Clear—faintly luminous in magnetic meridian.
"	......	11.30	41.30	*1172.50	......	11.45	45.00	*1124.50	*29.94	*45.00	*38.00				
12	......	10.30	41.30	1172.00	......	11	45.00	1133.00	29.90	45.00	45.50	......	....	ESE	Clouded.
13	9	......	41.30	1173.00	9.15	......	44.30	1132.00	29.92	46.00	45.00	......	....	"	Clouded.
14	9	......	41.00	1172.75	9.30	......	45.00	1132.00	29.95	46.00	45.00	●	....	SE	Clouded.
"	......	11.30	41.30	*1171.50	......	12	45.00	*1133.00	"	"	"	......	8 P.M. 10′ E. 8¾ P.M. 25′ E. 9 P.M. 10′ E.	"	Pale white Aurora this evening in N. & N.W.
15	noon		41.15	1172.00	......	1	45.30	1131.50	29.70	46.50	44.00	......	....	"	Clouded.
17	9	......	41.15	1174.50	9.30	......	45.00	1133.75	29.20	48.00	51.00	......	....	SE	Clouded, and some rain.
18	9	......	41.45	1175.50	9.30	......	45.15	1134.00	29.00	50.00	52.00	......	....	SW	Clouds and mists, with some rain.
19	......	2.30	41.30	1175.50	......	4	45.00	1134.50	29.20	51.00	50.50	......	....	SSE	Watery clouds and damp.
20	......	10.15	41.30	1176.00	......	10.20	45.00	1135.00	28.40	51.50	50.00	......	....	SE	Rain.
21	9	......	42.00	1176.00	9.45	......	45.00	1134.50	28.35	52.00	50.00	☽	....	"	Rain and strong wind.
"	......	10.15	41.30	*1176.50	......	10.45	45.00	*1134.50	*28.50	*53.00	*50.00	......	10 P.M. 15′ E.	"	Nearly calm.
25	......	7	41.00	1171.00	......	7.30	45.00	1130.50	30.00	46.00	35.00	......	....	NNW	Some snow fell in the night.
26	......	5.45	40.30	1168.50	......	6.15	45.00	1129.50	30.20	45.00	35.00	......	....	N	Fine weather, and a little snow on the ground.
28	......	7.30	41.00	1170.00	......	8	44.30	1129.00	29.63	45.00	39.00	○	....	NW	Cloudy with showers.
29	......	7.30	41.00	1169.00	......	8	44.30	1130.00	29.63	44.00	39.00	......	....	SSE	Rain and sleet.
31	......	8	41.00	1170.50	......	8.30	45.00	1130.00	28.17	44.50	40.00	......	....	SW	Much rain to-day.

4	......	9.45	41.00	1170.00	......	10.15	45.00	1130.50	29.55	45.00	39.00	☾	....	WNW	Clear.
5	......	7.30	41.00	1170.00	......	7.45	45.00	1130.50	29.85	44.50	41.00	......	....	"	Clear.
7	......	7	41.00	1177.00	......	7.30	45.00	1136.50	29.43	50.00	52.00	......	....	SSW	Clouded, and a little rain.
8	9	......	41.15	1177.50	9.30	......	45.00	1137.00	29.50	55.00	55.50	......	....	"	Rain.
9	......	9.15	40.30	1174.50	......	9.45	44.30	1134.00	29.90	56.50	55.00	......	....	"	Clouded, and some rain.
10	8.15	......	41.00	1174.50	8.30	......	45.00	1135.50	29.90	55.00	57.00	......	....	SW	Fine weather.
"	......	10	40.30	*1174.00	......	9.30	44.30	*1135.00	*30.00	*55.00	*52.00	......	Slight variation, E.	"	Faintly luminous in N.W.
11	10.15	......	41.15	1175.50	10.30	......	45.00	1136.50	30.15	55.00	52.00	......	....	WSW	Clouded and damp.
12	9	......	41.15	1176.50	9.30	......	45.00	1137.00	30.20	53.00	53.50	●	....	"	Clouded and damp.
14	......	6.45	41.00	1175.00	......	7	45.00	1135.50	30.00	51.00	50.50	......	....	S	Clouded, and some rain.
15	9.15	......	41.30	1175.50	10	......	45.00	1136.00	29.85	50.50	50.00	......	11 P.M. 25′ E. $11\frac{3}{4}$ P.M. 0′ E.	"	Clouded, and some rain—at 11 P.M. horizon in N.N.W. a little luminous, but much increased at $11\frac{3}{4}$ to 12.
16	8.15	......	41.30	1176.00	8.30	......	45.00	1136.00	29.80	51.00	48.00	......	....	NW	Fine morning, afterwards clouded.
17	8	......	41.15	1173.75	8.30	......	45.30	1134.50	30.10	48.50	48.00	......	7 P.M. 30′ E. $7\frac{1}{4}$ P.M. 15′ E. $7\frac{3}{4}$ P.M. 0′ E.	"	Fine weather, slightly luminous in N.N.W. at 7 P.M.
18	8	......	41.00	1174.00	8.20	......	45.30	1133.00	30.25	48.00	45.00	......	....	W	Clear weather.
19	9.45	......	41.15	1174.00	10.30	......	45.30	1134.50	30.20	49.50	51.00	☽	....	NW	Clouded.
21	......	7.15	41.00	1173.00	......	7.45	45.30	1133.20	30.05	47.00	47.00	......	....	WNW	Fine day—some clouds.
22	8	......	41.15	1174.50	8.30	......	45.45	1135.40	30.00	48.00	51.00	......	....	N	Some rain and damp.
23	8	......	41.00	1174.50	8.20	......	45.15	1134.60	30.37	47.00	44.00	......	....	NE	Clouded, but dry weather.
24	8	......	40.30	1174.00	8.30	......	45.30	1135.00	30.15	47.50	50.00	......	....	SW	Clouded—some showers.
25	8	......	41.00	1174.00	8.20	......	46.00	1135.50	29.87	48.00	47.00	......	....	NW	Clouded—damp.
26	......	6	41.30	1175.00	......	6.20	46.00	1136.50	29.35	49.00	48.00	○	....	WSW	Much rain.
28	......	9.30	41.00	1173.50	......	10	46.00	1134.50	29.73	48.00	42.00	......	....	NNW	Clear this evening and dry.
Mean for the month				1173.75	......	......	......	1134.17	29.78	48.83	47.37				

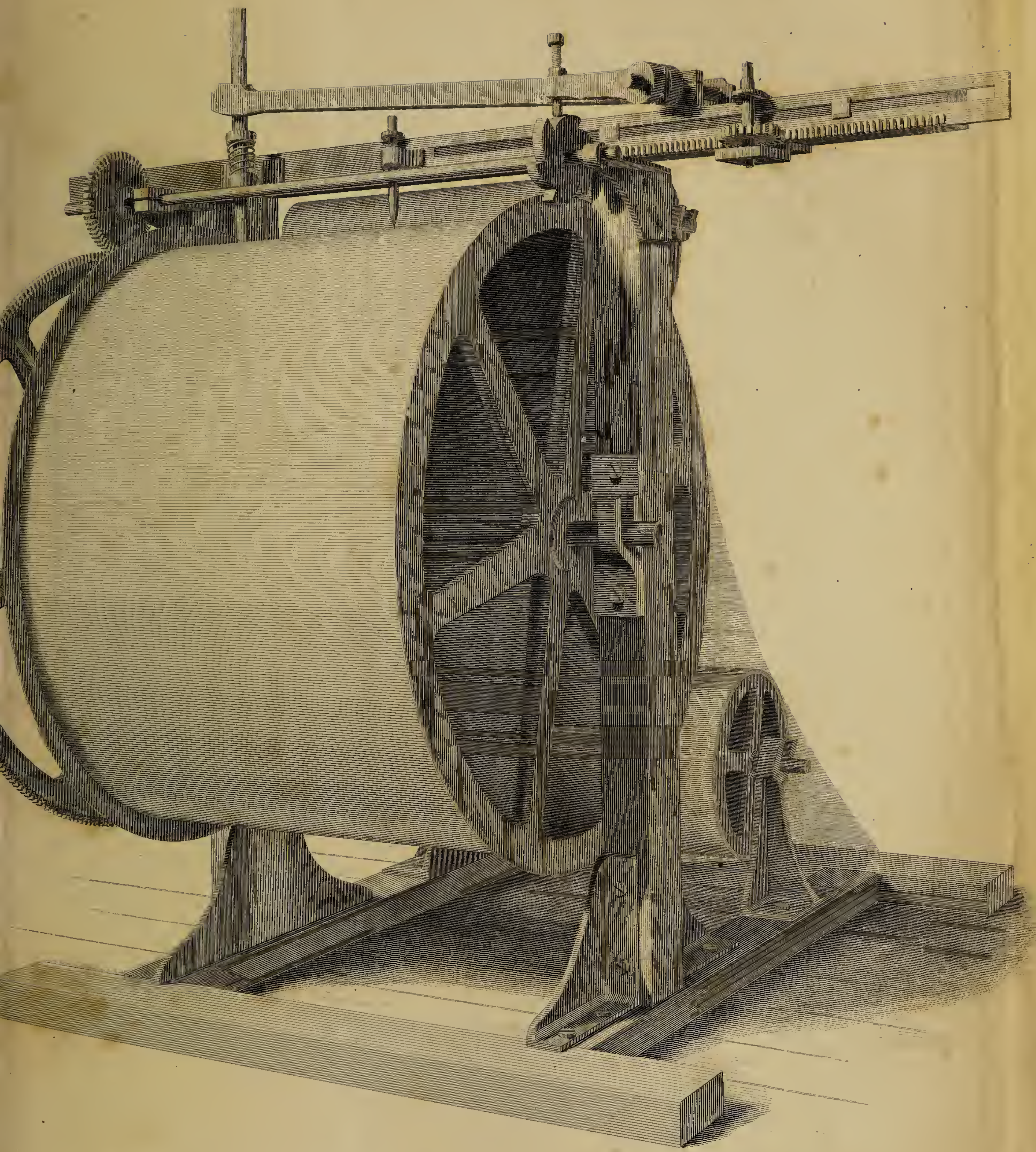
Note 1.—I have added 8″ to each of the numbers marked thus †, in consequence of a slight alteration having been made in the dip of needle No. 1, on the 8th of November, which appeared to add at least eight seconds to the time of its performing 80 vibrations.

2.—The numbers in the above Table marked thus *, are not comprised in the monthly averages, as I thought it best to include in the mean, only one observation for each day, and most of the results so marked are rather anomalous, or were obtained about the time when the needle varied considerably to the eastward. I may here remark that I had four needles, from which I noted the variations.

3.—There is a considerable difference in the arcs described by the two needles: this I attribute to the circumstance of No. 1 being not so heavy as No. 2, the latter having of course required a more considerable counterpoise, in order to give an elevation to its south pole equal to the depression of the north pole of No. 1.

No. 2 weighs, including the support and counterpoise	....	898 grains.
No. 1 weighs, including the support and counterpoise	....	854 ——
Difference	....	44 grains.

On adding 202 grains to the weight of No 1, and retaining it at the same dip, instead of 41° 15′, it described an arc of 50°, after having performed eighty vibrations. The difference in the times of the two needles, I attribute to No. 2 being more strongly magnetic than No. 1.



spective View of a Graphic Registrer of the Tides and Winds.

Eng.d by T. Bradley.

No II

A Roller on which the Paper is wound

B The Drawing Cylinder

C Roller for keeping the Paper in contact with the Drawing Cylinder

D Toothed wheel connected with the Clock

E Camb Wheel for raising the Hammer

F The Hammer

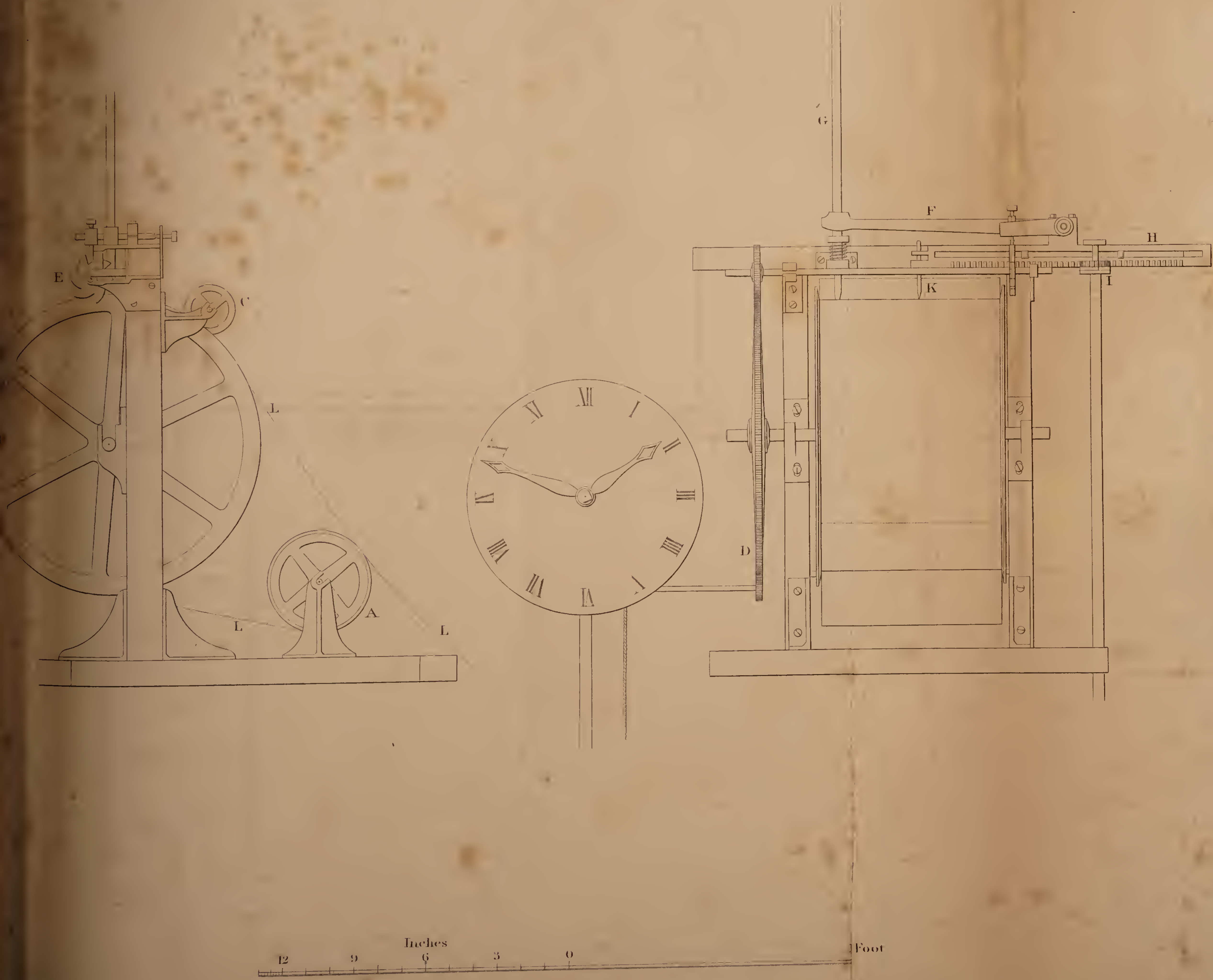
G The Spindle with the Arrow upon it which is connected with the Weathercock

H The Rack

I The Pinion connected with the Float

K The Pencil

LLL The Paper

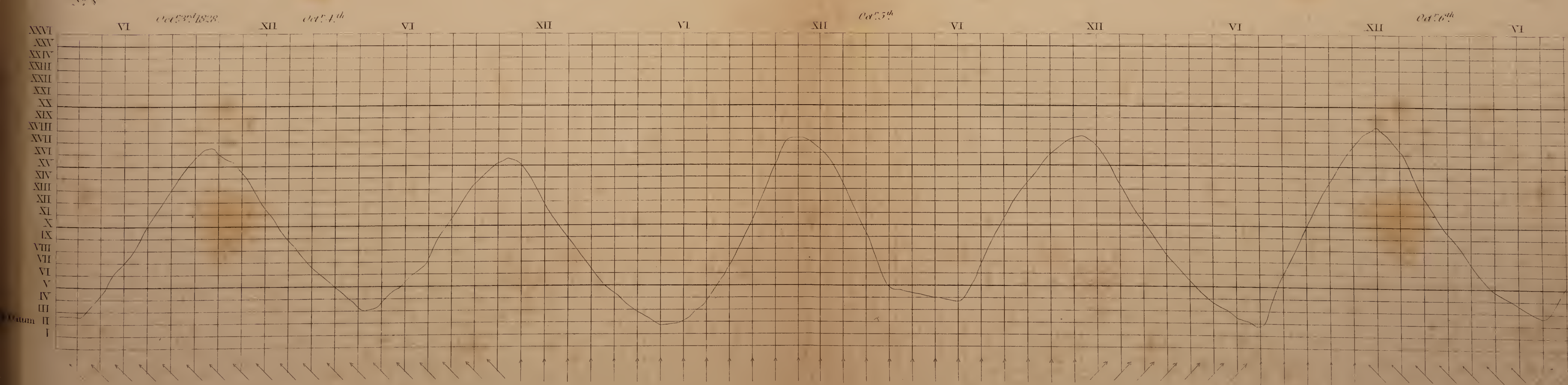


No III

Engd by F. Bradley

No. IV

No. V



Engd by T. Brewer

XII. *Description of a Graphical Registrer of Tides and Winds.* *By* HENRY R. PALMER, *Civil Engineer.* *Communicated by* JOHN WILLIAM LUBBOCK, *Esq. V.P. and Treas. R.S.*

Read March 10, 1831.

IN the exercise of my profession, particularly in reference to questions relating to harbours and tidal rivers, I have frequently had occasion for a train of observations on the rate of the rising and falling of the tides, continued through at least one whole series of them. Such observations have usually been made at intervals of fifteen minutes, and in order to exhibit the rates of rising and falling in the different periods of the same tide, and the differences of the several tides, I have represented them in the form of a chart.

The Plate No. IV. exhibits one of those charts which was made previously to the commencement of a new entrance to the London Docks. The use of it was to ascertain for what length of time there would be a given depth of water about the said entrance, and in the channel of the river near to it, during the different tides.

Having for some time past directed my attention to the nature of the effect that will be produced on the tides of the river Thames in the port of London by the removal of London Bridge, I have thought it very important that the changes should be progressively marked as they occur, that the ultimate consequences may with more ease and certainty be demonstrated.

In order to continue a series of constant, and, as far as possible, of unerring observations through so great a length of time as must necessarily elapse during the progress of these changes, I have realised an idea which I have for many years entertained, and for which the works now carrying on at the London Docks, under my direction, afford me an excellent opportunity of carrying into effect, viz. To construct a machine which being acted upon jointly by a timepiece, and a float resting on the water, shall represent every tide in succession

by such lines as those in the chart before referred to; while by connecting a type in the machine, with a weather-cock, above the house in which it is placed, an hourly register of the winds would also be obtained.

The performance of such a machine must if well arranged be evidently free from those inaccuracies and doubts which the frequent and long-continued observations of individuals, through nights as well as days, must be liable to. It will require only the occasional attention of a superintendant to correct the time, and supply it with paper.

The following is a description of the first that has been made for me, and which will shortly be placed at the mouth of the new entrance to the London Docks.

Reference to the Plates.

No. 1. A perspective view of the machine.

No. 2. An elevation and side view.

No. 3. The well and float, with the relative situation of the machine.

No. 4. A chart of tides in the river Thames observed in 1828.

No. 5. The same tides represented according to the form to be effected by the machine.

A number of parallel and equidistant lines, representing feet in height, are engraved, and printed on dry paper, whose sides are carefully cut parallel, and the ends joined until such a length is formed as to serve for three or four weeks consumption.

The long sheet thus produced is wound upon a brass roller, which is placed near the lower part of a cylinder one foot in diameter, so that the paper may pass round that cylinder, and be in contact with it through about three fourths of its circumference.

The contact of the paper is preserved by a roller pressing upon it by its own weight near the upper part of the cylinder.

On the axis of the cylinder is a toothed wheel, which is to be acted upon by a clock, and hence follows the motion of the paper in the direction of its length, moving equal distances in equal times. By means of the same toothed wheel, motion is given to a Cam Wheel having six teeth, and the velocity so regulated that it makes one revolution in six hours. Each tooth in the Cam Wheel

raises, in its turn, a hammer, whose fall strikes an impression on the paper once in every hour, so that the spaces passed through are measured as they occur, and leave no error that might otherwise arise from the expansion or contraction of the materials. It is obvious that when the paper has been taken off the machine, vertical lines must be ruled across the paper through the hourly points made by the hammer, and any of them may be subdivided into less measures of time if required. Immediately over the axis of the cylinder, and parallel with it, is a rack, which carries a pencil, and is acted upon by a pinion, which receives its motion from a float resting upon the water; so that as the tide rises and falls, the pencil moves backwards and forwards at a proportional rate, although through diminished space, and thus by the combined motions of the clock and the tide, a line is produced on the paper which represents both.

The impression hourly struck by the hammer is the figure of an arrow with a cross in the centre, the point of intersection being that through which the hour lines are drawn. The figure is cut on the end of an upright punch, which is connected with a weather-cock on the top of the house in which the machine is placed, and hence the direction of the arrow will always correspond with that of the wind, which becomes hourly registered.

The marking point for the tidal line is of steel, which makes such an impression on the paper as will be easily traced with a camel's hair pencil when the paper is removed, as will also the impression made by the wind arrow. A pencil, usually called the metallic pencil, may be employed for the original line if preferred.

The float which rests on the water is a hollow plate-iron vessel, suspended by a chain which passes twice round a light cast-iron barrel, and then descends, having a counter weight attached to it. The chain is of such a length, that both ends of it are always resting on the ground, so that the weight of chain on each side of the barrel is always equal.

The float is placed in a well communicating with the river, and to prevent that undulating motion which would be produced by the motion of the surface of the river during high winds, the water is received into the well through a fine wire gauze.

The motion is communicated from the float in the well to the machine in

the house by a light horizontal shaft revolving under the surface of the ground, and communicating by bevil wheels with an upright shaft which acts immediately upon the pencil rack *.

The references to the details are given on the several Plates.

Considering how many interests may be affected by the expected change in the circumstances of the river Thames, by the removal of London Bridge, it appears important that a similar gauge should be established at some point *above* the bridge, so that by the corresponding charts of the two machines, the facts will be so clearly exhibited as to defy that contradictory evidence which interested persons might hereafter produce, as well as furnish valuable data upon which remedial expedients might be founded.

Since the completion of the machine above described, my attention has been directed by Mr. Lubbock to the application of similar means for observing accurately the times of high water, not only as a scientific inquiry upon the tides generally, but with a view to the construction of accurate tide tables. Perceiving the value of that gentleman's suggestion, I devoted immediate attention to the subject, and trust, from the progress I have made, that the object will be accomplished in a satisfactory manner.

To indicate sensibly the time of high water to within any small portion of time (such as a minute), required a representation on a much larger scale than that above described, and therefore a distinct cylinder became necessary. It not being required to register the absolute heights by this machine, I have determined on registering only the upper part of the tide, that is, that portion which is contained in thirty minutes, of which the highest point will be one. Instead of representing the upper part of the tide by a continuous line, it will indicate the relative place and the time of it by punctures, one of which will be struck in each minute; so that the real time of any one puncture being known, the observer has only to count the number of punctures from that whose time is known, to the highest on the scale, and that will denote the time sought.

To accomplish this, the instrument by which the marks are made on the paper will not move as the pencil does in the first machine, but, keeping their

* To avoid derangement of the rate of the clock's motion, a weight is to be attached to the registrer, equal in force to the resistance of the friction of those parts the clock has to move.

places, the cylinder with the paper will move, and thus by a very simple motion from the clock, the real time will be expressed once in the thirty minutes around one of the minute points.

To avoid the risk of inattention on the part of the individual to whom the operation of the machine may be intrusted, the machinery connected with the float will put the registering apparatus into gear a short time before the time of high water, and the clock will put it out of gear when the operation has been continued for thirty minutes.

To obtain the exact time of high water, or even to ascertain it with moderate precision, does not appear practicable but by means similar to those described. The motion of the surface of the water, occasioned by the wind, and the vibrating motion of the general mass upon the turn of the tide, are obvious reasons: but by having the float placed in a protected situation, and the varying height *marked in fact by the time*, we may expect to arrive at a sufficient accuracy for the purposes required.

When this machine has been completed, a full description of it shall be forwarded to the Royal Society, and I hope it will hereafter do me the honour to accept either the original or copies of the series of observations upon the tides of the Thames, which both machines may exhibit.

XIII. *On the Errors in the Course of Vessels, occasioned by Local Attraction; with some Remarks on the recent Loss of His Majesty's ship Thetis. By* Peter Barlow, *Esq. F.R.S. Cor. Mem. of the Inst. France, Imp. Acad. Sciences St. Petersburgh, Acad. Sciences Brussels, &c. &c.*

Read April 21, 1831.

On presenting the following remarks to the attention of the Royal Society, I feel it necessary, first, to apologize for their not possessing that degree of scientific novelty which is generally expected in such communications,—and secondly, that in enforcing my argument, I may perhaps seem to give more value to my own investigations than is consistent with good taste. I must, however, either in some measure do this, or leave the evil untouched, which I feel it my duty to endeavour to remove; and therefore, trusting to a liberal interpretation of my motives, I shall state without reserve such facts as appear necessary to establish the object I have in view, and it is hoped the want of novelty will be compensated by the importance of the subject under consideration.

That a ship's compass is subject to a deviation from its true direction in consequence of the attraction of the iron used in the construction and appointments of the vessel, is now too generally admitted to require any argument, although I believe there may be still some few officers who are sceptical on this point; I have at least been seriously assured by one of rank and long standing, "that there certainly was no local attraction when he was at sea." Now there is really more in this observation than one would imagine, for there can be no question that forty years back the error arising from this disturbing force was very inconsiderable to what it is at present; every year in fact increasing the amount, and rendering a correction of the error more and more necessary. This increase is occasioned by the immense quantity of iron now employed in the construction of a ship of war and its appointments. At the period above

alluded to, iron ballast and iron tanks were perhaps scarcely known; now we have, besides these, iron knees, iron cables, and above all iron capstans, besides various other articles of the same material, which together form such an attracting mass, that if we cannot allow that there was no local attraction forty years back, we must at least admit that it was certainly then very inconsiderable to what it is at present, and that navigation by compass was at that time comparatively on equal terms with nautical astronomy; but since that period, the errors of the one have been gradually removed by improvements in instruments, the introduction of chronometers, and the correction of astronomical tables and data, whereas the compass still remains the same uncouth machine, and the disturbing forces to which it is exposed have been increased in, perhaps, a fourfold proportion. At all events the disturbing force is now considerable, and the deflection it causes in the needle, under some circumstances, very great; and as this effect is perpetually varying as the course of the vessel is changed, as it is also as she changes her latitude, though the course should remain the same, a constant attention to the amount of this error seems to be indispensable, at least in those circumstances where the whole safety of the vessel is dependent on the certainty of the courses steered, which is in fact always the case in a dark night, and when land is near.

It is almost impossible to give a very popular idea of the direction and amount of this deflecting force. It may however be stated, that in this latitude, and in all those northern latitudes where the dip is considerable, the greatest deflections take place, on an east or on a west course, diminishing both ways to the north and south, where it vanishes; and in all these cases the direction of the deflection is always to the right or left of a person, looking forward in the vessel, accordingly as the course of the vessel is to the right or left, that is to the east or west of the meridian, and it is exactly the reverse in a high southern latitude or with a considerable southern dip. But as we approach the equator, where the dip is small, the deflections at the east and west points vanish, and we have then four points of greatest attraction, viz. the N.E., N.W., S.E., and S.W., the direction of the deflection changing as we pass through the N., S., E., and West points; but the deflection at its maximum is much less in these latitudes than in those where the dip is more considerable, all other things being the same.

These rules, which are given upon the supposition that the compass is situated in its usual place, aft in the vessel, will furnish a general idea of the direction of the deflection. With regard to the actual amount, it is of course different in different vessels, varying in these latitudes from 5° to 12° or 14° with an easterly or westerly course; which become greater as we increase our latitude, but diminish (without however vanishing) at the equator, whence it again increases as we approach the southern pole.

The following are some results, several of which I have assisted in taking; they rest, as will be seen, on the best authorities, and will give a good general idea of the maximum amount.

Ship.	Commander.	Place.	Local Attraction. °	′
Conway	Captain Basil Hall	Portsmouth	4	32
Leven	Captain Owen	Northfleet	6	7
Baracouter	Captain Cutfield	Northfleet	14	30
Hecla	Captain Sir E. Parry	Northfleet	7	27
Fury	Captain Hopner	Northfleet	6	22
Griper	Captain Clavering	Nore	13	36
Adventure	Captain King	Plymouth	7	48
Gloucester	Captain Stuart	Channel	9	30

Giving a mean of 8° 44′ at the east and west points in these latitudes.

The latter observation by Captain Stuart is from the remark-book of the Gloucester, and from which I beg to give the following extract:

"1830. 30th August. From not having had a favourable opportunity of ascertaining the ship's magnetism or local attraction in steering down the British Channel, I only allowed the true variation as found, but observed the ship was invariably drawn to the southward of her intended place, notwithstanding the greatest care being taken in steering her. But on taking an amplitude of the sun at setting on the 1st September, I found the variation to be 34° W. (when the ship's head was west), which difference from the true variation in the Channel 24° 30′ W. will account for the ship being so drawn to the southward of her intended track."

It would be quite superfluous to give further evidence of the existence of

this disturbing power; and little, I conceive, need be said to show how much such deviations from the estimated course of a vessel, in channels and narrow seas, are calculated to lead to the most disastrous events. To take the last case, for example, where the deviation is 9° 30′, and for the deviation in miles the general expression (dist. $\times$ 2 sin $\frac{1}{2}$ deviation), we shall find that after running ten miles, the vessel would be more than a mile and a half to the southward of her reckoning; in a distance of twenty miles, three miles and a quarter to the southward; in thirty miles, five miles to the southward, and so on as the distance increases.

Now it requires no knowledge of navigation to estimate the fatal consequences that might attend such an error in a narrow channel and in a dark night, if it were wholly unknown or disregarded. We see also how very easy it is, after an accident has occurred, to imagine a current (unknown of course to exist before), to account for the disaster. The Gloucester, for example, in the above instance was constantly "drawn to the southward;" and this might have been set down to the effect of a current, had it not been proved to be local attraction.

That a ship is sometimes involved in an unknown or unusual current, which may lead her into an error of reckoning, no one can for a moment deny; but I do at the same time maintain, that unless a proper attention be paid to the local attraction, a vessel is as it were in a perpetual current, setting sometimes in one direction, and sometimes in another, sufficient to baffle the most experienced pilot; and I further maintain, that science and humanity both require, before we admit the plea of unknown currents to explain the cause of every disaster, that it be sufficiently ascertained how far allowance has been made, or a correction obtained for that current, which it is now well known a vessel carries with her through every league of her voyage.

Let us now turn to the late melancholy wreck of His Majesty's ship Thetis. It appears from the account given of this disaster in the United Service Journal, that "the Thetis sailed from Rio Janeiro on the 4th of December, with a million of dollars on board, besides other treasure, and every prospect of a fine passage, stretching away to the S.E. The next day, the wind coming rather favourable, they tacked, thinking themselves clear of land; and so confident were they, that the top-mast studding sails were ordered to be set, the ship run-

ning at the rate of nine knots; and the first intimation they had of being near land, was the jib-boom striking against a high perpendicular cliff, when the bowsprit broke short off, the shock sending all three masts over the side;" and thus in a moment perished twenty-five valuable lives, and a fine vessel, with her cargo, worth nearly a quarter of a million sterling.

Here then we have a case of a ship leaving port one day, with every prospect of a fine passage, which had so far lost her reckoning on the evening of the next day, as to be wrecked on a rock not more than seventy miles from her point of departure, which was supposed to be some miles to her west.

I have no desire to prejudge the cause of this unfortunate misreckoning; I wish only the true cause should be ascertained. In the letter of the commander of the Thetis to the admiral on the station, he says, that "from all the precautionary measures taken, nothing but the strongest currents, and the thick hazy weather, and hard rain, can be pleaded in extenuation."

I most sincerely hope that amongst "the precautionary measures taken," that of correcting or making proper allowance for the local attraction was included; for I have no hesitation in asserting, if such precaution was not taken, that this omission would be quite sufficient to account for the accident. It is obvious from the general principles which I have stated in the preceding part of this paper, that at Rio, where the dip of the needle is about 22° south, (the course steered on the 4th of December having been S.E., and on the 5th of December necessarily somewhere between the east and north,) the local attraction would be constantly drawing the vessel over to the westward, and there can be no question that her being more to the westward than her reckoning, was, from whatever source it might have proceeded, the cause of the disastrous event.

Without therefore in any way prejudging the case, I have only to express a hope that some inquiry may be made, to ascertain whether any and what allowance or correction was made for the local attraction of the vessel.

It is impossible now, if the local attraction of the Thetis has not been before taken, to know its amount; but we can ascertain, at least approximatively, what would have been the deflection of the Gloucester under similar circumstances; and as the amount of attraction in this vessel is nearly the mean of all those I have given in a former page, it may be interesting to see the result.

It has been stated that the maximum attraction or deflection is much less (all other things being the same) with a small dip than with a large one; the proportion being, "that the tangents of the angles of deflections are inversely as the cosines of the dip:" now, the dip at Rio Janeiro being about 22°, and in London $69\frac{1}{2}°$, we have

$$\cos 22° : \cos 69\tfrac{1}{2}° : : \tan 9° 30' : \tan 3° 33'.$$

That is, the Gloucester leaving Rio under similar circumstances to the Thetis, would on any course about the S.E. or N.E. (the former being stated as the course of the latter vessel on the 4th, and the other her probable course after tacking on the 5th of December), be constantly deflected about $3\frac{1}{2}°$* out of her supposed course, and as the sine of 3° 33′ is about $\frac{1}{16}$th part of the radius, it is obvious, taking only the error due to the 5th of December, and reckoning the distance run at eighty miles, that the ship would pass five miles nearer to Cape Frio than her reckoning, an error quite sufficient to account for the fatal catastrophe which has occurred to the Thetis; for it appears that a distance of only so many fathoms would have nearly carried her clear of the land. I do not include the error due to the first day, because its tendency would only be to carry the vessel to the southward about the same quantity, which of itself could have produced no evil.

After all, let it be remembered that this is a supposititious case, and that my object in stating it is merely to show what might happen if the deflection from local attraction were disregarded, and thereby to prove the propriety and necessity of ascertaining whether in the case of the Thetis, and in all similar cases, the proper correction was made, before the apology of currents can be admitted.

I urge this the more particularly, because I fear this source of error is too much disregarded, and as I think it probable that in the several investigations which have been held to inquire into cases of vessels lost in a similar way for the last ten years, since I have been interested on this subject, no question has been asked whether or not the error of the compass had been corrected; and thus vessel after vessel is lost, and current after current is imagined to ac-

* I say about $3\frac{1}{2}°$, because much depends upon the direction of the centre of attraction in the vessel.

count for the loss, while an actually existing and known cause is allowed to remain uncorrected and disregarded.

That the remedy for this evil, which I have been so fortunate as to discover, is simple and universal, is, I believe, generally admitted; indeed, after being submitted to trial by two of our most scientific officers from 57° south latitude to 80° north latitude, and having been found to be effective to the most extreme point, it is impossible that any doubt should remain on that head.

It must be, therefore, that the error itself is disregarded, and it would consequently be rendering an essential service to the navy, when any loss is sustained from neglecting this necessary precaution, that it should be traced to its proper source; and it is with this view that I have drawn together these few remarks. If I overrate the importance of the error or the value of the remedy, my apology must be the opinions which have been given on the subject by many distinguished naval commanders, both English and foreign, and the high marks of approbation with which my investigations have been acknowledged by the Royal Society, and other learned societies of Europe. To which I may also add my anxiety, that where science can be brought to facilitate the progress of navigation, and to contribute to its security, it may not be allowed to be neglected in the British navy.

XIV. *On the Meteorological Observations made at the Apartments of the Royal Society during the Years* 1827, 1828, *and* 1829. *By* J. W. Lubbock, *Esq. V.P. & Treas. R.S.*

Read April 14, 1831.

THE phenomena which principally deserve attention connected with the science of meteorology, are:

1. The annual and diurnal variations of the barometer and thermometer, due to the action of the sun.

2. The variations of the barometer due to the moon, and dependent on her age.

3. The comparative temperature and barometrical pressure at different points of the earth's surface, the isothermal lines, and lines of equal barometrical pressure.

4. The influence of the direction of the wind on the temperature and barometrical pressure.

5. Phenomena connected with the electrical state of the air, the aurora borealis, &c.

In order to determine the annual variations of the barometer, I have taken the mean of the observations in each month, made at the apartments of the Royal Society, during the years 1827, 1828, 1829, and 1830. The results are given in the following Table, which shows the differences from the mean*.

The two first columns result from these observations, reduced to 32° Fahr., and corrected for capillarity.

The four other columns are deduced from Table 3 in the valuable work of M. Bouvard "Sur les Observations Méteorologiques." (Mémoires de l'Academie des Sciences. Vol. vii. p. 312.)

* The mean result being given for each year separately in the Philosophical Transactions, of course it was only necessary for me to add these together, and take the fourth. Since the reading of the paper, the observations of 1830 have been added and taken into account.

	Obs^ns at Somerset House.		Obs^ns at the Paris Observatory.			
	9 A.M.	3 P.M.	9 A.M.	12.	3 P.M.	9 P.M.
January	+.006	+.005	+.085	+.067	+.072	+.069
February ..	+.064	+.070	+.071	+.070	+.065	+.063
March	—.004	—.009	—.006	—.004	—.007	—.005
April	—.044	—.143	—.044	—.047	—.053	—.042
May	+.002	—.025	—.044	—.043	—.047	—.046
June	+.006	—.031	+.037	+.040	+.040	+.064
July	—.017	—.022	+.008	+.004	+.009	+.008
August	—.005	—.001	+.018	+.016	+.014	+.013
September ..	—.039	—.048	+.016	+.014	+.015	+.019
October	+.117	+.116	—.063	—.060	—.062	—.056
November ..	+.036	+.025	—.021	—.015	—.012	—.011
December ..	—.006	+.005	—.047	—.042	—.035	—.039
Mean	29.861	29.840	29.778	29.767	29.748	29.762

Thus the mean height of the barometer at 9 A.M. for January is 29.861 + .006 = 29.867.

It may be remarked, that according to this Table, the annual variations appear to be independent of the diurnal variations. The Paris observations present much greater regularity than those made here, which results perhaps from their greater number. In order to determine the diurnal variations of the barometer, it is necessary that the observations should be repeated much more frequently in the course of the day than is done here at present. The mean height of the barometer here at 9 in the morning is greater by .021 inch (or about $\frac{1}{50}$th of an inch), than at 3 in the afternoon; and so regular is this diurnal variation, that considering the mean of each month separately for the years 1827, 1828, 1829, and 1830, there are only two cases in which the mean height is greater at 3 in the afternoon than at 9 in the morning. The corresponding difference at Paris is .030 inch *.

In order to determine the fluctuations of the barometer due to the moon, it would have been desirable to possess many more observations; but, unfortunately, previous to 1827, the observations of the barometer at Somerset House seem not to have been made at stated times of the day, a condition which appears to me absolutely necessary, in order that meteorological observations

* There is a very interesting paper on the annual and diurnal variations of the barometer, by M. CARLINI, in the 20th volume of the Memorie della Società Italiana. Fasc. 1mo. (Memorie di Matematica.)

may be applied to this or any other useful purpose, except that of serving at the time to prognosticate the weather, or but imperfectly to determine the correction due to the direction of the wind.

I was therefore obliged to confine myself to the years 1827, 1828, and 1829. The method which I have adopted with respect to these aerial tides is similar to that which I have used in order to determine the phenomena of the tides in the river Thames, and consists in classifying all the heights of the barometer, and taking their mean, which correspond to a particular age of the moon, defined by the circumstance of her transit taking place in a given half-hour of the day. Thus all the days in the years 1827, 1828, and 1829, were found when the moon passed the meridian between 12 and half past 12, and the mean of the transits taken, which of course is nearly a quarter past 12; the heights of the barometer were then taken on the same days, and the mean taken; and thus all the transits of the moon which occurred during the years 1827, 1828, and 1829, were taken, and the corresponding observed heights of the barometer selected and compared with them. The height of the attached thermometer was also taken, and the mean height of the barometer corrected afterwards by the mean height of the attached thermometer, so as to reduce it to 32° Fahr.

Although the transits of the moon were at first classed for every half-hour, I afterwards combined them for every hour, in order to make use of a greater number of observations in obtaining results. The mean transit thus found, scarcely differed from the half-hour, which is therefore taken as the time of the moon's transit in the following Table, in which the results are exhibited.

Table.

No. of Obs.	Time of Moon's transit.	Moon's age.	9 o'clock A.M. Barom.	9 o'clock A.M. Attach. Therm.	9 o'clock A.M. Barom. red. to 32° Fahr.	9 o'clock A.M. Diff. of Barom. from mean.	3 o'clock P.M. Barom.	3 o'clock P.M. Attach. Therm.	3 o'clock P.M. Barom. red. to 32° Fahr.	3 o'clock P.M. Diff. of Barom. from mean.
	h. m.	days.	inches.		inches.	inch.	inches.		inches.	inch.
85	0 30 or 12 30	.6 or 15.3	29.983	54.3	29.926	+.063	29.956	55.9	29.894	+.055
87	1 30 or 13 30	1.8 or 16.6	29.951	55.5	29.891	+.028	29.913	57.4	29.859	+.010
90	2 30 or 14 30	3.0 or 17.8	29.916	54.0	29.860	—.003	29.898	56.0	29.836	—.003
89	3 30 or 15 30	4.3 or 19.1	29.892	53.8	29.836	—.027	29.883	55.5	29.823	—.016
88	4 30 or 16 30	5.5 or 20.3	29.904	53.3	29.849	—.014	29.891	54.8	29.832	—.007
92	5 30 or 17 30	6.8 or 21.5	9.896	53.8	29.840	—.023	29.892	56.1	29.830	—.009
86	6 30 or 18 30	7.9 or 22.7	29.924	54.0	29.868	+.005	29.897	56.1	29.835	—.004
93	7 30 or 19 30	10.4 or 25.2	29.899	53.9	29.843	—.020	29.884	55.7	29.823	—.016
87	8 30 or 20 30	10.4 or 25.2	29.890	53.9	29.834	—.029	29.870	55.6	29.809	—.030
86	9 30 or 21 30	11.7 or 26.4	29.910	53.6	29.855	—.008	29.901	55.8	29.839	.000
88	10 30 or 22 30	12.9 or 27.7	29.911	52.9	29.857	—.006	29.913	55.2	29.854	+.015
88	11 30 or 23 30	14.2 or 28.9	29.938	53.1	29.874	+.011	29.919	55.2	29.860	+.021
	Mean		29.919	53.9	29.863		29.901	55.8	29.839	

The following Table results from Table VI. of M. Bouvard, (p. 316,) reduced to English feet.

	Variation of Barometer from mean.	
	9 A.M. inch.	3 P.M. inch.
Day of the Syzygy	—.004	—.008
First day after the Syzygy	—.010	—.006
Second day after the Syzygy	—.013	—.009
Second day before Quadrature	+.008	+.005
First day before Quadrature	+.024	+.032
Day of the Quadrature	+.025	+.017
First day after the Quadrature	—.002	—.001
Second day after the Quadrature	—.000	+.012
Second day before the Syzygy	—.019	—.018
First day before the Syzygy	—.009	—.019
Mean height	29·781	29·748

The results afforded by the observations at Somerset House differ widely from those above obtained by M. Bouvard from the observations at the Paris Observatory; according to the former, the barometer is highest at new and full moon and lowest at quadrature; according to the latter, the contrary is the case.

The extent of the fluctuations of the barometer due to the moon according to the former is about .08 or nearly $\frac{1}{10}$th of an inch, according to the latter only ·05 or $\frac{1}{20}$th of an inch.

They agree in this, that the fluctuations take place nearly in the same manner in the morning and in the afternoon; whence it follows, that the period of the principal inequality of the height of the barometer due to the action of the moon is not the same as that of the ocean; for if it were so, as the observations are made at a distance of six hours, the maximum in the morning would correspond to the minimum in the afternoon.

Laplace enumerates among the most important causes of the fluctuation of the pressure of the atmosphere, the rising and falling of the ocean due to the action of the sun and moon, the ocean serving as the basis or support of the atmosphere. But with that deference which is due to the authority of so great

a mathematician, I must confess that this cause does not appear to me adequate to produce any sensible effect; for in the open sea the variations of the height of the water due to the tides, where this cause would be most felt, do not exceed three or four feet, and any considerable rise of the tide is in general confined within very narrow limits, as in channels and between the banks of rivers. Lastly, I have endeavoured to ascertain how far the barometer is affected by the direction of the wind; and the following Table gives the results which I have obtained with this view. The fluctuation due to this is much greater than that due to any other cause; and it is therefore very important that this correction should be carefully ascertained, in order that it may be applied when observations of the barometer are classed, in order to determine any other inequality. The barometer is lowest, as might be expected, in the rainy quarters, as S.W. and W.S.W.

Table showing results deduced from the Meteorological Observations made at Somerset House during the years 1827, 1828, and 1829, classed according to the direction of the Wind.

Direction of the Wind at 9 A.M.	No. of Observations.	9 o'clock A.M.		3 o'clock P.M.		Dew point at 9 A.M. in degrees of Fahr.	Rain in inches, read off at 9 A.M.
		Barometer.	Attach. Therm.	Barometer.	Attach. Therm.		
N	117	30.009	47.4	30.014	49.3	41	.019
NNE	88	30.068	52.2	30.043	54.0	45	.032
NE	35	29.923	49.3	29.912	51.1	42	
ENE	16	30.005	49.4	29.941	52.8	45	.100
E	74	29.915	53.2	29.889	55.8	47	.021
ESE	68	29.951	52.5	29.916	54.2	45	.022
SE	30	29.793	57.8	29.772	59.2	52	.053
SSE	39	29.742	55.8	29.680	57.9	51	.052
S	73	29.808	54.4	29.795	56.4	49	.033
SSW	88	29.815	56.7	29.782	58.5	51	.038
SW	103	29.884	54.8	29.835	56.6	48	.069
WSW	143	29.900	54.5	29.898	56.7	48	.059
W	83	29.944	57.0	29.936	59.1	49	.059
WNW	18	29.889	57.2	29.887	60.0	50	.053
NW	65	29.936	55.6	29.802	57.4	46	.054
NNW	56	29.978	54.8	30.014	56.8	46	.080
Mean of Total		29.918	54.7	29.893	55.7	47	.044

TABLE deduced from the preceding, showing the variations of the Barometer reduced to 32° FAHR.

Direction of the Wind.	No. of Observations.	9 A.M.	3 P.M.
N	117	+.114	+.140
NNE	88	+.158	+.155
NE	35	+.021	+.032
ENE	16	+.100	+.056
E	74	—.002	—.005
ESE	68	+.038	+.029
SE	30	—.128	—.128
SSE	39	—.178	—.218
S	73	—.107	—.098
SSW	88	—.106	—.113
SW	103	—.034	—.056
WSW	143	—.015	+.003
W	83	+.023	+.036
WNW	18	—.032	+.013
NW	65	+.013	—.093
NNW	56	+.018	+.118
Mean		29.863	29.835

I shall not attempt to enter into any discussion of the influence of electrical phenomena upon the weather; no observations with reference to this part of the subject have yet been made here.

I have to acknowledge the very kind assistance of Mr. DEACON, (to whom I have been indebted before,) in forming the Tables which accompany this paper. I have not discussed the circumstances under which the observations have been made which serve for the foundation of the results which are here presented, although I fear that the instruments employed are unworthy of the Society and of the care bestowed upon the observations by Mr. HUDSON. This discussion would have been necessary if my object had been to determine the mean temperature or the mean barometrical pressure at London; but as I have only endeavoured to ascertain the fluctuations of the barometer due to certain causes, whose periods are independent of any errors that may arise from the construction and condition of the instruments, those errors are of little importance in the preceding investigation.

Since this paper was read, Mr. Hudson has made some observations with a view to determine the diurnal variation of the barometer; they were begun on the 26th of April, and have been continued to the present time, June 13th. The results are exhibited in the following Table, which seem to indicate a minimum about 6 o'clock P.M.

Mean Times of Observation.	Number of Observations.	Barometer.	Attached Thermometer.	Barometer corrected, and reduced to 32°.	Difference of Barometer from Mean.
A.M. h. m.					
9 0	49	29.900	61.4	29.821	+.025
10 4	43	29.885	62.8	29.803	+.008
11 2	47	29.895	62.4	29.814	+.019
12 3	38	29.902	63.3	29.819	+.024
P.M.					
1 4	43	29.866	63.5	29.783	—.012
2 3	44	29.887	63.8	29.803	+.008
3 1	49	29.880	63.4	29.797	+.002
4 3	40	29.868	64.1	29.783	—.012
5 2	44	29.856	63.5	29.773	—.022
6 4	39	29.837	63.6	29.753	—.042
7 3	37	29.877	63.2	29.794	—.001
8 4	36	29.858	61.5	29.779	—.016
9 3	38	29.863	61.0	29.785	—.010
10 2	33	29.896	60.8	29.819	+.024
11 2	34	29.884	60.9	29.806	+.010
11 57	34	29.874	60.4	29.790	—.005
Mean		29.877	62.6	29.795	

METEOROLOGICAL JOURNAL,

KEPT BY THE ASSISTANT SECRETARY

AT THE APARTMENTS OF THE

ROYAL SOCIETY,

BY ORDER OF

THE PRESIDENT AND COUNCIL.

MDCCCXXXI.

METEOROLOGICAL JOURNAL FOR JULY, 1830.

1830. July.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	External Thermometer. Fahrenheit. 3 P.M.	External Thermometer. Self-registering. Lowest.	External Thermometer. Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
♃ 1	29.943	71.8	29.809	71.4	60	68.3	66.6	53.7	69.7		ESE	Fine—cloudy—light brisk wind.
♀ 2	29.728	69.0	29.656	69.2	61	65.4	67.4	58.8	67.6	0.333	SSE	Fine—light clouds—showery—light brisk wind.
♄ 3	29.484	70.4	29.580	68.3	54	62.5	66.0	56.3	68.7	0.156	NNE	Fair—cloudy—light wind. Thunder and lightning with rain at 1 P. M.
☉ 4	29.788	66.7	29.820	69.7	58	63.3	71.8	55.7	72.7	0.181	W	Fine and clear—cloudy.
☽ 5	30.003	64.3	30.045	68.0	56	59.6	66.2	55.3	67.7		NNE	Overcast—light wind.
○ ♂ 6	30.029	70.7	29.930	70.0	57	66.8	60.8	57.3	69.3		SSW	Cloudy—brisk wind.
☿ 7	29.654	69.3	29.625	69.8	60	62.7	64.0	58.7	69.0	0.019	SSW	Cloudy—light showers.
♃ 8	29.733	73.7	29.670	68.2	46	62.3	65.0	53.7	67.3	0.036	NW var.	Fine and clear—light clouds and wind.
♀ 9	29.464	66.4	29.524	66.2	47	59.8	60.5	49.7	65.3		WNW	Fair—cloudy.
♄ 10	29.782	67.7	29.857	66.9	50	60.5	65.2	52.3	66.3	0.019	W	Lightly cloudy.
☉ 11	29.836	62.3	29.723	64.3	55	57.7	62.4	53.5	64.7		E	Cloudy—light showers.
☽ 12	29.583	65.7	29.805	67.2	61	62.7	64.8	57.7	66.4	0.056	W	Fine—cloudy—showery.
♂ 13	30.156	72.7	30.164	69.2	50	65.4	71.2	48.8	72.3	0.044	W	Fine and clear—light clouds.
☿ 14	30.120	67.6	30.060	68.8	58	68.5	73.5	54.7	74.5		SSE	Fine—nearly cloudless.
♃ 15	29.994	70.3	29.992	70.7	63	67.4	69.8	58.7	70.8		S	Fine—light clouds.
♀ 16	30.038	74.3	30.037	71.6	53	67.7	68.4	57.6	73.7		SSW	Fine—light clouds.
♄ 17	30.079	74.3	29.972	69.8	56	65.4	64.2	52.8	66.7		SSE var.	A. M. Clear. P. M. Overcast. At night, high wind and rain.
☉ 18	29.824	67.4	29.836	67.7	63	63.7	62.3	61.4	67.3	0.083	SSE	A. M. early, wind and rain.—Continued rain.
☽ 19	29.997	72.8	30.039	70.5	56	66.7	69.2	54.3	70.7	0.333	W	Fine—light clouds—showery.
● ♂ 20	30.143	69.7	30.133	70.5	56	64.7	64.0	55.7	69.0	0.033	SW	Lightly cloudy.
☿ 21	30.228	68.6	30.228	70.6	60	65.2	73.4	56.6	75.7		W	Fine—light clouds.
♃ 22	30.248	66.7	30.209	70.2	60	64.3	70.4	60.4	72.8		W	Overcast.
♀ 23	30.155	74.4	30.129	73.7	61	69.7	73.7	62.3	77.3		SW	Overcast. Fine A. M.
♄ 24	30.090	74.0	30.116	74.2	62	70.3	75.0	64.4	75.7		SW	Fine and clear—light clouds.
☉ 25	30.265	74.3	30.255	75.6	62	69.7	79.8	59.6	81.5		SW	Fine—lightly cloudy.
☽ 26	30.280	79.3	30.246	78.3	56	77.2	81.6	66.3	85.2		S	Fine—lightly cloudy.
♂ 27	30.373	80.3	30.344	78.8	60	76.7	81.8	66.6	83.7		E	Clear and cloudless.
☿ 28	30.362	77.7	30.313	77.6	67	74.5	80.4	60.3	82.6		N	Clear and cloudless.
♃ 29	30.197	78.3	30.117	79.6	68	75.7	80.8	64.3	83.3		NNE	Fine—lightly cloudy.
♀ 30	29.977	78.7	29.911	81.0	69	76.5	82.8	66.8	85.8	0.014	W	Fine and clear—lightly cloudy.
♄ 31	30.065	74.3	30.073	78.7	66	71.7	78.7	66.7	79.7		W	A. M. Overcast. P. M. Fine and clear.
	Mean 29.988	Mean 71.4	Mean 29.975	Mean 71.5	Mean 58	Mean 66.9	Mean 70.4	Mean 58.1	Mean 73.0	Sum 1.307		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. .... { 9 A.M. 29.890   3 P.M. 29.877 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge .... =83 feet 2½ in.

.... above the mean level of the Sea (presumed about)................ =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR AUGUST, 1830.

1830. August.	9 o'clock, A.M.		3 o'clock, P.M.		Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer.				Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
	Barom.	Attach. Therm.	Barom.	Attach. Therm.		Fahrenheit. 9 A.M.	Fahrenheit. 3 P.M.	Self-registering. Lowest.	Self-registering. Highest.			
☉ 1	30.026	74.6	29.951	75.1	55	71.6	75.4	60.6	76.3		SSW	Fine and clear—light breeze. Evening lowering.
☽ 2	29.844	71.8	29.862	74.3	59	67.2	71.7	58.7	72.2		SSW	A.M. Showery. P.M. Fine and clear—light clouds.
♂ 3	30.023	72.4	30.052	72.7	55	67.3	70.5	55.5	71.3	0.281	W	Fine and clear—cloudy. Evening lowering.
○ ☿ 4	30.071	67.8	29.982	73.4	61	64.4	73.7	58.7	75.7		ESE	A.M. Cloudy. P.M. Fine and clear—light clouds.
♃ 5	29.918	71.7	29.963	73.3	58	67.3	72.7	64.3	73.7		W	Fine—lightly cloudy.
♀ 6	30.032	69.7	30.035	70.7	51	63.7	68.3	53.3	68.7		WSW	Clear—cloudy—light wind.
♄ 7	29.948	67.3	29.870	70.3	48	64.6	68.7	55.7	70.5		SSW	Fine—lightly cloudy.
☉ 8	29.723	64.8	29.737	69.8	56	62.7	69.8	57.6	70.7		ESE	Fine—lightly cloudy.
☽ 9	29.842	67.7	29.788	70.8	68	64.7	69.8	56.3	72.3		ESE	Overcast.
♂ 10	29.664	66.4	29.639	69.0	61	62.7	65.2	57.6	66.3		ENE	Overcast.
☿ 11	29.729	65.5	29.714	70.3	60	61.7	71.8	57.8	73.4		SSW	A.M. Hazy. P.M. Fine—at night, rain.
♃ 12	29.770	70.7	29.802	71.1	57	65.8	69.8	57.3	70.7	0.072	SSW	Fine and clear—light clouds and wind.
♀ 13	29.700	66.1	29.687	66.8	60	60.3	63.8	58.6	63.5	0.028	WSW	A.M. Rain. P.M. Fine. Thunder at 3½h.
♄ 14	29.892	66.3	29.734	66.8	55	61.3	60.0	53.3	66.7	0.736	WSW	A.M. Fair. P.M. Heavy rain.
☉ 15	29.760	65.4	29.808	68.7	54	59.7	66.7	52.3	69.3	0.597	WNW	Fine and clear—cloudy.
☽ 16	29.957	65.3	29.942	66.7	51	59.3	64.3	48.7	67.7		W	Fine—light haze and clouds.
♂ 17	29.978	63.3	29.955	65.6	49	56.7	61.6	46.8	63.8	0.038	W	Fine and clear—lightly cloudy. Evening, rain.
● ☿ 18	30.155	62.7	30.185	64.3	47	56.3	62.6	46.8	63.6	0.169	N	Fine and clear—light clouds and wind.
♃ 19	30.175	59.7	30.118	63.9	47	56.7	62.3	47.7	65.2	0.014	NNW	Fine and clear—light clouds.
♀ 20	29.975	58.4	29.957	63.7	44	54.7	59.3	47.6	62.0		NNW	Fine—lightly cloudy.
♄ 21	30.014	60.5	29.986	64.0	50	56.0	63.5	50.0	65.5		NNE	A.M. Overcast. P.M. Fine.
☉ 22	30.031	62.0	30.024	64.3	50	58.7	66.7	47.0	68.5		S	Fine—light clouds. At night, light rain.
☽ 23	29.994	64.0	29.958	67.2	57	62.0	65.8	58.0	68.0		NW	Overcast.
♂ 24	29.883	61.4	29.888	62.0	56	61.4	68.8	56.6	70.8		SW	Fine—light clouds. At night, rain.
☿ 25	29.807	65.2	29.770	63.0	56	62.8	65.0	56.6	69.8		WSW	Overcast—light wind—showery.
♃ 26	29.849	64.2	29.873	66.9	55	61.0	64.6	55.8	65.5		SW	Fine and clear—light wind.
♀ 27	29.782	64.6	29.625	67.8	58	60.5	66.0	54.0	69.0		S	A.M. Clear. P.M. Overcast. At night, rain.
♄ 28	29.394	64.4	29.457	66.5	53	58.6	63.5	54.8	64.6	0.250	S	A.M. Overcast—showery. P.M. Clear. Heavy rain at 1h.
☉ 29	29.931	64.5	29.996	66.2	50	59.0	62.8	49.8	65.6	0.500	WSW	A.M. Clear. P.M. Rain. Thunder with rain at 1h. 45m.
☽ 30	30.223	61.5	30.192	64.6	44	50.5	63.4	46.6	64.4		WSW	Fine—light haze.
♂ 31	30.291	59.8	30.261	64.2	48	55.0	66.0	46.0	67.0		S	Fine and clear.
	Mean 29.916	Mean 65.5	Mean 29.897	Mean 67.9	Mean 53	Mean 61.1	Mean 66.6	Mean 53.9	Mean 68.5	Sum 2.685		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. .... 9 A.M. 29.834 | 3 P.M. 29.808

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge .... =83 feet 2¼ in.

.... above the mean level of the Sea (presumed about) .... =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR SEPTEMBER, 1830.

1830. Septem.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	External Thermometer. Fahrenheit. 3 P.M.	External Thermometer. Self-registering. Lowest.	External Thermometer. Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
☿ 1	30.342	61.8	30.291	65.8	49	58.5	67.6	50.0	68.8	0.250	SW	Fine and clear.
○ ♃ 2	30.200	62.6	30.085	67.8	52	57.8	67.0	50.6	69.8		WSW	Fine—lightly overcast.
♀ 3	29.835	63.6	29.812	65.8	51	60.5	63.2	53.4	65.4		S var.	Fine—lightly overcast.
♄ 4	29.954	62.2	29.966	65.2	53	58.0	62.8	53.2	65.8		N	Fine and clear—light clouds.
☉ 5	29.704	61.2	29.697	65.2	55	57.4	64.6	54.2	65.2		WNW	Overcast—light rain A.M.
☽ 6	29.562	61.4	29.495	64.9	53	58.2	62.4	52.0	63.6		SW	Overcast. Thunder and lightning, with heavy rain at 4½ P.M.
♂ 7	29.748	61.4	29.779	63.6	53	57.6	59.8	54.0	62.5	0.250	W var.	Fair—lightly overcast.
☿ 8	30.004	61.0	30.079	64.8	53	57.8	63.0	53.6	64.2		SE var.	Fair—lightly overcast.
♃ 9	29.865	60.5	29.792	62.8	50	57.6	59.0	49.2	61.3		S	Overcast. At night, rain.
♀ 10	29.657	59.7	29.701	63.2	56	56.4	61.8	54.0	63.0	0.375	W	Overcast. A.M. early, rain.
♄ 11	29.777	57.5	29.756	61.8	47	53.6	60.2	45.8	61.4		SW	Fine—light clouds.
☉ 12	29.388	59.0	29.326	61.3	50	56.2	58.4	51.0	60.6		S	Fine—light clouds—showers.
☽ 13	29.489	58.2	29.548	61.4	49	54.0	60.0	47.5	62.0	0.250	SW	Fine—light clouds.
♂ 14	29.445	59.6	29.460	62.5	51	57.5	61.3	50.0	65.2		S	Fine—light clouds. A.M. early, rain.
☿ 15	29.663	59.2	29.657	62.2	49	57.5	62.0	49.2	63.4		SE	Fine—lightly overcast. At night, rain.
♃ 16	29.606	61.4	29.710	63.5	54	59.0	63.1	56.5	65.6		WSW	Fine and clear.
● ♀ 17	29.686	58.5	29.641	61.6	53	54.8	59.0	51.5	59.8		E	Overcast. Light rain A.M.
♄ 18	29.594	59.6	29.681	62.0	52	55.8	59.4	52.0	63.5	0.250	SW	Lightly overcast.
☉ 19	29.962	58.4	29.866	62.3	49	55.2	60.2	47.2	63.2		S	A.M. Fine and clear. P.M. Rain.
☽ 20	29.641	60.0	29.609	62.8	50	56.5	61.6	52.0	63.8		SW var.	Fine and clear.
♂ 21	29.200	57.8	29.251	61.3	52	53.0	56.0	51.0	58.2	0.250	ESE	A.M. Rain. P.M. Fine.
☿ 22	29.574	55.2	29.677	59.2	46	50.6	59.5	43.0	60.2	0.250	WSW	Fine and clear.
♃ 23	29.325	57.4	29.410	60.5	52	58.4	60.5	50.0	62.5		S var.	Fine—showery.
♀ 24	29.675	56.3	29.594	58.8	46	53.8	52.6	47.0	57.8		S var.	Nearly cloudless. P.M. Hail with high wind.
♄ 25	29.736	56.0	29.800	58.8	45	53.6	58.5	46.8	59.8	0.250	WSW	Fine and clear—light wind.
☉ 26	30.212	54.0	30.282	57.6	45	50.5	58.5	43.8	60.8		SW	Fine—light clouds.
☽ 27	30.420	54.8	30.380	58.4	47	53.4	59.6	46.5	61.2		SE	Fine—lightly overcast.
♂ 28	30.279	57.6	30.154	59.8	49	57.6	61.8	53.2	62.8		SW	Fair—lightly overcast.
☿ 29	30.037	56.2	30.025	59.0	51	53.3	56.8	50.0	58.2	0.250	S	Fair—lightly overcast.
♃ 30	30.109	53.8	30.016	57.6	47	48.8	55.8	44.0	56.2		SSW	Fair—lightly overcast.
	Mean 29.789	Mean 58.9	Mean 29.785	Mean 62.0	Mean 50	Mean 55.8	Mean 60.5	Mean 50.1	Mean 62.0	Sum 2.375		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. .... { 9 A.M. 29.724 | 3 P.M. 29.712 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge....................... =83 feet 2¼ in.

.... above the mean level of the Sea (presumed about) .... =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR OCTOBER, 1830.

1830. Oct.	9 o'clock, A.M.		3 o'clock, P.M.		Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer.				Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
						Fahrenheit.		Self-registering.				
	Barom.	Attach. Therm.	Barom.	Attach. Therm.		9 A.M.	3 P.M.	Lowest.	Highest.			
♀ 1	30.079	54.5	30.071	58.6	46	53.5	58.8	44.2	60.5		S var.	Lightly cloudy.
○ ♄ 2	30.119	57.2	30.049	59.5	49	57.6	59.8	53.4	62.0		SE	Lightly cloudy. Rain at night.
☉ 3	30.008	59.0	30.034	60.6	56	59.4	59.8	56.2	61.0		S	Overcast—light rain.
☽ 4	30.152	58.9	30.257	61.8	52	56.5	60.2	51.8	62.8		SSW	Fine—light haze.
♂ 5	30.439	55.0	30.411	59.0	45	50.9	57.6	43.8	58.6		S var.	Fine—light clouds.
☿ 6	30.404	55.0	30.344	56.6	47	51.3	53.4	45.2	54.6		WSW	Fair—hazy.
♃ 7	30.385	55.2	30.383	58.4	49	53.5	59.6	48.0	60.8		WSW	Fair—hazy.
♀ 8	30.495	55.6	30.470	57.8	50	52.8	60.2	48.3	61.5		W	Fair—light clouds.
♄ 9	30.538	56.3	30.534	59.3	46	54.8	57.6	52.0	58.2		NW	Overcast—hazy.
☉ 10	30.547	56.3	30.493	58.8	51	54.2	59.4	50.9	59.4		NNE	Lightly cloudy.
☽ 11	30.411	53.7	30.352	57.5	50	51.4	54.6	45.7	55.3		E	Fine—light haze.
♂ 12	30.366	54.7	30.317	56.8	49	50.7	55.4	49.3	61.2		NNE	Overcast—hazy.
☿ 13	30.404	54.0	30.374	58.1	47	51.2	57.2	46.8	59.5		E	Fine and clear.
♃ 14	30.340	53.3	30.289	56.0	44	49.4	53.8	46.0	55.2		E	Fine and clear.
♀ 15	30.273	50.2	30.190	53.8	38	46.3	53.5	40.2	53.5		ESE	Fine and clear.
● ♄ 16	30.279	49.1	30.277	53.0	41	42.6	53.0	39.0	53.5		NNE	A.M. Fog. P.M. Fine and clear.
☉ 17	30.374	50.2	30.348	53.0	42	47.8	53.4	41.5	53.2		ESE	A.M. Fog. P.M. Fair—hazy.
☽ 18	30.290	50.2	30.196	53.6	43	48.0	55.2	42.4	56.0		E	Fine and clear.
♂ 19	30.035	52.6	29.994	56.2	46	53.5	59.8	47.4	61.2		ESE	Fair—light clouds.
☿ 20	30.052	56.2	30.051	59.0	51	57.8	62.2	53.8	64.8		S	Fine and clear.
♃ 21	30.210	59.0	30.210	61.5	53	58.5	63.6	54.6	66.0		S	Fine—light clouds.
♀ 22	30.303	60.2	30.264	63.5	55	60.0	65.0	54.8	66.5		S	Fine—brisk wind.
♄ 23	30.418	59.8	30.467	61.2	50	55.2	57.6	53.8	58.6		W var.	Fine—lightly overcast.
☉ 24	30.487	57.3	30.352	59.5	47	51.6	63.2	49.5	57.0		SW	Cloudy. A.M. Thick haze.
☽ 25	30.083	57.9	29.920	59.8	52	55.8	59.5	51.2	59.6		S	Overcast. Rain A.M.
♂ 26	30.021	55.6	30.148	56.4	45	48.9	50.0	46.2	52.2	0.708	S	Fine—light clouds.
☿ 27	30.372	49.4	30.278	53.3	35	41.0	51.2	35.2	54.5		SW	Fair—hazy. Rain at night.
♃ 28	29.940	54.1	29.886	57.4	49	54.8	59.6	40.2	60.6	0.073	SW	Cloudy, Rain at night.
♀ 29	29.695	53.2	29.671	55.2	43	48.8	52.5	45.5	53.4	0.100	WSW	A.M. Fine and clear. P.M. Cloudy.
♄ 30	30.014	49.2	30.033	51.0	37	42.4	46.8	39.0	51.5		S var.	Fine—light clouds.
☉ 31	29.916	50.7	29.923	53.8	50	51.7	57.9	41.3	58.3		SW	A.M. Foggy. P.M. Nearly cloudless—light wind.
	Mean 30.240	Mean 54.6	Mean 30.212	Mean 57.4	Mean 47	Mean 52.0	Mean 57.1	Mean 47.0	Mean 58.4	Sum 0.881		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. .... 9 A.M. 30.186 3 P.M. 30.151

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge........................ =83 feet $2\frac{1}{2}$ in.

.... above the mean level of the Sea (presumed about) .... =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR NOVEMBER, 1830.

1830. Nov.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	Fahrenheit. 3 P.M.	Self-registering. Lowest.	Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
☽ 1	30.127	54.3	30.130	56.2	53	53.4	57.0	49.6	57.7		SW	Fine and clear.
♂ 2	30.141	54.0	30.085	56.8	50	51.7	56.5	49.3	57.3		WSW	Overcast.
☿ 3	29.927	54.8	29.843	56.6	52	52.4	55.5	50.7	55.7		S	Fine—light clouds.
♃ 4	29.856	55.3	29.831	56.8	50	49.7	55.6	48.3	55.6		SW	Fine—lightly cloudy. At night, strong wind.
♀ 5	29.961	54.4	29.873	57.0	51	50.8	54.6	47.3	55.7	0.047	SSW	Fine—lightly overcast. At night, strong wind.
♄ 6	29.570	56.7	29.397	58.4	55	55.7	58.4	50.5	58.5	0.008	SSE	Cloudy. High wind through the night.
☉ 7	29.136	56.5	29.246	56.7	51	51.5	51.6	50.8	51.3	0.208		A.M. Rain. P.M. Fine—light clouds.
☽ 8	29.654	49.6	29.732	52.5	45	45.4	49.2	41.7	49.2	0.047	W	Fine—lightly cloudy.
♂ 9	29.902	47.7	29.881	50.4	34	40.4	48.2	34.7	48.7		WSW	Cloudless—light haze.
☿ 10	29.775	50.2	29.681	53.3	46	49.6	53.2	39.3	53.6	0.014	SW	A.M. Cloudless. P.M. Light rain. At night, high brisk wind, with showers.
♃ 11	29.533	52.7	29.542	53.5	48	49.6	50.8	47.8	51.2	0.458	SW	A.M. Fine. P.M. Light rain.
♀ 12	....	......	30.055	51.0	...	......	48.3	......	......			Fine—light haze.
♄ 13	29.845	48.9	29.651	51.0	40	46.5	48.9	39.0	50.8		ESE	Fine—light clouds.
☉ 14	29.626	51.2	29.606	53.3	46	50.5	52.8	45.5	53.8		S	Fine—lightly overcast. At night, rain.
● ☽ 15	29.602	50.2	29.655	52.6	44	47.8	51.8	42.2	53.4	0.180	S var.	Fine—light clouds. At night, rain.
♂ 16	29.312	53.2	29.065	54.4	47	53.6	54.8	46.8	54.8		E var.	Rain—brisk wind.
☿ 17	29.483	50.6	29.550	52.5	44	45.5	48.8	41.8	51.6	0.033	S	Fine—light clouds. Hail storm at 2½h P.M., and rain at night.
♃ 18	29.784	47.8	29.875	47.8	39	40.5	45.2	38.5	44.6	0.250	SW	Strong fog.
♀ 19	30.206	46.8	30.210	48.0	35	36.8	42.4	34.6	42.2		SW	Fine—hazy.
♄ 20	30.006	46.4	29.932	48.5	37	43.6	47.2	35.5	47.4		SE	Overcast. Rain P.M.
☉ 21	29.936	47.2	29.893	51.2	41	45.0	50.8	40.2	51.8		S	Fine—lightly overcast. At night, rain.
☽ 22	29.751	50.2	29.889	51.2	45	48.7	48.2	44.0	51.5	0.094	WSW	Fine—light clouds.
♂ 23	30.152	47.5	30.212	49.0	40	42.0	46.8	40.0	47.2		SW	Fine—hazy.
☿ 24	30.418	44.6	30.422	45.1	34	37.8	40.6	35.8	40.4		W	Hazy.
♃ 25	30.386	41.8	30.358	43.8	32	36.0	40.5	33.0	40.5		S	Fine—hazy.
♀ 26	30.126	43.2	29.994	44.5	35	38.6	40.0	34.8	43.0		SE	Overcast.
♄ 27	29.750	42.3	29.671	44.0	35	37.4	39.8	34.8	41.0		SE	Fine—light clouds.
☉ 28	29.493	42.5	29.554	44.8	37	42.0	45.2	36.0	45.2	0·361	SE	Overcast. Light rain P.M.
☽ 29	29.787	44.8	29.835	45.2	40	42.8	43.6	41.2	43.6		ESE	Overcast. Light rain A.M.
○ ♂ 30	29.924	45.4	29.911	46.2	41	43.5	44.4	41.6	44.2		ESE	Overcast—light rain.
	Mean 29.833	Mean 49.3	Mean 29.819	Mean 51.1	Mean 43	Mean 45.8	Mean 49.0	Mean 41.9	Mean 49.7	Sum 1.700		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. .... 9 A.M. 29.793 | 3 P.M. 29.774

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge .... = 83 feet 2½ in.

.... above the mean level of the Sea (presumed about) = 95 feet.

The external Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR DECEMBER, 1830.

1830. Decem.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	External Thermometer. Fahrenheit. 3 P.M.	External Thermometer. Self-registering. Lowest.	External Thermometer. Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
☿ 1	30.029	45.3	30.010	45.2	38	42.6	42.0	41.0	42.2		ESE	Overcast—light fog.
♃ 2	29.867	44.0	29.740	43.5	36	39.8	37.5	38.3	39.4		ESE	Overcast.
♀ 3	29.602	41.8	29.600	42.4	36	37.8	39.6	35.5	39.2	0.047	ESE	Foggy. Light rain A.M.
♄ 4	29.752	41.8	29.776	42.4	33	38.8	40.0	37.0	39.6		SE	Lightly foggy.
⊙ 5	29.756	42.3	29.590	43.6	32	39.8	42.3	38.0	43.4		ESE	Fine—light clouds.
☽ 6	29.118	43.0	29.080	44.5	36	44.5	46.3	35.2	46.5		ESE	Overcast—light rain.
♂ 7	29.150	45.6	29.166	46.5	41	44.6	45.8	43.5	46.6		E	Overcast.
☿ 8	29.201	46.8	29.176	47.0	41	45.6	45.0	44.0	45.0		ESE	Overcast. Light rain P.M.
♃ 9	28.897	46.2	28.869	48.0	39	45.6	47.8	42.5	47.2	0.294	ESE	Overcast. Rain A.M.
♀ 10	29.035	47.4	29.119	47.9	40	43.0	43.3	41.8	43.3	0.186	SW	Light fog.
♄ 11	29.314	43.5	29.307	45.4	33	36.8	41.8	34.4	42.0		SW	Fine—lightly cloudy.
⊙ 12	29.404	41.0	29.635	42.7	32	33.5	39.2	31.0	39.2		SW	Fine—lightly cloudy.
☽ 13	30.190	37.2	30.277	38.8	26	31.0	35.2	28.8	35.0		S	Fine—light haze.
♂ 14	30.398	36.5	30.386	38.8	24	32.8	40.0	28.3	40.2		S	Overcast and foggy. Light rain P.M.
● ☿ 15	30.471	38.8	30.382	40.3	31	37.0	39.5	32.0	39.8		SW	Foggy.
♃ 16	30.361	39.2	30.368	39.9	32	35.5	35.2	34.0	36.2	0.136	E	Fine—lightly cloudy. Light rain A.M.
♀ 17	30.025	37.7	29.960	39.6	29	32.6	38.2	27.9	38.0		SW	Fine—light clouds. Snow A.M.
♄ 18	30.162	37.0	30.176	38.5	28	33.6	36.0	31.8	36.0	0.186	N var.	Fine—light clouds.
⊙ 19	30.186	37.2	29.971	38.7	29	33.0	37.5	30.0	42.2		SW	Overcast.
☽ 20	29.491	39.8	29.558	41.4	28	39.0	41.8	30.2	41.8		S var.	Fine—light clouds.
♂ 21	29.823	38.7	29.784	40.0	30	35.6	40.0	33.8	44.2		SW	Fine—hazy.
☿ 22	29.060	41.8	29.400	43.8	37	44.9	48.2	34.8	48.3		SW	Light rain.
♃ 23	29.427	40.1	29.459	40.0	31	34.8	33.6	32.0	34.2	0.033	SSW	Hazy. Snow early A.M.
♀ 24	29.439	32.7	29.314	32.4	13	20.4	23.7	18.6	23.7		NW	Fine—lightly cloudy.
♄ 25	29.284	28.6	29.275	29.7	16	18.4	24.5	15.8	24.5		WSW	Lightly overcast and hazy.
⊙ 26	29.292	25.8	29.260	29.7	19	22.6	30.0	16.8	30.0		W	Lightly cloudy and hazy.
☽ 27	29.167	29.7	29.165	31.7	28	28.5	31.3	21.7	32.2		E	Light clouds and haze.
♂ 28	29.041	32.4	29.134	33.3	31	33.0	34.2	27.3	34.2		E	A.M. Snow. P.M. Light rain.
○ ☿ 29	29.656	32.6	29.657	34.8	30	31.0	36.3	28.6	36.3		WSW	Fair—light clouds.
♃ 30	29.453	35.6	29.282	37.0	33	36.7	38.2	29.4	45.2		ESE	Overcast and foggy. Light rain P.M.
♀ 31	29.185	40.4	29.374	41.8	42	45.5	44.2	35.7	45.4		S var.	Lightly overcast and cloudy—brisk unsteady wind.
	Mean 29.588	Mean 39.0	Mean 29.589	Mean 40.3	Mean 31	Mean 36.0	Mean 38.7	Mean 32.2	Mean 39.4	Sum 0.882		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. .... 9 A.M. 29.574 — 3 P.M. 29.572

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge .... = 83 feet $2\frac{1}{4}$ in.

....above the mean level of the Sea (presumed about) .... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCXXXI.

PART II.

LONDON:

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MDCCCXXXI.

CONTENTS.

PHILOSOPHICAL TRANSACTIONS.

XV.—*Researches in Physical Astronomy. By* J. W. Lubbock, *Esq. V.P. and Treas. R.S.*

Read May 19, 1831.

On the Theory of the Moon.

THE method pursued by Clairaut in the solution of this important problem of Physical Astronomy, consists in the integration of the differential equations furnished by the principles of dynamics, upon the hypothesis that in the gravitation of the celestial bodies the force varies inversely as the square of the distance, and in which the true longitude of the moon is the independent variable; the time is thus obtained in terms of the true longitude, and by the reversion of series the longitude is afterwards obtained in terms of the time, which is necessary for the purpose of forming astronomical tables. But while on the one hand this method possesses the advantage, that the disturbing function can be developed with somewhat greater facility in terms of the true longitude of the moon than in terms of the mean longitude, yet on the other hand, the differential equations in which the true longitude is the independent variable are far more complicated than those in which the time is the independent variable. The latter equations are used in the planetary theory; so that the method of Clairaut has the additional inconvenience, that while the lunar theory is a particular case of the problem of the three bodies, one system of equations is used in this case, and another in the case of the planets.

The method of Clairaut has been adopted, however, by Mayer, by Laplace, and by M. Damoiseau. The last-mentioned author has arranged his results with remarkable clearness, so that any part of his processes may be easily verified by any one who does not shrink from this gigantic undertaking; and the immense labour which this method requires, when all sensible quantities

are retained, may be seen in his invaluable memoir. Mr. Brice Bronwin has recently communicated to the Society a lunar theory, in which the same method is adopted.

Having reflected much upon the difficulties of this problem, I am led to believe that the integration of the differential equations in which the time is the independent variable, is at least as easy as the method hitherto, I think, solely employed, and I now have the honour to submit to the Society a lunar theory founded upon this integration, which is in fact merely an extension of the equations given in my Researches in Physical Astronomy, already printed, by embracing those terms which, in consequence of the magnitude of the eccentricity of the moon's orbit, are sensible; and the suppression of those, on the other hand, which are insensible on account of the great distance of the sun, the disturbing body. By means of the Table which I have given (Table II.), the developments may all be effected at once with the greatest facility.

The first column contains the indices, which I have employed to distinguish the inequalities. The numbers in the second column are the indices affixed by M. Damoiseau, in the Mém. sur la Théor. de la Lune, p. 547. to the inequalities of longitude.

$t^* = nt - n_{\prime} t, \quad x = cnt - \varpi, \quad z = n_{\prime} t - \varpi_{\prime}, \quad y = gnt - \nu.$

0	..	0	21	45	$2t-3x$	42	73	$2t-3x-z$
1	30	$2t$†	22	46	$2t+3x$	43	..	$2t+3x+z$
2	1	x	23	21	$2x+z$	44	26	$3x-z$
3	31	$2t-x$‡	24	53	$2t-2x-z$	45	..	$2t-3x+z$
4	32	$2t+x$	25	54	$2t+2x+z$	46	..	$2t+3x-z$
5	16	z §	26	20	$2x-z$	47	..	$2x+2z$
6	33	$2t-z$	27	51	$2t-2x+z$	48	75	$2t-2x-2z$
7	34	$2t+z$	28	52	$2t+2x-z$	49	..	$2t+2x+2z$
8	2	$2x$	29	23	$x+2z$	50	..	$2x-2z$
9	35	$2t-2x$	30	59	$2t-x-2z$	51	..	$2t-2x+2z$
10	36	$2t+2x$	31	..	$2t+x+2z$	52	..	$2t+2x-2z$
11	19	$x+z$	32	22	$x-2z$	53	..	$x+3z$
12	41	$2t-x-z$	33	61	$2t-x+2z$	54	..	$2t-x-3z$
13	42	$2t+x+z$	34	60	$2t+x-2z$	55	..	$2t+x+3z$
14	18	$x-z$	35	..	$3z$	56	..	$x-3z$
15	39	$2t-x+z$	36	..	$2t-3z$	57	..	$2t-x+3z$
16	40	$2t+x-z$	37	..	$2t+3z$	58	..	$2t+x-3z$
17	17	$2z$	38	9	$4x$	59	..	$4z$
18	43	$2t-2z$	39	67	$2t-4x$	60	..	$2t-4z$
19	44	$2t+2z$	40	..	$2t+4x$	61	..	$2t+4z$
20	4	$3x$	41	27	$3x+z$	62	3	$2y$

* Inconvenience arises from using the letter t in this acceptation. I have done so in order to conform to the notation of M. Damoiseau. † Variation. ‡ Evection. § Annual Equation.

63	37	$2t-2y$	105	84	$t+z$	146	....	y
64	38	$2t+2y$	106	85	$t-2x$	147	....	$2t-y$
65	5	$x-2y$	107	86	$t+2x$	148	....	$2t+y$
66	6	$x+2y$	108	91	$t-x-z$	149	....	$x-y$
67	49	$2t-x-2y$	109	92	$t+x+z$	150	....	$x+y$
68	47	$2t-x+2y$	110	89	$t-x+z$	151	....	$2t-x-y$
69	48	$2t+x-2y$	111	....	$t+x-z$	152	....	$2t-x+y$
70	50	$2t+x+2y$	112	....	$t-2z$	153	....	$2t+x-y$
71	24	$z-2y$	113	....	$t+2z$	154	....	$2t+x+y$
72	25	$z+2y$	114	....	$t-2y$	155	....	$z-y$
73	57	$2t-z-2y$	115	....	$t+2y$	156	....	$z+y$
74	56	$2t-z+2y$	116	100	$3t$	157	....	$2t-z-y$
75	55	$2t+z-2y$	117	101	$3t-x$	158	....	$2t-z+y$
76	58	$2t+z+2y$	118	102	$3t+x$	159	....	$2t+z-y$
77	7	$2x-2y$	119	103	$3t-z$	160	....	$2t+z+y$
78	8	$2x+2y$	120	104	$3t+z$	161	....	$2x \quad y$
79	65	$2t-2x-2y$	121	....	$3t-2x$	162	....	$2x+y$
80	63	$2t-2x+2y$	122	....	$3t+2x$	163	....	$2t-2x-y$
81	64	$2t+2x-2y$	123	....	$3t-x-z$	164	....	$2t-2x+y$
82	..	$2t+2x+2y$	124	....	$3t+x+z$	165	....	$2t+2x-y$
83	..	$x+z-2y$	125	....	$3t-x+z$	166	....	$2t+2x+y$
84	..	$x+z+2y$	126	....	$3t+x-z$	167	....	$x+z-y$
85	..	$2t-x-z-2y$	127	....	$3t-2z$	168	....	$x+z+y$
86	..	$2t-x-z+2y$	128	....	$3t+2z$	169	....	$2t-x-z-y$
87	..	$2t+x+z-2y$	129	....	$3t-2y$	170	....	$2t-x-z+y$
88	..	$2t+x+z+2y$	130	....	$3t+2y$	171	....	$2t+x+z-y$
89	..	$x-z-2y$	131	120	$4t$	172	....	$2t+x+z+y$
90	..	$x-z+2y$	132	121	$4t-x$	173	....	$x-z-y$
91	..	$2t-x+z-2y$	133	122	$4t+x$	174	....	$x-z+y$
92	..	$2t-x+z+2y$	134	123	$4t-z$	175	....	$2t-x+z-y$
93	..	$2t+x-z-2y$	135	124	$4t+z$	176	....	$2t-x+z+y$
94	..	$2t+x-z+2y$	136	125	$4t-2x$	177	....	$2t+x-z-y$
95	..	$2z-2y$	137	126	$4t+2x$	178	....	$2t+x-z+y$
96	..	$2z+2y$	138	131	$4t-x-z$	179	....	$2z-y$
97	..	$2t-2z-2y$	139	....	$4t+x+z$	180	....	$2z+y$
98	..	$2t-2z+2y$	140	129	$4t-x+z$	181	....	$2t-2z-y$
99	..	$2t+2z-2y$	141	....	$4t+x-z$	182	....	$2t-2z+y$
100	..	$2t+2z+2y$	142	....	$4t-2z$	183	....	$2t+2z-y$
101	80	t*	143	....	$4t+2z$	184	....	$2t+2z+y$
102	81	$t-x$	144	127	$4t-2y$	185	....	$t-y$
103	82	$t+x$	145	....	$4t+2y$	186	....	$t+y$
104	83	$t-z$						

$$\cos 2t \cos 2t = \frac{1}{2} \cos 4t + \frac{1}{2}$$

[131] [0]

$$\cos 2t \cos x = \frac{1}{2} \cos (2t + x) + \frac{1}{2} \cos (-2t + x)$$

[4] [−3]

Hence the multiplication of cos $2t$ by cos $2t$ produces the arguments 131 and 0, similarly the multiplication of cos x by cos $2t$ produces the arguments 4 and -3; proceeding in this way the following Table was formed, by writing down the indices instead of the arguments themselves.

* Parallactic inequality.

Table I.

Showing the arguments which result from the combination of the arguments 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 17, 20, 35, 62, 101, 146 and 147, with the arguments 1, 2, 3, &c. by addition and subtraction.

0	1	2	3	4	5	6	7	8	9	10	11	12	14	17	20	35	62	101	146	147	
1	131 0	4 3	132 2	133 − 2	7 6	134 5	135 − 5	10 9	136 8	137 − 8	13 12	138 11	16 15	19 18	22 21	37 36	64 63	116 101	148 147	 146	1
2	4 − 3	8 0	1 −9	10 − 1	11 14	16 − 12	13 − 15	20 − 2	3 − 21	22 − 4	23 − 5	6 − 24	26 5	29 32	38 − 8	53 56	66 65	103 −102	150 149	153 −151	2
3	132 − 2	1 9	136 0	131 − 8	15 12	138 − 14	140 − 11	4 21	 2	133 − 20	7 24	 5	6 27	33 30	10 39	57 54	68 67	117 102	152 151	...... −149	3
4	133 2	10 1	131 8	137 0	13 16	141 11	139 14	22 3	132 20	...... − 2	25 6	134 23	28 7	31 34	40 9	55 58	70 69	118 103	154 153	 150	4
5	7 − 6	11 −14	15 − 12	13 − 16	17 0	1 − 18	19 − 1	23 −26	27 − 24	25 − 28	29 − 2	3 − 30	2 −32	35 − 5	41 −44	59 −17	72 71	105 −104	156 155	159 −157	5
6	134 − 5	16 12	138 14	141 − 11	1 18	142 0	131 − 17	28 24	 26	...... − 23	4 30	 2	34 3	7 36	46 42	19 60	74 73	119 104	158 157	...... −155	6
7	135 5	13 15	140 11	139 − 14	19 1	131 17	143 0	25 27	 23	...... − 26	31 3	132 29	4 33	37 6	43 45	61 18	76 75	120 105	160 159	 156	7
8	10 − 9	20 2	4 − 21	22 − 3	23 26	28 − 24	25 27	38 0	1 − 39	...... − 1	41 14	16 − 42	44 11	47 50	....	....	78 77	107 −106	162 161	165 −163	8
9	136 − 8	3 21	...... − 2	132 − 20	27 24	...... − 26	...... − 23	1 39	 0	131 − 38	15 42	...... − 14	12 45	51 48	....	....	80 79	121 106	164 163	...... −161	9
10	137 8	22 4	133 20	 2	25 28	 23	 26	40 1	131 38	 0	43 16	141 41	46 13	49 52	....	....	82 81	122 107	166 165	 162	10
11	13 − 12	23 5	7 − 24	25 − 6	29 2	4 − 30	31 − 3	41 −14	15 − 42	43 − 16	47 0	1 − 48	8 17	53 14	....	....	84 83	109 −108	168 167	171 −169	11
12	138 − 11	6 24	...... − 5	134 − 23	3 30	...... − 2	132 − 29	16 42	 14	141 − 41	1 48	 0	18 9	15 54	....	....	86 85	123 108	170 169	...... −167	12
13	139 11	25 7	135 23	 5	31 4	133 29	 2	43 15	140 41	..... − 14	49 1	131 47	10 19	55 16	....	....	88 87	124 109	172 171	 168	13
14	16 − 15	26 − 5	6 − 27	28 − 7	2 32	34 − 3	4 − 33	44 −11	12 − 45	46 − 13	8 −17	18 − 9	50 0	11 56	....	....	90 89	111 −110	174 173	177 −175	14
15	140 − 14	7 27	 5	135 − 26	33 3	132 − 32	...... − 2	13 45	 11	139 − 44	19 9	136 17	1 51	57 12	....	....	92 91	125 110	176 175	...... −173	15
16	141 14	28 6	134 26	...... − 5	4 34	 2	133 32	46 12	138 44	...... − 11	10 18	142 8	52 1	13 58	....	....	94 93	126 111	178 177	 174	16
17	19 − 18	29 −32	33 − 30	31 34	35 5	7 − 36	37 − 6	47 −50	51 − 48	49 − 52	53 −14	15 − 54	11 −56	59 0	....	....	96 95	113 −112	180 179	183 −181	17
18	142 − 17	34 30	 32	...... − 29	6 36	...... − 5	134 − 35	52 48	 50	...... − 47	16 54	 14	58 12	1 60	....	....	92 97	127 112	182 181	...... −179	18
19	143 17	31 33	 29	...... − 32	37 7	135 35	 5	49 51	 47	...... − 50	55 15	140 53	13 57	61 1	....	....	100 99	128 113	184 183	 180	19
20	22 − 21	38 8	10 − 39	40 − 9	41 44	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	20
21	...... − 20	9 39	...... − 8	136 − 38	45 42	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....	21

TABLE I. (Continued.)

0	1	2	3	4	5	6	7	8	9	10	11	12	14	17	20	35	62	101	146	147	
22	......	40	......	......	43	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	22
	20	10	38	8	46	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
23	25	41	13	43	47	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	23
	− 24	11	− 42	− 12	8	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
24	......	12	......	138	9	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	24
	− 23	42	− 11	− 41	48	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
25	......	43	139	......	49	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	25
	23	13	41	11	10	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
26	28	44	16	46	8	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	26
	− 27	14	− 45	− 15	50	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
27	......	15	......	140	51	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	27
	− 26	45	− 14	− 44	9	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
28	......	46	141	......	10	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	28
	26	16	44	14	52	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
29	31	47	19	49	53	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	29
	− 30	17	− 48	− 18	11	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
30	......	18	......	142	12	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	30
	− 29	48	− 17	− 47	54	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
31	......	49	143	......	55	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	31
	29	19	47	17	13	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
32	34	50	18	52	14	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	32
	− 33	− 17	− 51	− 19	56	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
33	......	19	......	143	57	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	33
	− 32	51	17	− 50	15	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
34	......	52	142	......	16	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	34
	32	18	50	− 17	58	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
35	37	53	57	58	59	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	35
	− 36	− 56	− 54	− 58	17	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
36	......	58	......	......	18	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	36
	− 35	54	56	− 53	60	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
37	......	55	......	......	61	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	37
	35	57	53	− 56	19	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
62	64	66	68	70	72	74	76	78	80	82	84	86	90	96	......	......	...	115	...	148	62
	− 63	− 65	− 67	− 69	− 71	− 73	− 75	− 77	− 79	− 81	− 83	− 85	−89	− 95	......	......	0	−114	146	......	
63	144	69	......	......	75	......	......	81	......	......	87	......	93	99	......	......	1	129	147	......	63
	− 62	67	65	− 66	73	71	− 72	79	77	− 78	85	83	91	97	......	......	...	114	...	−146	
64	145	70	......	......	76	......	......	82	......	......	88	......	94	100	......	......	...	130	...	......	64
	62	68	66	− 65	74	72	− 71	80	78	− 77	86	84	92	98	......	......	1	115	148	......	
65	69	77	63	81	83	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	65
	− 68	− 62	77	− 64	89	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
66	70	78	64	82	84	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	66
	− 67	62	− 79	− 63	90	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
67	......	63	......	144	91	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	67
	− 66	79	− 62	− 78	85	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
68	......	64	......	145	92	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	68
	− 65	80	62	− 77	86	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
69	......	81	144	......	87	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	69
	65	63	77	− 62	93	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	

TABLE I. (Continued.)

0	1	2	3	4	5	6	7	8	9	10	11	12	14	17	20	35	62	101	146	147	
70	......	82	145	......	88	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	70
	66	64	78	62	94	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
71	75	83	91	87	95	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	71
	– 74	– 90	– 86	– 94	– 62	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
72	76	84	92	88	96	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	72
	– 73	– 89	– 85	– 93	62	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
73	......	93	......	......	63	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	73
	– 72	85	89	– 84	97	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
74	......	94	......	......	64	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	74
	– 71	86	90	– 83	98	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
75	......	87	......	......	99	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	75
	71	91	83	– 90	63	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
76	......	88	......	......	100	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	76
	72	92	84	– 89	64	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
101	116	103	117	118	105	119	120	107	121	122	109	123	111	113	...	...	115	1	186	......	101
	–101	102	–102	–103	104	–104	–105	106	–106	–107	108	–108	110	112	...	...	114	0	185	–185	
102	117	101	121	116	110	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	102
	–103	106	–101	–107	108	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
103	118	107	116	122	109	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	103
	–102	101	–106	–101	111	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
104	119	111	123	126	101	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	104
	–105	108	–110	–109	112	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
105	120	109	125	124	113	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	105
	–104	110	–108	–111	101	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
116	......	118	......	......	120	......	......	122	......	......	124	......	126	128	...	...	130	131	...	......	116
	101	117	103	102	119	105	104	121	107	106	123	109	125	127	...	...	129	1	...	186	
117	......	116	......	......	125	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	117
	102	121	101	106	123	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
118	......	122	......	......	124	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	118
	103	116	107	101	126	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
119	......	126	......	......	116	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	119
	104	123	111	108	127	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
120	......	124	......	......	128	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	120
	105	125	109	110	116	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
131	......	133	......	......	135	......	......	137	......	......	139	......	141	143	...	...	145	......	...	......	131
	1	132	4	3	134	7	6	136	10	9	138	13	140	142	...	...	144	116	...	148	
132	......	131	......	......	140	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	132
	3	136	1	9	138	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
133	......	137	......	......	139	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	133
	4	131	10	1	141	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
134	......	141	......	......	131	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	134
	6	138	16	12	142	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
135	......	139	......	......	143	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	135
	7	140	13	15	131	......	......	......	......	......	......	......	......	......	...	...	......	......	...	......	
146	148	150	152	154	156	158	160	162	164	166	168	170	174	180	...	...	......	186	62	1	146
	–147	–149	–151	–153	–155	–157	–159	–161	163	–165	–167	–169	–173	–179	...	...	–146	–185	0	– 63	
147	......	153	......	......	159	......	......	164	......	......	171	......	177	183	...	...	148	......	1	144	147
	–146	151	149	–150	157	155	–156	163	161	–162	169	167	175	181	...	...	......	185	63	0	

Table I. (Continued.)

0	1	2	3	4	5	6	7	8	9	10	11	12	14	17	20	35	62	101	146	147	
148	 146	154 152	 150	...... −149	160 158	 156	...... −155	166 164	 162	...... −161	172 170	 168	178 176	184 182	... ...	... ...	 147	 186	64 1	131 62	148
149	153 −152	161 −146	147 −164	165 −148	167 173	....	....	....	....	....	....	....	....	....	... ...	... ...	....	....	... ...	....	149
150	154 −151	162 146	148 −163	166 −147	168 174	....	....	....	....	....	....	....	....	....	... ...	... ...	....	....	... ...	....	150
151	...... −150	147 163	...... −146	...... −162	175 169	....	....	....	....	....	....	....	....	....	... ...	... ...	....	....	... ...	....	151
152	...... −149	148 164	 146	...... −161	176 170	....	....	....	....	....	....	....	....	....	... ...	... ...	....	....	... ...	....	152
153	 149	165 147	 161	...... −146	171 177	....	....	....	....	....	....	....	....	....	... ...	... ...	....	....	... ...	....	153
154	 150	166 148	 162	 146	172 178	....	....	....	....	....	....	....	....	....	... ...	... ...	....	....	... ...	....	154
155	159 −158	167 −174	175 −170	...... −178	179 −146	....	....	....	....	....	....	....	....	....	... ...	... ...	....	....	... ...	....	155
156	160 −157	168 −173	176 −169	172 −177	180 146	....	....	....	....	....	....	....	....	....	... ...	... ...	....	....	... ...	....	156
157	...... −156	177 169	 173	...... −168	147 181	....	....	....	....	....	....	....	....	....	... ...	... ...	....	....	... ...	....	157
158	...... −155	178 170	 174	...... −167	148 182	....	....	....	....	....	....	....	....	....	... ...	... ...	....	....	... ...	....	158
159	 155	171 175	 167	...... −174	183 147	....	....	....	....	....	....	....	....	....	... ...	... ...	....	....	... ...	....	159
160	 156	172 176	 168	...... −173	184 148	....	....	....	....	....	....	....	....	....	... ...	... ...	....	....	... ...	....	160

	38	59	
1	40 39	61 60	1

Table II.

Showing the arguments which, by their combination with the arguments 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 17, 20, 35, 62, 101, 146, 147, produce the arguments, 12, 3, &c. in the left hand column. This Table is formed from the preceding, by making the numbers in the left hand column in that Table change places with the rest. A full stop is placed after the figure where it does not occupy the same *cell* as in the preceding Table.

0	1	2	3	4	5	6	7	8	9	10	11	12	14	17	20	35	62	101	146	147	
1	0 131	3 4	2 132	133. – 2	6 7	5	...... – 5	9 10	8 ...	...... – 8	12 13	11	15 16	18 19	... ...	... ...	63 64	101 116	147 148	146 ...	1
2	4. – 3	0 8	– 9. 1	10. – 1	14 11	16. – 12	– 15. 13	...... – 2	... 3	...... – 4	...... – 5	 6	5 ...	....	... ...	... ...	... ...	... ...	... ...	... ...	2
3	132. – 2	9 1	0	131. – 8	12 15	...... – 14	...... – 11	 4	2 ...	....	 7	5	... 6	....	... ...	... ...	... ...	... ...	... ...	... ...	3
4	2 133	1 10	8 131	0	16 13	11	14	3	... ...	...... – 2	6	....	7 ...	....	... ...	... ...	... ...	... ...	... ...	... ...	4
5	7. – 6	– 14. 11	15. – 12	– 16. 13	0 17	– 18. 1	19. – 1	....	... ...	....	...... – 2	 3	... 2	...... – 5	... ...	... ...	... ...	... ...	... ...	... ...	5
6	134. – 5	12 16	14	...... – 11	18 1	....	131. – 17	....	... ...	....	 4	2	3 ...	 7	... ...	... ...	... ...	... ...	... ...	... ...	6
7	5 135	15 13	11	...... – 14	1 19	17 131	....	....	... ...	....	3	....	... 4	6	... ...	... ...	... ...	... ...	... ...	... ...	7
8	10. – 9	2 20	– 21. 4	22. – 3	26 23	....	....	....	... 1	...... – 1	14	 16	11 ...	....	... – 2	... ...	... ...	... ...	... ...	... ...	8
9	...... – 8	21 3	...... – 2	132. – 20	24 27	....	....	 1	... ...	 131	 15	...... –14	... 12	....	... 4	... ...	... ...	... ...	... ...	.. ...	9
10	8	4 22	20 133	2	28 25	....	....	1	... 131	....	16	....	13 ...	....	3 ...	... ...	... ...	... ...	... ...	... ...	10
11	13. – 12	5 23	– 24. 7	25. – 6	2 29	 4	...... – 3	...... – 14	... 15	...... – 16	....	 1	17 8	14	... ...	... ...	... ...	... ...	... ...	... ...	11
12	...... – 11	24 6	...... – 5	134. – 23	30 3	...... – 2	....	 16	14 ...	....	 1	....	9 18	 15	... ...	... ...	... ...	... ...	... ...	... ...	12
13	11	7 25	23 135	5	4 31	....	2	15	... ...	...... – 14	1	 131	19 10	16	... ...	... ...	... ...	... ...	... ...	... ...	13
14	16. – 15	26. – 5	– 27. 6	28. – 7	32 2	...... – 3	 4	...... – 11	... 12	...... – 13	– 17. 8	18. – 9	... ...	 11	... ...	... ...	... ...	... ...	... ...	... ...	14
15	...... – 14	27 7	5	135. – 26	3 33	....	...... – 2	 13	11 ...	....	9 19	17	... 1	12	... ...	... ...	... ...	... ...	... ...	... ...	15
16	14	6 28	26 134	...... – 5	34 4	2	....	12	... ...	...... – 11	18 10	8	1 ...	 13	... ...	... ..	... ...	... ...	... ...	... ...	16
17	19. – 18	– 32. 29	33 – 30	– 34. 31	5 35	 7	...... – 6	....	... ...	....	...... – 14	 15	... 11	0	... ...	... – 5	... ...	... ...	.. ...	... ...	17
18	...... – 17	30 34	32	...... – 29	36 6	...... – 5	....	....	... ...	....	 16	14	12 ...	 1	... ...	... 7	... ...	... ...	... ...	... ...	18
19	17	33 31	29	...... – 32	7 37	....	5	....	... ...	....	15	....	... 13	1	... ...	6 ...	... ...	... ..	... ...	... ...	19
20	22. – 21	8	 10	...... – 9	... ...	....	....	2	... 4	...... – 3	....	....	... ...	....	0 ...	... ...	... ...	... ...	... ...	... ...	20

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	1	2	3	4	5	6	7	8	9	10	11	12	14	17	20	35	62	101	146	147	
21	......	......	......	......	...	......	......	......	......	......	...	......	......	...	...	...	...	...	...	...	21
	– 20	9	– 8	......	...	......	......	3	– 2	......	...	......	......	...	1	...	...	...	...	...	
22	20	10	......	8	...	......	......	4	......	2	...	......	......	...	1	...	...	...	...	...	22
	......	......	......	......	...	......	......	......	......	......	...	......	......	...	...	...	...	...	...	...	
23	25.	11	......	......	8	......	......	5	......	......	2	......	......	...	...	...	...	...	...	...	23
	– 24	......	13	– 12	...	10	– 9	......	7	– 6	...	4	......	...	...	...	...	...	...	...	
24	......	......	......	......	...	......	......	......	......	......	...	......	......	...	...	...	...	...	...	...	24
	– 23	12	– 11	......	9	– 8	......	6	– 5	......	3	– 2	......	...	...	...	...	...	...	...	
25	23	13	......	11	10	......	8	7	......	5	4	......	......	...	...	...	...	...	...	..	25
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26	28.	14	......	......	...	......	......	......	......	......	...	......	2	...	...	...	...	...	...	...	26
	– 27	......	16	– 15	8	– 9	10	– 5	6	– 7	...	......	......	...	...	...	...	...	...	...	
27	......	......	......	......	9	......	......	......	5	......	...	......	......	...	..	...	...	...	...	...	27
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28	26	16	......	14	...	8	......	6	......	......	...	......	4	...	...	...	...	...	...	...	28
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29	31.	17	......	......	11	......	......	......	......	......	5	......	......	2	...	...	...	...	...	...	29
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	– 29	18	– 17	......	12	– 11	......	......	......	......	6	– 5	......	3	...	...	...	...	...	...	
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32	34.	......	......	......	...	......	......	......	......	......	...	......	......	...	...	...	...	...	...	...	32
	– 33	– 17	18	– 19	14	– 15	16	......	......	......	...	......	– 5	2	...	...	...	...	...	...	
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39	......	......	......	......	...	......	......	......	......	......	...	......	......	...	...	...	...	...	...	...	39
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	......	24	– 23	......	21	......	......	12	– 11	......	9	– 8	......	...	6	...	...	...	...	...	
43	......	25	......	23	22	......	......	13	......	11	10	......	......	...	7	...	...	...	...	...	43
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	.. ...	......	28	– 27	20	......	......	......	16	– 15	...	......	......	...	–5	...	...	...	..	...	

TABLE II. (Continued.)

0	1	2	3	4	5	6	7	8	9	10	11	12	14	17	20	35	62	101	146	147	
45	....	 27	...... – 26	....	21	... ...	... ...	 15	...... – 14	....	... ...	....	 9	... ...	... 7	... ...	... ...	... ...	... ...	... ...	45
46	....	28	....	26	 22	... ...	... ...	16	....	14	... ...	....	10	... ...	6 ...	... ...	... ...	... ...	... ...	... ...	46
47	....	29	 31	...... – 30	23	... ...	... ...	17	 19	...... – 18	11 ...	 13	....	8 ...	... ...	... ...	... ...	... ...	... ...	... ...	47
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52	....	34	....	32	 28	... ...	... ...	18	....	...... – 17	... ...	....	16	... 10	... ...	... ...	... ...	... ...	... ...	... ...	52
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54	....	 36	...... – 35	....	 30	... ...	... ...	....	....	....	... 18	...... – 17	....	... 12	... ...	... 3	... ...	... ...	... ...	... ...	54
55	....	37	....	....	31	... ...	... ...	....	....	....	19 ...	....	....	13 ...	... ...	4 ...	... ...	... ...	... ...	... ...	55
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57	....	 37	35	....	33	... ...	... ...	....	....	....	... ...	....	 19	15 ...	... ...	3 ...	... ...	... ...	... ...	... ...	57
58	....	36	....	35. – 35	 34	... ...	... ...	....	....	....	... ...	....	18	... 16	... ...	... 4	... ...	... ...	... ...	... ...	58
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62	64. – 63	66. – 65	68. – 67	70. – 69.	72. – 71	... ...	... ...	....	....	....	... ...	....	....	... ...	... ...	.. ...	... ...	... ...	146 ...	... 148	62
63	...... – 62	67 69	65	...... – 66	73 75	... ...	... ...	....	....	....	... ...	....	....	... ...	... ...	... ...	... 1	... ...	... 147	... – 146	63
64	62	68 70	66	...... – 65	74 76	... ...	... ...	....	....	....	... ...	....	....	... ...	... ...	... ...	1 ...	... ...	148 ...	... ...	64
65	69. – 68	...... – 62	 63	...... – 64.	....	... ...	... ...	....	....	....	... ...	....	....	... ...	... ...	... ...	... 2	... ...	... ...	... ...	65
66	70. – 67	62	 64	...... – 63	....	... ...	... ...	....	....	....	... ...	....	....	... ..	... ...	... ...	2 ...	... ...	... ...	... ...	66
67	76 – 66	 63	...... – 62	....	....	... ...	... ...	....	....	....	... ...	....	....	... ...	... ...	... ...	... 3	... ...	... ...	... ...	67
68	...... – 65	 64	62	....	....	... ...	... ...	....	....	....	... ...	....	....	... ...	.. ...	... ...	3 ...	... ...	... ...	... ...	68

TABLE II. (Continued.)

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70	66	64 ...	... ...	62	....	....	....	....	....	....	....	....	....	... ...	... ...	... ...	4 ...	... ...	... ...	... ...	70
71	75. – 74	... ...	... ...	....	...... – 62	 63	...... – 64	....	....	....	....	....	....	... ...	... ...	... ...	... 5	... ...	... ...	... ...	71
72	76. – 73	... ...	... ...	....	62	 64	...... – 63	....	....	....	....	....	....	... ...	... ...	... ...	5 ...	... ...	... ...	... ...	72
73	...... – 72	... ...	... ...	....	 63	...... – 62	....	....	....	....	....	....	....	... ...	... ...	... ...	... 6	... ...	... ...	... ...	73
74	...... – 71	... ...	... ...	....	 64	62	....	....	....	....	....	....	....	... ...	... ...	... ...	6 ...	... ...	... ...	... ...	74
75	71	... ...	... ...	....	63	....	...... – 62	....	....	....	....	....	....	... ...	... ...	... ...	... 7	... ...	... ...	... ...	75
76	72	... ...	... ...	....	64	....	62	....	....	....	....	....	....	... ...	... ...	... ...	7 ...	... ...	... ...	... ...	76
77	....	65 ...	69. 65	...... – 68	....	....	....	...... – 62	 63	...... – 64	....	....	....	... ...	... ...	... ...	... 8	... ...	... ...	... ...	77
78	....	66 ...	... 70	...... – 67	....	....	....	62	 64	...... – 63	....	....	....	... ...	... ...	... ...	8 ...	... ...	... ...	... ...	78
79	....	... 67	... –66	....	....	....	....	 63	...... – 62	....	....	....	....	... ...	... ...	... ...	... 9	... ...	... ...	... ...	79
80	....	... 68	... ...	....	....	....	....	 64	62	....	....	....	....	... ...	... ...	... ...	9 ...	... ...	... ...	... ...	80
81	....	69 ...	... ...	65	....	....	....	63	....	...... – 62	....	....	....	... ...	... ...	... ...	... 10	... ...	... ...	... ...	81
82	....	70 ...	... ...	66	....	....	....	64	....	62	....	....	....	... ...	... ...	... ...	10 ...	... ...	... ...	... ...	82
83	....	71 ...	... 75	...... – 74	65	....	....	....	....	....	...... – 62	 63	....	... ...	... ...	... ...	... 11	... ...	... ...	... ...	83
84	....	72 ...	... 76	...... – 73	66	....	....	....	....	....	62	 64	....	... ...	... ...	... ...	11 ...	... ...	... ...	... ...	84
85	....	... 73	... –72	....	 67	....	....	....	....	....	 63	...... – 62	....	... ...	... ...	... ...	... 12	... ...	... ...	... ...	85
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88	....	76 ...	... ...	72	70	....	....	....	....	....	64	....	....	... ...	... ...	... ...	13 ...	... ...	... ...	... ...	88
89	....	... –72	... 73	...... – 76	 65	....	....	....	....	....	....	....	...... – 62	... ...	... ...	... ...	... 14	... ...	... ...	... ...	89
90	....	... –71	... 74	...... – 75	 66	....	....	....	....	....	....	....	62	... ...	... ...	... ...	14 ...	... ...	... ...	... ...	90
91	....	... 75	71 ...	....	67	....	....	....	....	....	....	....	 63	... ...	... ...	... ...	... 15	... ...	... ...	... ...	91
92	....	... 76	72 ...	....	68	....	....	....	....	....	....	....	 64	... ...	... ...	... ...	15 ...	... ...	... ...	... ...	92

Table II. (Continued.)

0	1	2	3	4	5	6	7	8	9	10	11	12	14	17	20	35	62	101	146	147	
93	....	73 ...	....	...... − 72	... 69	....	....	... ...	....	....	... ...	....	63 ...	....	... ...	... ...	... 16	....	... ...	... ...	93
94	....	74 ...	....	...... − 71	... 70	....	....	... ...	....	....	... ...	....	64 ...	....	... ...	... ...	16 ...	....	... ...	... ...	94
95	....	... ...	....	....	71 ...	....	....	... ...	....	....	... ...	....	... ...	...... − 62	... ...	... ...	... 17	....	... ...	... ...	95
96	....	... ...	....	....	72 ...	....	....	... ...	....	....	... ...	....	... ...	62	... ...	... ...	17 ...	....	... ...	... ...	96
97	....	... ...	....	....	... 73	....	....	... ...	....	....	... ...	....	... ...	 63	... ...	... ...	... 18	....	... ...	... ...	97
98	....	... ...	....	....	... 74	....	....	... ...	....	....	... ...	....	... ...	 64	... ...	... ...	18 ...	....	... ...	... ...	98
99	....	... ...	....	....	75 ...	....	....	... ...	....	....	... ...	....	... ...	63	... ...	... ...	... 19	....	... ...	... ...	99
100	....	... ...	....	....	76 ...	....	....	... ...	....	....	... ...	....	... ...	64	... ...	... ...	19 ...	....	... ...	... ...	100
101	116. −101	102 103	117. −102	118. −103	104 105	....	....	... ...	....	....	... ...	....	.. ...	....	... ...	... ...	... ...	 1	... ...	... ...	101
102	117. −103	... 101	...... −101	 116	... ...	....	....	... ...	....	....	... ...	....	... ...	....	... ...	... ...	... ...	3. − 2	... ...	... ...	102
103	118. −102	101 ...	 116	...... −101	... ...	....	....	... ...	....	....	... ...	....	... ...	....	... ...	... ...	... ...	2 4	... ...	... ...	103
104	119. −105	... ...	....	....	... 101	...... −101	 116	... ...	....	....	... ...	....	... ...	....	... ...	... ...	... ...	6. − 5	... ...	... ...	104
105	120. −104	... ...	....	....	101 ...	 116	...... −101	... ...	....	....	... ...	....	... ...	....	... ...	... ...	... ...	5 7	... ...	... ...	105
106	....	... 102	...... −103	 117	... ...	....	....	... 101	...... −101	 116	... ...	....	... ...	....	... ...	... ...	... ...	9. − 8	... ...	... ...	106
107	....	103 ...	 118	...... −102	... ...	....	....	101 ...	 116	...... −101	... ...	....	... ...	....	... ...	... ...	... ...	8 10	... ...	... ...	107
108	....	... 104	...... −105	 119	... 102	....	....	... ...	....	....	... 101	...... −101	... ...	....	... ...	... ...	... ...	12. − 11	... ...	... ...	108
109	....	105 ...	 120	...... −104	103 ...	....	....	... ...	....	....	101 ...	 116	... ...	....	... ...	... ...	... ...	11 13	... ...	... ...	109
110	....	... 105	...... −104	 120	102 ...	....	....	... ...	....	....	... ...	....	... 101	....	... ...	... ...	... ...	15. − 14	... ...	... ...	110
111	....	104 ...	 119	...... −105	... 103	....	....	... ...	....	....	... ...	....	101 ...	....	... ...	... ...	... ...	14 16	... ...	... ...	111
112	....	... ...	....	....	... 104	....	....	... ...	....	....	... ...	....	... ...	 101	... ...	... ...	... ...	18. − 17	... ...	... ...	112
113	....	... ...	....	....	105 ...	....	....	... ...	....	....	... ...	....	... ...	101	... ...	... ...	... ...	17 19	... ...	... ...	113
114	....	... ...	....	....	... ...	....	....	... ...	....	....	... ...	....	... ...	....	... ...	... ...	... 101	63. − 62	... ...	... ...	114
115	....	... ...	....	....	... ...	....	....	... ...	....	....	... ...	....	... ...	....	... ...	... ...	101 ...	62 64	... ...	... ...	115
116	101	117 118	103	102	119 120	....	....	... ...	....	....	... ...	....	... ...	....	... ...	... ...	... ...	1 131	... ...	... ...	116

Table II. (Continued.)

0	1	2	3	4	5	6	7	8	9	10	11	12	14	17	20	35	62	101	146	147	
117	102	......	101	......	......	......	......	......	......	......	......	......	......	......	......	......	......	3	...	...	117
	......	116	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
118	103	116	......	101	......	......	......	......	......	......	......	......	......	......	......	......	......	4	...	...	118
	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
119	104	......	......	......	......	101	......	......	......	......	......	......	......	......	......	......	......	6	...	...	119
	......	......	......	......	116	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
120	105	......	....	......	116	......	101	......	......	......	......	......	......	......	......	......	......	7	...	...	120
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	......	117	......	......	......	......	......	116	......	......	......	......	......	......	......	......	......	......	...	...	
122	.....	118	......	103	......	......	......	116	......	101	......	......	......	.....	......	......	......	10	...	...	122
	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
123	......	......	104	......	......	......	......	......	......	......	......	101	......	......	......	......	......	12	...	...	123
	......	119	......	......	117	......	......	......	......	......	116	......	......	......	......	......	......	......	...	...	
124	......	120	......	105	118	......	......	......	......	......	116	......	......	......	......	......	......	13	...	...	124
	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
125	......	......	105	......	117	......	......	......	......	......	......	......	......	......	......	......	......	15	...	...	125
	......	120	......	......	......	......	......	......	.. ...	......	......	......	116	......	......	......	......	......	...	...	
126	......	119	......	104	......	......	......	......	......	......	......	......	116	......	......	......	......	16	...	...	126
	......	......	......	......	118	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
127	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	18	...	...	127
	......	......	......	......	119	......	......	......	......	......	......	......	......	116	......	......	......	......	...	...	
128	......	......	......	......	120	......	......	......	......	......	......	......	......	116	......	......	......	19	...	...	128
	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
129	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	63	...	...	129
	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	116	......	...	...	
130	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	116	64	...	...	130
	......	......	......	......	......	......	......	......	......	......	......	... ..	......	......	......	......	......	......	...	...	
131	1	132	4	3	134	7	6	......	10	9	......	13	......	......	......	......	......	116	...	148	131
	......	133	......	......	135	......	......	......	......	......	.. ...	......	......	......	......	......	......	......	...	...	
132	3	......	1	9	......	15	12	......	4	......	......	7	......	......	......	......	......	......	...	...	132
	......	131	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
133	4	131	10	1	......	13	16	......	......	3	......	......	......	......	......	......	......	......	...	...	133
	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
134	6	......	16	12	......	1	18	......	......	......	......	4	......	......	......	......	......	......	...	...	134
	......	......	......	......	131	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
135	7	......	13	15	131	19	1	......	......	......	......	......	......	......	......	......	......	......	...	...	135
	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
136	9	... ..	3	21	......	......	......	......	1	......	......	15	......	......	......	......	......	......	...	...	136
	......	132	......	......	......	......	......	131	......	......	......	......	......	......	......	......	......	......	...	...	
137	10	133	......	4	......	......	......	131	......	1	......	......	......	......	......	.. ...	......	......	...	...	137
	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
138	12	......	6	......	......	3	......	......	16	......	......	1	......	......	......	......	......	......	...	...	138
	......	134	......	......	132	......	......	......	......	......	131	......	......	......	......	......	......	......	...	...	
139	13	135	......	7	133	......	4	......	......	15	131	......	......	......	......	......	......	......	...	...	139
	......	... ..	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	......	...	...	
140	15	......	7	......	132	......	3	......	13	......	......	19	......	......	......	......	......	......	...	...	140
	.....	135	......	......	......	......	......	......	......	......	......	......	131	......	......	.. ...	......	......	...	...	

TABLE II. (Continued.)

0	1	2	3	4	5	6	7	8	9	10	11	12	14	17	62	101	146	147	
141	16	134	......	6	......	......	......	......	......	12	...	10	131	......	......	...	......	......	141
	......	......	......	......	133	......	......	......	......	......	...	...	...	......	......	...	......	......	
142	18	......	......	......	......	6	......	......	......	......	...	16	...	......	......	...	......	......	142
	......	......	......	......	134	......	......	......	......	......	...	...	...	131	......	...	......	......	
143	19	......	......	......	135	......	7	......	......	......	...	...	...	131	......	...	......	......	143
	......	......	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	......	
144	......	......	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	147	144
	......	......	......	......	......	......	......	......	......	......	...	...	...	......	131	...	......	......	
145	......	......	......	......	......	......	......	......	......	......	...	...	...	......	131	...	......	......	145
	......	......	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	......	
146	148.	150.	152.	154.	156.	......	......	......	......	......	...	...	...	......	......	...	......	– 63.	146
	–147	–149	–151	–153	–155	......	......	......	......	......	...	...	...	......	–146	...	62	1	
147	......	151	149	......	157	......	......	......	......	......	...	...	...	......	......	...	63	......	147
	–146	153	......	–150	159	......	......	......	......	......	...	...	...	......	148	...	1	......	
148	146	152	150	......	158	......	......	......	......	......	...	...	...	......	147	...	1	62	148
	......	154	......	–149	160	......	......	......	......	......	...	...	...	......	......	...	64	131	
149	153.	......	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	......	149
	–152	–146	147	–148	......	......	......	......	......	......	...	...	...	......	......	...	2	– 3	
150	154.	146	148.	......	......	......	......	......	......	......	...	...	...	......	......	...	2	......	150
	–151	......	......	–147	......	......	......	......	......	......	...	...	...	......	......	...	......	4	
151	......	......	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	......	151
	–150	147	–146	......	......	......	......	......	......	......	...	...	...	......	......	...	3	– 2	
152	......	......	146	......	......	......	......	......	......	......	...	...	...	......	......	...	3	......	152
	–149	148	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	......	
153	149	147	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	2	153
	......	......	......	–146	......	......	......	......	......	......	...	...	...	......	......	...	4	......	
154	150	148	......	146	......	......	......	......	......	......	...	...	...	......	......	...	4	......	154
	......	......	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	......	
155	159.	......	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	......	155
	–158	......	......	......	–146	147	–148	......	......	......	...	...	...	......	......	...	5	– 6	
156	160.	......	......	......	146	......	......	......	......	......	...	...	...	......	......	...	5	......	156
	–157	......	......	......	......	148	–147	......	......	......	...	...	...	......	......	...	......	7	
157	......	......	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	......	157
	–156	......	......	......	147	–146	......	......	......	......	...	...	...	......	......	...	6	– 5	
158	......	......	......	......	......	146	......	......	......	......	...	...	...	......	......	...	6	......	158
	–155	......	......	......	148	......	......	......	......	......	...	...	...	......	......	...	......	......	
159	155	......	......	......	147	......	......	......	......	......	...	...	...	......	......	...	......	5	159
	......	......	......	......	......	......	–146	......	......	......	...	...	...	......	......	...	7	......	
160	156	......	......	......	148	......	146	......	......	......	...	...	...	......	......	...	7	......	160
	......	......	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	......	
161	......	149	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	......	161
	......	......	153	–152	......	......	......	–146	147	–148	...	...	...	......	......	...	8	– 9	
162	......	150	......	......	......	......	......	146	......	......	...	...	...	......	......	...	8	......	162
	......	......	154	–151	......	......	......	......	148	–147	...	...	...	......	......	...	......	10	
163	......	......	......	......	......	......	......	......	......	......	...	...	...	......	......	...	......	......	163
	......	151	–150	......	......	......	......	147	146	......	...	...	...	......	......	...	9	– 8	
164	......	......	......	......	......	......	......	147	146	......	...	...	...	......	......	...	9	......	164
	......	152	–149	......	......	......	......	148	......	......	...	...	...	......	......	...	......	......	

TABLE II. (Continued.)

0	1	2	3	4	5	6	7	8	9	10	11	12	14	17	62	101	146	147	
165	...	153	......	149	......	......	......	......	......	......	......	......	......	......	...	......	...	8	165
	...	......	......	......	......	......	......	......	......	−146	......	......	......	......	...	......	10	......	
166	...	154	......	150	......	......	......	148	......	146	......	......	......	......	...	......	10	......	166
	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
167	...	155	......	......	149	......	......	......	......	......	......	......	......	......	...	......	...	......	167
	...	......	159	−158	......	......	......	......	......	......	−146	147	......	......	...	......	11	−12	
168	...	156	......	......	150	......	......	......	......	......	146	......	......	......	...	......	11	......	168
	...	......	160	−157	......	......	......	......	......	......	......	148	......	......	...	......	...	13	
169	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	169
	...	157	−156	......	151	......	......	......	......	......	147	−146	......	......	...	......	12	−11	
170	...	......	......	......	......	......	......	......	......	......	......	146	......	......	...	......	12	......	170
	...	158	−155	......	152	......	......	......	......	......	148	......	......	......	...	......	...	......	
171	...	159	......	......	153	......	......	......	......	......	147	......	......	......	...	......	...	11	171
	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	......	13	......	
172	...	160	......	156	154	......	......	......	......	......	148	......	......	......	...	......	13	......	172
	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
173	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	173
	...	−156	157	−160	149	......	......	......	......	......	......	......	−146	......	...	......	14	−15	
174	...	......	......	......	......	......	......	......	......	......	......	......	146	......	...	......	14	......	174
	...	−155	158	−159	150	......	......	......	......	......	......	......	......	......	...	......	...	16	
175	...	......	155	......	151	......	......	......	......	......	......	......	......	......	...	......	...	......	175
	...	159	......	......	......	......	......	......	......	......	......	......	147	......	...	......	15	−14	
176	...	......	156	......	152	......	......	......	......	......	......	......	......	......	...	......	15	......	176
	...	160	......	......	......	......	......	......	......	......	......	......	148	......	...	......	...	......	
177	...	157	......	......	......	......	......	......	......	......	......	......	147	......	...	......	...	14	177
	...	......	......	−156	153	......	......	......	......	......	......	......	......	......	...	......	16	......	
178	...	158	......	......	......	......	......	......	......	......	......	......	148	......	...	......	16	......	178
	...	......	......	−155	154	......	......	......	......	......	......	......	......	......	...	......	...	......	
179	...	......	......	......	155	......	......	......	......	......	......	......	......	......	...	......	...	......	179
	...	......	......	......	......	......	......	......	......	......	......	......	......	−146	...	......	17	−18	
180	...	......	......	......	156	......	......	......	......	......	......	......	......	146	...	......	17	......	180
	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	......	...	19	
181	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	181
	...	......	......	......	157	......	......	......	......	......	......	......	......	147	...	......	18	−17	
182	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	......	18	......	182
	...	......	......	......	158	......	......	......	......	......	......	......	......	148	...	......	...	......	
183	...	......	......	......	159	......	......	......	......	......	......	......	......	147	...	......	...	17	183
	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	......	19	......	
184	...	......	......	......	160	......	......	......	......	......	......	......	......	148	...	......	19	......	184
	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	......	...	......	
185	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	147.	...	......	185
	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	−146	...	......	
186	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	146	...	......	186
	...	......	......	......	......	......	......	......	......	......	......	......	......	......	...	148	...	......	

	38	59			38	59	
39	......	......	39	60	......	......	60
	1	......			......	1	
40	1	......	40	61	......	1	61
	......	......			......	......	

Table II. may be used in forming the developments required in the method employed by MM. LAPLACE and DAMOISEAU; for this purpose it is only necessary to make $t = \lambda' - \lambda_{\prime}$ instead of $n\,t - n_{\prime}t$

$$x = c\,\lambda' - \varpi \quad . \quad . \quad . \quad c\,n\,t - \varpi$$

$$z = c_{\prime}\lambda_{\prime} - \varpi_{\prime} \quad . \quad . \quad . \quad c_{\prime}n_{\prime}t - \varpi_{\prime}$$

$$\text{and } y = g\,\lambda' - \nu \quad . \quad . \quad . \quad g\,n\,t - \nu$$

The notation throughout is the same as that used Phil. Trans. 1830, p. 328, with the exception of the indices of the arguments.

In the elliptic movement;

$$a^5 r^{-5} = 1 + 5\,e^2\left(1 + \frac{21}{8}e^2\right) + 5\,e\left(1 + \frac{27}{8}e^2\right)\cos x + 10\,e^2\left(1 + \frac{31}{12}e^2\right)\cos 2\,x$$
$$+ \frac{145}{8}e^3\cos 3\,x + \frac{745}{48}e^4\cos 4\,x$$

$$a^4 r^{-4} = 1 + 3\,e^2 + 4\,e\cos x + 7\,e^2\cos 2\,x$$

$$a^3 r^{-3} = 1 + \frac{3}{2}e^2\left(1 + \frac{5}{4}e^2\right) + 3e\left(1 + \frac{9}{8}e^2\right)\cos x + \frac{9}{2}e^2\left(1 + \frac{7}{9}e^2\right)\cos 2\,x$$
$$+ \frac{53}{8}e^3\cos 3\,x + \frac{77}{8}e^4\cos 4\,x$$

$$a^2 r^{-2} = 1 + \frac{e^2}{2}\left(1 + \frac{3}{4}e^2\right) + 2\,e\left(1 + \frac{3}{8}e^2\right)\cos x + \frac{5}{2}e^2\left(1 + \frac{2}{15}e^2\right)\cos 2\,x$$
$$+ \frac{13}{4}e^3\cos 3\,x + \frac{103}{24}e^4\cos 4\,x$$

$$a\,r^{-1} = 1 + e\left(1 - \frac{e^2}{8}\right)\cos x + e^2\left(1 - \frac{e^2}{3}\right)\cos 2\,x + \frac{9}{8}e^3\cos 3\,x + \frac{4}{3}e^4\cos 4\,x$$

$$\frac{r}{a} = 1 + \frac{e^2}{2} - e\left(1 - \frac{3\,e^2}{8}\right)\cos x - \frac{e^2}{2}\left(1 - \frac{2\,e^2}{3}\right)\cos 2\,x - \frac{3\,e^3}{8}\cos 3\,x - \frac{e^4}{3}\cos 4\,x$$

$$\frac{r^2}{a^2} = 1 + \frac{3\,e^2}{2} - 2\,e\left(1 - \frac{e^2}{8}\right)\cos x - \frac{e^2}{2}\left(1 - \frac{e^2}{3}\right)\cos 2\,x - \frac{e^3}{4}\cos 3\,x - \frac{e^4}{6}\cos 4\,x$$

$$\frac{r^3}{a^3} = 1 + 3\,e^2\left(1 + \frac{e^2}{8}\right) - 3\,e\left(1 + \frac{3}{8}e^2\right)\cos x - \frac{5}{8}e^4\cos 2\,x + \frac{e^3}{8}\cos 3\,x + \frac{e^4}{8}\cos 4\,x$$

$$\frac{r^4}{a^4} = 1 + 5\,e^2 - 4\,e\cos x + e^2\cos 2\,x$$

$$\begin{aligned}\frac{a}{r} = {} & r_0 \\ & + r_1\cos 2\,t \\ & + e\,r_2\cos x \\ & + e\,r_3\cos(2\,t - x) \\ & + e\,r_4\cos(2\,t + x) \\ & + e_{\prime}\,r_5\cos z \\ & + e_{\prime}\,r_6\cos(2\,t - z) + \text{\&c. \&c.}\end{aligned}$$

$$\begin{aligned}
\lambda = & \; n t \\
& + \lambda_1 \cos 2 t \\
& + e \lambda_2 \cos x \\
& + e \lambda_3 \cos (2 t - x) \\
& + e \lambda_4 \cos (2 t + x) \\
& + e_{\prime} \lambda_5 \cos z \text{ \&c. \&c.}
\end{aligned}$$

The quantities λ correspond to the quantities b in M. DAMOISEAU's notation.

$$\begin{aligned}
s = & \; \gamma s_{146} \sin y \\
& + \gamma s_{147} \sin (2 t - y) \\
& + \gamma s_{148} \sin (2 t + y) \\
& + e \gamma s_{149} \sin (x - y) \text{ \&c. \&c.} \\
& \gamma = \tan \iota
\end{aligned}$$

$$R = m_{\prime} \left\{ \frac{\grave{r} r_{\prime} \cos (\lambda - \lambda_{\prime})}{r_{\prime}^3} - \frac{1}{\left\{ r^2 - 2 \grave{r} r_{\prime} \cos (\grave{\lambda} - \lambda_{\prime}) + r_{\prime}^2 \right\}^{\frac{1}{2}}} \right\}$$

$$= m_{\prime} \left\{ - \frac{1}{r_{\prime}} + \frac{r^2}{2 r_{\prime}^3} - \frac{3}{8} \frac{\left\{ 2 \grave{r} r_{\prime} \cos (\grave{\lambda} - \lambda_{\prime}) - r^2 \right\}^2}{r_{\prime}^5} - \frac{15}{48} \frac{\left\{ 2 \grave{r} r_{\prime} \cos (\grave{\lambda} - \lambda_{\prime}) - r^2 \right\}^3}{r_{\prime}^7} \right\}$$

$$= m_{\prime} \left\{ - \frac{1}{r_{\prime}} + \frac{r^2}{2 r_{\prime}^3} - \frac{3}{2} \frac{\grave{r}^2 r_{\prime}^2}{r_{\prime}^5} \cos (\grave{\lambda} - \lambda_{\prime})^2 + \frac{3}{2} \frac{r^2 \grave{r} r_{\prime}}{r_{\prime}^5} \cos (\lambda - \lambda_{\prime}) - \frac{5}{2} \frac{\grave{r}^3 r_{\prime}^3}{r_{\prime}^7} \cos (\grave{\lambda} - \lambda_{\prime})^3 \right\}$$

$$\begin{aligned}
= m_{\prime} \Bigg\{ - \frac{1}{r_{\prime}} & - \frac{\grave{r}^2}{4 r_{\prime}^3} \left\{ 1 + 3 \cos (2 \grave{\lambda} - 2 \lambda_{\prime}) - 2 s^2 \right\} \\
& - \frac{\grave{r}^3}{8 r_{\prime}^4} \left\{ 3 (1 - 4 s^2) \cos (\grave{\lambda} - \lambda_{\prime}) + 5 \cos (3 \grave{\lambda} - 3 \lambda_{\prime}) \right\} \Bigg\}
\end{aligned}$$

$$\grave{r} r_{\prime} {\textstyle\substack{\cos \\ \sin}} (\grave{\lambda} - \lambda_{\prime}) = r r_{\prime} \left\{ \cos^2 \frac{\iota}{2} {\textstyle\substack{\cos \\ \sin}} (\lambda - \lambda_{\prime}) + \sin^2 \frac{\iota}{2} {\textstyle\substack{\cos \\ \sin}} (\lambda + \lambda_{\prime} - 2 \nu) \right\}$$

$$\begin{aligned}
=^{*} a a_{\prime} \cos^2 \frac{\iota}{2} \Bigg\{ & \left(1 - \frac{e^2}{2} - \frac{e^4}{64}\right) \left(1 - \frac{e_{\prime}^2}{2} - \frac{e_{\prime}^4}{64}\right) {\textstyle\substack{\cos \\ \sin}}\, t - \frac{3 e}{2} \left(1 - \frac{e_{\prime}^2}{2}\right) {\textstyle\substack{\cos \\ \sin}} (t - x) \\
& + \frac{e}{2} \left(1 - \frac{3}{4} e^2\right) \left(1 - \frac{e_{\prime}^2}{2}\right) {\textstyle\substack{\cos \\ \sin}} (t + x) + \frac{3}{8} e^2 (1 - e^2) \left(1 - \frac{e_{\prime}^2}{2}\right) {\textstyle\substack{\cos \\ \sin}} (t + 2 x) \\
& + \frac{e^3}{3} {\textstyle\substack{\cos \\ \sin}} (t + 3 x) + \frac{125}{384} e^4 {\textstyle\substack{\cos \\ \sin}} (t + 4 x) + \frac{e^2}{8} \left(1 + \frac{e^2}{3}\right) \left(1 - \frac{e_{\prime}^2}{2}\right) {\textstyle\substack{\cos \\ \sin}} (t - 2 x) \\
& + \frac{e^3}{24} {\textstyle\substack{\cos \\ \sin}} (t - 3 x) + \frac{3}{128} e^4 {\textstyle\substack{\cos \\ \sin}} (t - 4 x) - \frac{3}{2} e_{\prime} \left(1 - \frac{e^2}{2}\right) {\textstyle\substack{\cos \\ \sin}} (t + z) \\
& + \frac{9}{4} e e_{\prime} {\textstyle\substack{\cos \\ \sin}} (t - x + z) - \frac{3}{4} e e_{\prime} \left(1 - \frac{3}{4} e^2\right) {\textstyle\substack{\cos \\ \sin}} (t + x + z) \\
& - \frac{9}{16} e^2 e_{\prime} {\textstyle\substack{\cos \\ \sin}} (t + 2 x + z) - \frac{e^3 e_{\prime}}{2} {\textstyle\substack{\cos \\ \sin}} (t + 3 x + z) - \frac{3}{16} e^2 e_{\prime} {\textstyle\substack{\cos \\ \sin}} (t - 2 x + z)
\end{aligned}$$

* See Phil. Trans. 1830, p. 343.

$$- \frac{e^3 e_{\prime} \cos}{16 \sin} (t - 3x + z) + \frac{e_{\prime}}{2}\left(1 - \frac{3}{4} e_{\prime}^2\right)\left(1 - \frac{e^2}{2}\right) \frac{\cos}{\sin} (t - z)$$

$$- \frac{3}{4} e\, e_{\prime}\left(1 - \frac{3}{4} e_{\prime}^2\right) \frac{\cos}{\sin} (t - x - z)$$

$$+ \frac{e\, e_{\prime}}{4}\left(1 - \frac{3}{4} e^2\right)\left(1 - \frac{3}{4} e_{\prime}^2\right) \frac{\cos}{\sin} (t + x - z) + \frac{3}{16} e^2 e_{\prime} \frac{\cos}{\sin} (t + 2x - z)$$

$$+ \frac{e^3 e_{\prime} \cos}{6 \sin} (t + 3x - z) + \frac{e^2 e_{\prime} \cos}{16 \sin} (t - 2x - z) + \frac{e^3 e_{\prime} \cos}{48 \sin} (t - 3x - z)$$

$$+ \frac{3}{8} e_{\prime}^2 (1 - e_{\prime}^2)\left(1 - \frac{e^2}{2}\right) \frac{\cos}{\sin} (t - 2z) - \frac{9}{16} e\, e_{\prime}^2 \frac{\cos}{\sin} (t - x - 2z)$$

$$+ \frac{3}{16} e\, e_{\prime}^2 \frac{\cos}{\sin} (t + x - 2z) + \frac{9}{64} e^2 e_{\prime}^2 \frac{\cos}{\sin} (t + 2x - 2z)$$

$$+ \frac{3}{64} e^2 e_{\prime}^2 \frac{\cos}{\sin} (t - 2x - 2z) + \frac{e_{\prime}^3 \cos}{3 \sin} (t - 3z) - \frac{e\, e_{\prime}^3 \cos}{2 \sin} (t - x - 3z)$$

$$+ \frac{e\, e_{\prime}^3 \cos}{6 \sin} (t + x - 3z) + \frac{125}{384} e_{\prime}^4 \frac{\cos}{\sin} (t - 4z)$$

$$+ \frac{e_{\prime}^2}{8}\left(1 + \frac{e_{\prime}^2}{3}\right)\left(1 - \frac{e^2}{2}\right) \frac{\cos}{\sin} (t + 2z) - \frac{3}{16} e\, e_{\prime}^2 \frac{\cos}{\sin} (t - x + 2z)$$

$$+ \frac{e\, e_{\prime}^2 \cos}{16 \sin} (t + x + 2z) + \frac{3}{64} e^2 e_{\prime}^2 \frac{\cos}{\sin} (t + 2x + 2z) + \frac{e^2 e_{\prime}^2 \cos}{64 \sin} (t - 2x + 2z)$$

$$+ \frac{e_{\prime}^3 \cos}{24 \sin} (t + 3z) - \frac{e\, e_{\prime}^3 \cos}{16 \sin} (t - x + 3z) + \frac{e\, e_{\prime}^3 \cos}{48 \sin} (t + x + 3z)$$

$$+ \frac{3}{128} e_{\prime}^4 \frac{\cos}{\sin} (t + 4z) \Big\}$$

$$+ a\, a_{\prime} \sin^2 \frac{\iota}{2} \left\{\left(1 - \frac{e^2 + e_{\prime}^2}{2}\right) \frac{\cos}{\sin} (t - 2y) - \frac{3}{2} e \frac{\cos}{\sin} (t + x - 2y) + \frac{e}{2} \frac{\cos}{\sin} (t - x - 2y)\right.$$

$$+ \frac{3}{8} e^2 \frac{\cos}{\sin} (t - 2x - 2y) + \frac{e^2}{8} \frac{\cos}{\sin} (t + 2x - 2y) - \frac{3}{2} e_{\prime} \frac{\cos}{\sin} (t + z - 2y)$$

$$+ \frac{9}{4} e\, e_{\prime} \frac{\cos}{\sin} (t + x + z - 2y) - \frac{3}{4} e\, e_{\prime} \frac{\cos}{\sin} (t - x + z - 2y)$$

$$+ \frac{e_{\prime}}{2} \frac{\cos}{\sin} (t - z - 2y) - \frac{3}{4} e\, e_{\prime} \frac{\cos}{\sin} (t + x - z - 2y) + \frac{e\, e_{\prime}}{4} \frac{\cos}{\sin} (t - x - z - 2y)$$

$r'^2 r_{\prime}^2 \cos (\lambda' - \lambda_{\prime})^2$

$$= a^2 a_{\prime}^2 \cos^4 \frac{\iota}{2}\left\{\frac{1}{2} + \left\{- \frac{1}{2} + \frac{9}{8} + \frac{1}{8}\right\}(e^2 + e_{\prime}^2) + \left\{\frac{1}{2} - \frac{9}{8} - \frac{1}{8} - \frac{9}{8} + \frac{81}{32} + \frac{9}{32}\right.\right.$$

$$\left. - \frac{1}{8} + \frac{9}{32} + \frac{1}{32}\right\} e^2 e_{\prime}^2 + \left\{\frac{7}{64} - \frac{3}{16} + \frac{9}{128} + \frac{1}{128}\right\}(e^4 + e_{\prime}^4)$$

[0]

$$+\left\{\frac{1}{2}+\left\{-\frac{1}{2}-\frac{3}{4}\right\}(e^2+e_{,}^2)+\left\{\frac{1}{2}+\frac{3}{4}+\frac{3}{4}+\frac{9}{16}+\frac{9}{16}\right\}e^2 e_{,}^2\right.$$

$$\left.+\left\{\frac{7}{64}+\frac{9}{16}+\frac{3}{64}\right\}(e^4+e_{,}^4)\right\}\cos 2t$$

[1]

$$+\left\{-\frac{3}{2}+\frac{1}{2}+\left\{\frac{3}{4}-\frac{5}{8}-\frac{3}{16}+\frac{3}{16}\right\}e^2\right.$$

$$\left.+\left\{\frac{3}{2}-\frac{1}{2}-\frac{27}{8}+\frac{9}{8}-\frac{3}{8}+\frac{1}{8}\right\}e_{,}^2\right\}e\cos x$$

[2] [5]*

$$+\left\{-\frac{3}{2}+\left\{\frac{3}{4}+\frac{1}{16}\right\}e^2+\left\{\frac{3}{2}+\frac{9}{8}+\frac{9}{8}\right\}e_{,}^2\right\}e\cos(2t-x)$$

[3] [7]

$$+\left\{-\frac{1}{2}+\left\{\frac{5}{8}-\frac{9}{16}\right\}e^2+\left\{-\frac{1}{2}-\frac{3}{8}-\frac{3}{8}\right\}e_{,}^2\right\}e\cos(2t+x)$$

[4] [6]

$$+\left\{\frac{3}{8}+\frac{1}{8}-\frac{3}{4}+\left\{-\frac{9}{16}-\frac{1}{48}+\frac{9}{16}-\frac{1}{16}+\frac{1}{6}\right\}e^2\right.$$

$$\left.+\left\{-\frac{3}{8}-\frac{1}{8}+\frac{3}{4}+\frac{27}{32}+\frac{9}{32}-\frac{27}{16}+\frac{3}{32}+\frac{1}{32}-\frac{3}{16}\right\}e_{,}^2\right\}e^2\cos 2x$$

[8] [17]

$$+\left\{\frac{9}{8}+\frac{1}{8}+\left\{-\frac{1}{48}+\frac{1}{48}\right\}e^2\right.$$

$$\left.+\left\{-\frac{9}{8}-\frac{1}{8}-\frac{3}{32}-\frac{27}{16}-\frac{3}{32}\right\}e_{,}^2\right\}e^2\cos(2t-2x)$$

[9] [19]

$$+\left\{\frac{1}{8}+\frac{3}{8}+\left\{-\frac{3}{16}-\frac{9}{16}-\frac{1}{2}\right\}e^2\right.$$

$$\left.+\left\{-\frac{1}{8}-\frac{3}{8}-\frac{9}{32}-\frac{3}{16}-\frac{9}{32}\right\}e_{,}^2\right\}e^2\cos(2t+2x)$$

[10] [18]

$$+\left\{-\frac{3}{4}-\frac{3}{4}+\frac{9}{4}+\frac{1}{4}+\left\{\frac{15}{16}+\frac{3}{8}-\frac{9}{8}-\frac{3}{32}-\frac{9}{32}\quad\frac{5}{16}\right.\right.$$

$$\left.\left.+\frac{3}{32}+\frac{9}{32}\right\}(e^2+e_{,}^2)\right\}e\,e_{,}\cos(x+z)$$

[11]

* The coefficient of argument 5 being the same, e and $e_{,}$ changing places, that coefficient is not written down, in order to avoid useless repetition.

$$+\left\{-\frac{3}{4}-\frac{3}{4}+\left\{\frac{3}{8}+\frac{3}{8}+\frac{1}{32}+\frac{1}{32}\right\}e^2\right.$$

$$\left.+\left\{\frac{15}{16}+\frac{15}{16}+\frac{27}{32}+\frac{27}{32}\right\}e_{\prime}^2\right\}e\,e_{\prime}\cos(2\,t-x-z)$$

[12] [13]

$$+\left\{\frac{9}{4}+\frac{1}{4}\quad\frac{3}{4}-\frac{3}{4}+\left\{-\frac{9}{8}-\frac{5}{16}+\frac{9}{32}+\frac{3}{8}+\frac{15}{16}\right.\right.$$

$$\left.\left.+\frac{3}{32}-\frac{9}{32}-\frac{3}{32}\right\}(e^2+e_{\prime}^2)\right\}e\,e_{\prime}\cos(x-z)$$

[14]

$$+\left\{\frac{9}{4}+\frac{9}{4}+\left\{-\frac{9}{8}-\frac{9}{8}-\frac{3}{32}-\frac{3}{32}\right\}(e^2+e_{\prime}^2)\right\}e\,e_{\prime}\cos(2\,t-x+z)$$

[15]

$$+\left\{\frac{1}{4}+\frac{1}{4}+\left\{-\frac{5}{16}-\frac{9}{32}-\frac{5}{16}-\frac{9}{32}\right\}(e_{\prime}+e_{\prime}^2)\right\}\cos(2\,t+x-z)$$

[16]

$$+\left\{\frac{1}{3}+\frac{1}{24}-\frac{9}{16}+\frac{1}{16}\right\}e\cos 3\,x$$

[20] [35]

$$+\left\{\frac{1}{24}-\frac{3}{16}\right\}e^3\cos(2\,t-3\,x)$$

[21] [37]

$$+\left\{\frac{1}{3}+\frac{3}{16}\right\}e^3\cos(2\,t+3\,x)$$

[22] [36]

$$+\left\{-\frac{9}{16}+\frac{1}{16}+\frac{9}{8}-\frac{3}{8}+\frac{3}{16}-\frac{3}{16}\right\}e^2\,e_{\prime}\cos(2\,x+z)$$

[23] [29]

$$+\left\{\frac{1}{16}+\frac{9}{8}+\frac{1}{16}\right\}e^2\,e_{\prime}\cos(2\,t-2\,x-z)$$

[24] [31]

$$+\left\{-\frac{9}{16}-\frac{3}{8}-\frac{9}{16}\right\}e^2\,e_{\prime}\cos(2\,t+2\,x+z)$$

[25] [30]

$$+\left\{-\frac{3}{16}+\frac{3}{16}-\frac{3}{8}+\frac{9}{8}-\frac{9}{16}+\frac{1}{16}\right\}e^2\,e_{\prime}\cos(2\,x-z)$$

[26] [32]

$$+\left\{-\frac{3}{16}-\frac{27}{8}-\frac{3}{16}\right\}e^2\,e_{\prime}\cos(2\,t-2\,x+z)$$

[27] [33]

$$+\left\{\frac{3}{16}+\frac{1}{8}+\frac{3}{16}\right\}e^2\,e_{\prime}\cos(2\,t+2\,x-z)$$

[28] [34]

$$+\left\{\frac{125}{384}+\frac{3}{128}-\frac{1}{2}+\frac{1}{48}+\frac{3}{64}\right\}e^4\cos 4\,x$$

[38] [59]

$$+\left\{\frac{1}{128}+\frac{3}{128}-\frac{1}{16}\right\} e^4 \cos(2t-4x)$$
[39] [61]

$$+\left\{\frac{9}{128}+\frac{125}{384}+\frac{1}{6}\right\} e^4 \cos(2t+4x)$$
[40] [60]

$$+\left\{-\frac{1}{2}+\frac{1}{48}+\frac{27}{32}+\frac{1}{32}-\frac{9}{32}+\frac{1}{6}-\frac{3}{32}-\frac{1}{16}\right\} e^3 \cos(3x+z)$$
[41] [53]

$$+\left\{\frac{1}{48}-\frac{3}{32}-\frac{3}{32}+\frac{1}{48}\right\} e^3 e_{\prime} \cos(2t-3x-z)$$
[42] [55]

$$+\left\{-\frac{1}{2}-\frac{9}{32}-\frac{9}{32}-\frac{1}{2}\right\} e^3 e_{\prime} \cos(2t+3x+z)$$
[43] [54]

$$+\left\{-\frac{1}{16}+\frac{1}{6}-\frac{9}{32}-\frac{3}{32}+\frac{27}{32}-\frac{1}{2}+\frac{1}{32}+\frac{1}{48}\right\} e^3 e_{\prime} \cos(3x-z)$$
[44] [56]

$$+\left\{-\frac{1}{16}+\frac{9}{32}+\frac{9}{32}-\frac{1}{16}\right\} e^3 e_{\prime} \cos(2t-3x+z)$$
[45] [57]

$$+\left\{\frac{1}{6}+\frac{3}{32}+\frac{3}{32}+\frac{1}{6}\right\} e^2 e_{\prime} \cos(2t+3x-z)$$
[46] [58]

$$+\left\{\frac{3}{64}+\frac{3}{64}-\frac{3}{32}-\frac{9}{32}+\frac{9}{64}+\frac{1}{64}-\frac{3}{32}+\frac{9}{16}-\frac{9}{32}\right\} e^2 e_{\prime}^2 \cos(2x+2z)$$
[47]

$$+\left\{\frac{9}{32}+\frac{3}{64}+\frac{27}{32}+\frac{3}{64}+\frac{1}{32}\right\} e^2 e_{\prime}^2 \cos(2t-2x+2z)$$
[48]

$$+\left\{\frac{9}{32}+\frac{3}{64}+\frac{1}{32}+\frac{3}{64}+\frac{27}{32}\right\} e^2 e_{\prime}^2 \cos(2t+2x+2z)$$
[49]

$$+\left\{\frac{9}{64}+\frac{1}{64}-\frac{9}{32}-\frac{3}{32}+\frac{3}{64}+\frac{3}{64}-\frac{9}{32}+\frac{9}{16}-\frac{3}{32}\right\} e^2 e_{\prime}^2 \cos(2x-2z)$$
[50]

$$+\left\{\frac{81}{32}+\frac{1}{64}+\frac{9}{32}+\frac{1}{64}+\frac{9}{32}\right\} e^2 e_{\prime}^2 \cos(2t-2x+2z)$$
[51]

$$+\left\{\frac{1}{32}+\frac{9}{64}+\frac{3}{32}+\frac{9}{64}+\frac{3}{32}\right\} e^2 e_{\prime}^2 \cos(2t+2x-2z)$$
[52]

$$+a^2 a_{\prime}^2 \sin^2\frac{\iota}{2}\cos^2\frac{\iota}{2}\left\{\left\{1+\left\{-1-\frac{3}{4}-\frac{3}{4}\right\}e^2+\left\{-1+\frac{9}{4}+\frac{1}{4}\right\}e_{\prime}^2\right\}\cos 2y\right.$$
[62]

$$\left.+\left\{1+\left\{-1+\frac{9}{4}+\frac{1}{4}\right\}e^2+\left\{-1-\frac{3}{4}-\frac{3}{4}\right\}e_{\prime}^2\right\}\cos(2t-2y)\right.$$
[63]

$$+\left\{-\frac{3}{2}-\frac{3}{2}\right\} e \cos (x-2 y) + \left\{\frac{1}{2}+\frac{1}{2}\right\} e \cos (x+2 y)$$
[65] [66]

$$+\left\{\frac{1}{2}-\frac{3}{2}\right\} e \cos (2 t-x-2 y)$$
[67]

$$+\left\{-\frac{3}{2}+\frac{1}{2}\right\} e \cos (2 t+x-2 y)+\left\{-\frac{3}{2}+\frac{1}{2}\right\} e_{,} \cos (z-2 y)$$
[69] [71]

$$+\left\{\frac{1}{2}-\frac{3}{2}\right\} e_{,} \cos (z+2 y)+\left\{\frac{1}{2}+\frac{1}{2}\right\} e_{,} \cos (2 t-z-2 y)$$
[72] [73]

$$+\left\{-\frac{3}{2}-\frac{3}{2}\right\} e_{,} \cos (2 t+z-2 y)+\left\{\frac{1}{8}+\frac{9}{4}+\frac{1}{8}\right\} e^{2} \cos (2 x-2\ \ )$$
[75] [77]

$$+\left\{\frac{3}{8}+\frac{1}{4}+\frac{3}{8}\right\} e^{2} \cos (2 x+2 y)$$
[78]

$$+\left\{\frac{3}{8}-\frac{3}{4}+\frac{1}{8}\right\} e^{2} \cos (2 t-2 x-2 y)+\left\{\frac{1}{8}-\frac{3}{4}+\frac{3}{8}\right\} e^{2} \cos (2 t+2 x-2 y)$$
[79] [81]

$$+\left\{\frac{9}{4}+\frac{9}{4}-\frac{3}{4}-\frac{3}{4}\right\} e e_{,} \cos (x+z-2 y)$$
[83]

$$+\left\{\frac{11}{4}+\frac{1}{4}-\frac{3}{4}-\frac{3}{4}\right\} e e_{,} \cos (x+z+2 y)$$
[84]

$$+\left\{\frac{1}{4}-\frac{3}{4}+\frac{1}{4}-\frac{3}{4}\right\} e e_{,} \cos (2 t-x-z-2 y)$$
[85]

$$+\left\{\frac{9}{4}-\frac{3}{4}+\frac{9}{4}-\frac{3}{4}\right\} e e_{,} \cos (2 t+x+z+2 y)$$
[87]

$$+\left\{-\frac{3}{4}-\frac{3}{4}+\frac{9}{4}+\frac{9}{4}\right\} e e_{,} \cos (x-z-2 y)$$
[89]

$$+\left\{-\frac{3}{4}-\frac{3}{4}+\frac{1}{4}+\frac{1}{4}\right\} e e_{,} \cos (x-z+2 y)$$
[90]

$$+\left\{-\frac{3}{4}+\frac{9}{4}-\frac{3}{4}+\frac{9}{4}\right\} e e_{,} \cos (2 t-x+z-2 y$$
[91]

$$+\left\{-\frac{3}{4}+\frac{1}{4}-\frac{3}{4}+\frac{1}{4}\right\} e e_{,} \cos (2 t+x-z-2 y$$
[93]

$$+\left\{-\frac{3}{4}+\frac{3}{8}\right\}e_{\prime}^{2}\cos(2z-2y)+\left\{-\frac{3}{4}+\frac{1}{8}\right\}e_{\prime}^{2}\cos(2z+2y)$$
[95] [96]

$$+\left\{\frac{1}{4}+\frac{3}{8}\right\}e_{\prime}^{2}\cos(2t-2z-2y)+\left\{\frac{9}{4}+\frac{1}{8}\right\}e_{\prime}^{2}\cos(2t+2z-2y)$$
[97] [99]

$$+a^2a_{\prime}^{2}\sin^4\frac{\iota}{2}\left\{\frac{1}{2}+\frac{1}{2}\cos(2t-2y)\right\}$$
[63]

$r'^2r^2\cos(\lambda'-\lambda)^2$

$$=a^2a_{\prime}^{2}\cos^4\frac{\iota}{2}\left\{\frac{1}{2}+\frac{3}{4}(e^2+e_{\prime}^{2})+\frac{9}{8}e^2e_{\prime}^{2}+\left\{\frac{1}{2}-\frac{5}{4}(e^2+e_{\prime}^{2})\right.\right.$$

$$\left.+\frac{23}{32}(e^4+e_{\prime}^{4})+\frac{25}{8}e^2e_{\prime}^{2}\right\}\cos 2t+\left\{-1+\frac{e^2}{8}-\frac{3}{2}e_{\prime}^{2}\right\}e\cos x$$
[1] [2] [5]

$$+\left\{-\frac{3}{2}+\frac{13}{16}e^2+\frac{15}{4}e_{\prime}^{2}\right\}e\cos(2t-x)+\left\{\frac{1}{2}-\frac{19}{16}e^2-\frac{5}{4}e_{\prime}^{2}\right\}e\cos(2t+x)$$
[3] [7] [4] [6]

$$+\left\{-\frac{1}{4}+\frac{1}{12}e^2-\frac{3}{8}e_{\prime}^{2}\right\}e^2\cos 2x$$
[8] [17]

$$+\left\{\frac{5}{4}-\frac{25}{8}e_{\prime}^{2}\right\}e^2\cos(2t-2x)+\left\{\frac{1}{2}-\frac{5}{4}e^2-\frac{5}{4}e_{\prime}^{2}\right\}e\cos(2t+2x)$$
[9] [19] [10] [18]

$$+\left\{1-\frac{1}{8}(e^2+e_{\prime}^{2})\right\}e\,e_{\prime}\cos(x+z)$$
[11]

$$+\left\{-\frac{3}{2}+\frac{13}{16}e^2+\frac{57}{16}e_{\prime}^{2}\right\}e\,e_{\prime}\cos(2t-x-z)$$
[12] [13]

$$+\left\{1-\frac{(e^2+e_{\prime}^{2})}{8}\right\}e\,e_{\prime}\cos(x-z)$$
[14]

$$+\left\{\frac{9}{2}-\frac{39}{16}(e^2+e_{\prime}^{2})\right\}e\,e_{\prime}\cos(2t-x+z)$$
[15]

$$+\left\{\frac{1}{2}-\frac{19}{16}(e^2+e_{\prime}^{2})\right\}e\,e_{\prime}\cos(2t+x-z)-\frac{e^3}{8}\cos 3x$$
[16 [20] [35]

$$-\frac{7e^3}{48}\cos(2t-3x)+\frac{25e^3}{48}\cos(2t+3x)+\frac{e^2e_{\prime}}{4}\cos(2x+z)$$
[21] [37] [22] [36] [23] [29]

$$+ \frac{5}{4} e^2 e_{\prime} \cos (2t - 2x - z) - \frac{25}{16} e^2 e_{\prime} \cos (2t + 2x + z) + \frac{e^2 e_{\prime}}{4} \cos (2x - z)$$
[24] [31] [25] [30] [26] [32]

$$- \frac{15}{4} e^2 e_{\prime} \cos (2t - 2x + z) + \frac{e^2 e_{\prime}}{2} \cos (2t + 2x - z)$$
[27] [33] [28] [34]

$$- \frac{e^4}{12} \cos 4x - \frac{e^4}{32} \cos (2t - 4x) + \frac{9}{16} e^4 \cos (2t + 4x) + \frac{e^3 e_{\prime}}{8} \cos (3x + z)$$
[38] [59] [39] [61] [40] [60] [41] [53]

$$- \frac{7}{48} e^3 e_{\prime} \cos (2t - 3x - z) - \frac{25}{16} e^2 e_{\prime} \cos (2t + 3x + z)$$
[42] [55] [43] [54]

$$+ \frac{e^3 e_{\prime}}{8} \cos (3x - z) + \frac{7}{16} e^3 e_{\prime} \cos (2t - 3x + z) + \frac{25}{48} e^3 e_{\prime} \cos (2t + 3x - z)$$
[44] [56] [45] [57] [46] [58]

$$+ \frac{e^2 e_{\prime}}{16} \cos (2x + 2z) + \frac{5}{4} e^2 e_{\prime}^2 \cos (2t - 2x - 2z)$$
[47] [48]

$$+ \frac{5}{4} e^2 e_{\prime} \cos (2t + 2x + 2z) + \frac{e^2 e_{\prime}^2}{16} \cos (2x - 2z)$$
[49] [50]

$$+ \frac{25}{8} e^2 e_{\prime}^2 \cos (2t - 2x + 2z) + \frac{e^2 e_{\prime}^2}{2} \cos (2t + 2x - 2z)$$
[51] [52]

$$+ a^2 a_{\prime}^2 \cos^2 \frac{\iota}{2} \sin^2 \frac{\iota}{2} \left\{ \left\{ 1 - \frac{5}{2} e^2 + \frac{3}{2} e_{\prime}^2 \right\} \cos 2y + \left\{ 1 + \frac{3}{2} e^2 - \frac{5}{2} e_{\prime}^2 \right\} \cos (2t - 2y) \right.$$
[62] [63]

$$- 3e \cos (x - 2y) + e \cos (x + 2y) - e \cos (2t - x - 2y)$$
[65] [66] [67]

$$- e \cos (2t + x - 2y) - e_{\prime} \cos (z - 2y) - e_{\prime} \cos (z + 2y)$$
[69] [71] [72]

$$+ e_{\prime} \cos (2t - z - 2y) - 3 e_{\prime} \cos (2t + z - 2y)$$
[73] [75]

$$+ \frac{5}{2} e^2 \cos (2x - 2y) + e^2 \cos (2x + 2y)$$
[77] [78]

$$- \frac{e^2}{4} \cos (2t - 2x - 2y) - \frac{e^2}{4} \cos (2t + 2x - 2y)$$
[79] [81]

$$+ 3\,e\,e_{\prime} \cos(x + z - 2\,y) - e\,e_{\prime} \cos(x + z + 2\,y)$$
[83] [84]

$$- e\,e_{\prime} \cos(2\,t - x - z - 2\,y) + 3\,e\,e_{\prime} \cos(2\,t + x + z - 2\,y)$$
[85] [87]

$$+ 3\,e\,e_{\prime} \cos(x - z - 2\,y) - e\,e_{\prime} \cos(x - z + 2\,y)$$
[89] [90]

$$+ 3\,e\,e_{\prime} \cos(2\,t - x + z - 2\,y) - e\,e_{\prime} \cos(2\,t + x - z - 2\,y)$$
[91] [93]

$$- \frac{3}{8} e_{\prime}^2 \cos(2\,z - 2\,y) - \frac{5}{8} e_{\prime}^2 \cos(2\,z + 2\,y)$$
[95] [96]

$$\left. + \frac{5\,e_{\prime}^2}{8} \cos(2\,t - 2\,z - 2\,y) + \frac{19}{8} e_{\prime}^2 \cos(2\,t + 2\,z - 2\,y) \right\}$$
[97] [99]

$$+ a^2 a_{\prime}^2 \sin^4 \frac{\iota}{2} \left\{ \frac{1}{2} + \frac{1}{2} \cos(2\,t - 2\,y) \right\}$$
[63]

$$\frac{r^2}{2\,r_{\prime}^3} = \frac{a^2}{a_{\prime}^3} \left\{ \frac{1}{2} + \frac{3}{4} e^2 + \frac{3}{4} e_{\prime}^2 + \frac{9}{8} e^2 e_{\prime}^2 + \frac{15}{16} e_{\prime}^4 - e \left\{ 1 - \frac{e^2}{8} + \frac{3}{2} e_{\prime}^2 \right\} \cos x \right.$$
[2]

$$+ \frac{3}{2} e_{\prime} \left\{ 1 + \frac{3}{2} e^2 + \frac{9}{8} e_{\prime}^2 \right\} \cos z - \frac{e^2}{4} \left\{ 1 - \frac{e^2}{3} + \frac{3\,e_{\prime}^2}{2} \right\} \cos 2\,x$$
[5] [8]

$$- \frac{3}{2} e\,e_{\prime} \left\{ 1 - \frac{e^2}{8} + \frac{9}{8} e_{\prime}^2 \right\} \cos(x + z)$$
[11]

$$- \frac{3}{2} e\,e_{\prime} \left\{ 1 - \frac{e^2}{8} + \frac{9}{8} e_{\prime}^2 \right\} \cos(x - z)$$
[14]

$$+ \frac{9}{4} e_{\prime}^2 \left\{ 1 + \frac{7}{9} e_{\prime}^2 + \frac{3}{2} e^2 \right\} \cos 2\,z - \frac{e^3}{8} \cos 3\,x - \frac{3}{8} e^2 e_{\prime} \cos(2\,x + z)$$
[17] [20] [23]

$$- \frac{3}{8} e^2 e_{\prime} \cos(2\,x - z) - \frac{9}{4} e\,e_{\prime}^2 \cos(x + 2\,z) - \frac{9}{4} e\,e_{\prime}^2 \cos(x - 2\,z)$$
[26] [29] [32]

$$+ \frac{53}{16} e_{\prime}^3 \cos 3\,z - \frac{e^4}{12} \cos 4\,x - \frac{3\,e^3 e_{\prime}}{16} \cos(3\,x + z)$$
[35] [38] [41]

$$-\frac{3}{16}e^3 e_, \cos(3x-z) - \frac{9}{16}e^2 e_,^2 \cos(2x+2z) - \frac{9}{16}e^2 e_, \cos(2x-2z)$$
[44] [47] [50]

$$-\frac{53}{16}e e_,^3 \cos(x+3z) - \frac{53}{16}e e_,^3 \cos(x-3z) + \frac{77}{16}e_,^4 \cos 4z$$
[53] [56] [59]

Terms in R multiplied by $-\frac{3}{2}\cos^4\frac{\iota}{2}\frac{a^2}{a_,^3}$

$$=^* \left\{\frac{1}{2}+\frac{3}{4}(e^2+e_,^2)+\frac{9}{8}e^2 e_,^2\right\}\left\{1+5e_,^2+\frac{105}{8}e_,^4\right\}$$
[0]

$$+\left\{-1+\frac{e_,^2}{8}-\frac{3}{2}e^2\right\}\left\{\frac{5}{2}e_,^2+\frac{135}{16}e_,^4\right\}-\frac{5}{4}e_,^4$$

$$+\left\{\left\{\frac{1}{2}-\frac{5}{4}(e^2+e_,^2)+\frac{23}{32}(e^4+e_,^4)+\frac{25}{8}e^2 e_,^2\right\}\left\{1+5e_,^2+\frac{105}{8}e_,^4\right\}\right.$$

$$+\left\{\frac{1}{2}-\frac{19}{16}e_,^2-\frac{5}{4}e^2-\frac{3}{2}+\frac{13}{16}e_,^2+\frac{15}{4}e^2\right\}\left\{\frac{5}{2}e_,^2+\frac{135}{16}e_,^4\right\}$$

$$\left.+\left\{\frac{5}{2}+\frac{25}{4}\right\}e_,^4\right\}\cos 2t$$
[1]

$$+\left\{-1+\frac{e^2}{8}-\frac{3}{2}e_,^2-5e_,^2+\frac{5}{2}e_,^2+\frac{5}{2}e_,^2\right\}e\cos x$$
[2]

$$+\left\{-\frac{3}{2}+\frac{13}{16}e^2+\frac{15}{4}e_,^2-\frac{15}{2}e_,^2-\frac{15}{4}e_,^2+\frac{45}{4}e_,^2\right\}e\cos(2t-x)$$
[3]

$$+\left\{\frac{1}{2}-\frac{19}{16}e^2-\frac{5}{4}e_,^2+\frac{5}{2}e_,^2+\frac{5}{4}e_,^2-\frac{15}{4}e_,^2\right\}e\cos(2t+x)$$
[4]

$$+\left\{-1+\frac{e_,^2}{8}-\frac{3}{2}e^2-5e_,^2+\frac{5}{2}+\frac{15}{4}e^2+\frac{15}{4}e_,^2+\frac{135}{16}e_,^2-\frac{5}{8}e_,^2-5e_,^2\right\}e_,\cos z$$
[5]

$$+\left\{\frac{1}{2}-\frac{19}{16}e_,^2-\frac{5}{4}e^2+\frac{5}{2}e_,^2+\frac{5}{4}e_,^2+\frac{5}{4}-\frac{25}{8}e^2-\frac{25}{8}e_,^2\right.$$

$$\left.+\frac{135}{32}e_,^2-\frac{15}{2}e_,^2\right\}e_,\cos(2t-z)$$
[6]

* This multiplication of $r^2 r_,^2 \cos(\lambda' - \lambda_,)^2$ by $r_,^5$ may be effected at once by means of Table II.

$$+\left\{-\frac{3}{2}+\frac{13}{16}e_{\prime}^2+\frac{15}{4}e^2-\frac{15}{2}e_{\prime}^2+\frac{5}{4}-\frac{25}{8}e^2-\frac{25}{8}e_{\prime}^2+\frac{135}{32}e_{\prime}^2+\frac{25}{8}e_{\prime}^2+\frac{5}{2}e_{\prime}^2\right\}e_{\prime}\cos(2t+z) \quad [7]$$

$$+\left\{-\frac{1}{4}+\frac{e^2}{12}-\frac{3}{8}e_{\prime}^2-\frac{5}{4}e_{\prime}^2+\frac{5}{8}e_{\prime}^2+\frac{5}{8}e_{\prime}^2\right\}e^2\cos 2x \quad [8]$$

$$+\left\{\frac{5}{4}-\frac{25}{8}e_{\prime}^2+\frac{25}{4}e_{\prime}^2+\frac{25}{8}e_{\prime}^2-\frac{75}{8}e_{\prime}^2\right\}e^2\cos(2t-2x) \quad [9]$$

$$+\left\{\frac{1}{2}-\frac{5}{4}e^2-\frac{5}{4}e_{\prime}^2+\frac{5}{2}e_{\prime}^2+\frac{5}{4}e_{\prime}^2-\frac{15}{4}e_{\prime}^2\right\}e^2\cos(2t+2x) \quad [10]$$

$$+\left\{1-\frac{e^2}{8}-\frac{e_{\prime}^2}{8}+5e_{\prime}^2-\frac{5}{2}+\frac{5e^2}{16}-\frac{15}{4}e_{\prime}^2-\frac{135}{16}e_{\prime}^2+\frac{5e_{\prime}^2}{8}+5e_{\prime}^2\right\}e\,e_{\prime}\cos(x+z) \quad [11]$$

$$+\left\{-\frac{3}{2}+\frac{13}{16}e^2+\frac{57}{16}e_{\prime}^2-\frac{15}{2}e_{\prime}^2-\frac{15}{4}e_{\prime}^2-\frac{15}{4}+\frac{65}{32}e^2+\frac{75}{8}e_{\prime}^2\right.$$

$$\left.-\frac{405}{32}e_{\prime}^2+\frac{45}{2}e_{\prime}^2\right\}e\,e_{\prime}\cos(2t\quad x-z) \quad [12] \quad +\left\{-\frac{3}{2}+\frac{13}{16}e_{\prime}^2+\frac{57}{16}e^2-\frac{15}{2}e_{\prime}^2+\frac{5}{4}\right.$$

$$\left.-\frac{95}{32}e^2-\frac{25}{8}e_{\prime}^2+\frac{135}{32}e_{\prime}^2+\frac{25}{8}e_{\prime}^2+\frac{5}{2}e_{\prime}^2\right\}e\,e_{\prime}\cos(2t+x+z) \quad [13]$$

$$+\left\{1-\frac{e^2}{8}-\frac{e_{\prime}^2}{8}+5e_{\prime}^2+\frac{5e_{\prime}^2}{8}-\frac{5}{2}+\frac{5}{16}e^2-\frac{15}{4}e_{\prime}^2-\frac{135}{16}e_{\prime}^2+5e_{\prime}^2\right\}e\,e_{\prime}\cos(x-z) \quad [14]$$

$$+\left\{\frac{9}{2}-\frac{39}{16}e^2-\frac{39}{16}e_{\prime}^2+\frac{45}{2}e_{\prime}^2-\frac{15}{4}+\frac{65}{32}e^2+\frac{75}{8}e_{\prime}^2-\frac{405}{32}e_{\prime}^2\right.$$

$$\left.-\frac{75}{8}e_{\prime}^2-\frac{15}{2}e_{\prime}^2\right\}e\,e_{\prime}\cos(2t-x+z) \quad [15]$$

$$+\left\{\frac{1}{2}-\frac{19}{16}e^2-\frac{19}{16}e_{\prime}^2+\frac{5}{2}e_{\prime}^2+\frac{5}{4}e_{\prime}^2+\frac{5}{4}-\frac{95}{32}e^2-\frac{25}{8}e_{\prime}^2+\frac{135}{32}e_{\prime}^2-\frac{15}{2}e_{\prime}^2\right\}e\,e_{\prime}\cos(2t+x-z) \quad [16]$$

$$+\left\{-\frac{1}{4}+\frac{e_{\prime}^2}{12}-\frac{3}{8}e^2-\frac{5}{4}e_{\prime}^2-\frac{5}{2}+\frac{5}{16}e_{\prime}^2-\frac{15}{4}e^2-\frac{135}{16}e_{\prime}^2-\frac{5}{16}e_{\prime}^2+5\right.$$

$$\left.+\frac{15}{2}e^2+\frac{15}{2}e_{\prime}^2+\frac{155}{12}e_{\prime}^2-\frac{145}{16}e_{\prime}^2\right\}e_{\prime}^2\cos 2z \quad [17]$$

$$+\left\{\frac{1}{2}-\frac{5}{4}e_{\prime}^2-\frac{5}{4}e^2+\frac{5}{2}e_{\prime}^2+\frac{125}{96}e_{\prime}^2+\frac{5}{4}-\frac{95}{32}e_{\prime}^2-\frac{25}{8}e^2+\frac{135}{32}e_{\prime}^2+\frac{5}{2}-\frac{25}{4}e^2-\frac{25}{4}e_{\prime}^2\right.$$

$$\left.+\frac{155}{24}e_{\prime}^2-\frac{435}{32}e_{\prime}^2\right\}e_{\prime}^2\cos(2t-2z) \quad [18]$$

$$+\left\{\frac{5}{4}-\frac{25}{8}e^2+\frac{25}{4}e_{,}^2-\frac{15}{4}+\frac{65}{32}e_{,}^2+\frac{75}{8}e^2-\frac{405}{32}e_{,}^2-\frac{35}{96}e_{,}^2+\frac{5}{2}-\frac{25}{4}e^2-\frac{25}{4}e_{,}^2\right.$$

$$\left.+\frac{155}{24}e_{,}^2+\frac{145}{32}e_{,}^2\right\}e_{,}^2\cos(2t+2z)$$

[19]

$$-\frac{e^3}{8}\cos 3x-\frac{7}{48}e^3\cos(2t-3x)+\frac{25}{48}e^3\cos(2t+3x)+\left\{\frac{1}{4}-\frac{5}{8}\right\}e^2e_{,}\cos(2x+z)$$

[20] [21] [22] [23]

$$+\left\{\frac{5}{4}+\frac{25}{8}\right\}e^2e_{,}\cos(2t-2x-z)-\left\{-\frac{3}{2}+\frac{5}{4}\right\}e^2e_{,}\cos(2t+2x+z)$$

[24] [25]

$$+\left\{\frac{1}{4}\quad\frac{5}{8}\right\}e^2e_{,}\cos(2x-z)+\left\{-\frac{15}{4}+\frac{25}{8}\right\}e^2e_{,}\cos(2t-2x+z)$$

[26] [27]

$$+\left\{\frac{1}{2}+\frac{5}{4}\right\}e^2e_{,}\cos(2t+2x-z)+\left\{\frac{1}{4}+\frac{5}{2}-5\right\}e\,e_{,}^2\cos(x+2z)$$

[28] [29]

$$+\left\{-\frac{3}{2}-\frac{15}{4}-\frac{15}{2}\right\}e\,e_{,}^2\cos(2t-x-2z)$$

[30]

$$+\left\{\frac{5}{4}-\frac{15}{4}+\frac{5}{2}\right\}e\,e_{,}^2\cos(2t+x+2z)+\left\{\frac{1}{4}+\frac{5}{2}-5\right\}e\,e_{,}^2\cos(x-2z)$$

[31] [32]

$$+\left\{-\frac{15}{4}+\frac{45}{4}-\frac{15}{2}\right\}e\,e_{,}^2\cos(2t-x+2z)+\left\{\frac{1}{2}+\frac{5}{4}+\frac{5}{2}\right\}e\,e_{,}^2\cos(2t+x-2z)$$

[33] [34]

$$+\left\{-\frac{1}{8}-\frac{5}{8}-5+\frac{145}{16}\right\}e_{,}^3\cos 3z+\left\{\frac{25}{48}+\frac{5}{4}+\frac{5}{2}+\frac{145}{32}\right\}e_{,}^3\cos(2t-3z)$$

[35] [36]

$$+\left\{-\frac{7}{48}+\frac{25}{8}-\frac{15}{2}+\frac{145}{32}\right\}e_{,}^3\cos(2t+3z)$$

[37]

$$-\frac{e^4}{12}\cos 4x-\frac{e^4}{32}\cos(2t-4x)+\frac{9}{16}e^4\cos(2t+4x)$$

[38] [39] [40]

$$+\left\{\frac{1}{8}-\frac{5}{16}\right\}e^3e_{,}\cos(3x+z)+\left\{-\frac{7}{48}-\frac{35}{96}\right\}e^3e_{,}\cos(2t-3x-z)$$

[41] [42]

$$+\left\{-\frac{25}{16}+\frac{125}{96}\right\}e^3e_{,}\cos(2t-3x-z)+\left\{\frac{1}{8}-\frac{5}{16}\right\}e^3e_{,}\cos(3x-z)$$

[43] [44]

$$+ \left\{ \frac{7}{16} - \frac{35}{96} \right\} e^3 e_{\prime} \cos (2t - 3x + z)$$
[45]

$$+ \left\{ \frac{25}{48} + \frac{125}{96} \right\} e^3 e_{\prime} \cos (2t + 3x - z) + \left\{ \frac{1}{16} + \frac{5}{8} - \frac{5}{4} \right\} e^2 e_{\prime}^2 \cos (2x + 2z)$$
[46] [47]

$$+ \left\{ \frac{5}{4} + \frac{25}{8} + \frac{25}{4} \right\} e^2 e_{\prime}^2 \cos (2t - 2x - 2z) + \left\{ \frac{5}{4} - \frac{15}{4} + \frac{5}{2} \right\} e^2 e_{\prime}^2 \cos (2t + 2x + 2z)$$
[48] [49]

$$+ \left\{ \frac{1}{16} + \frac{5}{8} - \frac{5}{4} \right\} e^2 e_{\prime}^2 \cos (2x - 2z) + \left\{ \frac{25}{8} - \frac{75}{8} + \frac{25}{4} \right\} e^2 e_{\prime}^2 \cos (2t - 2x + 2z)$$
[50] [51]

$$+ \left\{ \frac{1}{2} + \frac{5}{4} + \frac{5}{2} \right\} e^2 e_{\prime}^2 \cos (2t + 2x - 2z) + \left\{ \frac{1}{8} + \frac{5}{8} + 5 - \frac{145}{16} \right\} e e_{\prime}^3 \cos (x + 3z)$$
[52] [53]

$$+ \left\{ -\frac{25}{16} - \frac{15}{4} - \frac{15}{2} - \frac{435}{32} \right\} e e_{\prime}^3 \cos (2t - x - 3z)$$
[54]

$$+ \left\{ -\frac{7}{48} + \frac{25}{8} - \frac{15}{2} + \frac{145}{32} \right\} e e_{\prime}^3 \cos (2t + x + 3z)$$
[55]

$$+ \left\{ \frac{1}{8} + \frac{5}{8} + 5 - \frac{145}{16} \right\} e e_{\prime}^3 \cos (x - 3z) + \left\{ \frac{7}{16} - \frac{75}{8} + \frac{45}{2} - \frac{435}{32} \right\} e e_{\prime}^3 \cos (2t - x + 3z)$$
[56] [57]

$$+ \left\{ \frac{25}{48} + \frac{5}{4} + \frac{5}{2} - \frac{145}{32} \right\} e e_{\prime}^3 \cos (2t + x - 3z) - \left\{ -\frac{1}{12} - \frac{5}{16} - \frac{5}{4} - \frac{145}{16} + \frac{745}{96} \right\} e_{\prime}^4 \cos 4z$$
[58] [59]

$$+ \left\{ \frac{9}{16} + \frac{125}{96} + \frac{5}{2} + \frac{145}{32} + \frac{745}{192} \right\} e_{\prime}^4 \cos (2t - 4z)$$
[60]

$$+ \left\{ -\frac{1}{32} - \frac{35}{96} + \frac{25}{4} - \frac{435}{32} + \frac{745}{192} \right\} e_{\prime}^4 \cos (2t + 4z)$$
[61]

Terms in R multiplied by $-\frac{3}{2} \sin^2 \frac{\iota}{2} \cos^2 \frac{\iota}{2} \frac{a^2}{a_{\prime}^3}$

$$= \left\{ 1 - \frac{5}{2} e^2 + \frac{3}{2} e_{\prime}^2 + 5 e_{\prime}^2 - \frac{5}{2} e_{\prime}^2 - \frac{5}{2} e_{\prime}^2 \right\} \cos 2y$$
[62]

$$+ \left\{ 1 + \frac{3}{2} e^2 - \frac{5}{2} e_{\prime}^2 + 5 e_{\prime}^2 + \frac{5}{2} e_{\prime}^2 - \frac{15}{2} e_{\prime}^2 \right\} \cos (2t - 2y)$$
[63]

$$- 3e \cos (x - 2y) + e \cos (x + 2y) - e \cos (2t - x - 2y) - e \cos (2t + x - 2y)$$
[65] [66] [67] [69]

$$+\left\{-1+\frac{5}{2}\right\}e_{\prime}\cos(z-2y)+\left\{-1+\frac{5}{2}\right\}e_{\prime}\cos(z+2y)+\left\{1+\frac{5}{2}\right\}e_{\prime}\cos(2t-z-2y)$$
[71] [72] [73]

$$+\left\{-3+\frac{5}{2}\right\}e_{\prime}\cos(2t+z-2y)+\frac{5}{2}e^2\cos(2x-2y)+e^2\cos(2x+2y)$$
[75] [77] [78]

$$-\frac{e^2}{4}\cos(2t-2x-2y)-\frac{e^2}{4}\cos(2t+2x-2y)$$
[79] [81]

$$+\left\{3-\frac{15}{2}\right\}e\,e_{\prime}\cos(x+z-2y)+\left\{-1+\frac{5}{2}\right\}e\,e_{\prime}\cos(x+z+2y)$$
[83] [84]

$$+\left\{-1-\frac{5}{2}\right\}e\,e_{\prime}\cos(2t-x-z-2y)$$
[85]

$$+\left\{3-\frac{5}{2}\right\}e\,e_{\prime}\cos(2t+x+z-2y)+\left\{3-\frac{15}{2}\right\}e\,e_{\prime}\cos(2t-z-2y)$$
[87] [89]

$$+\left\{-1+\frac{5}{2}\right\}e\,e_{\prime}\cos(x-z+2y)+\left\{3-\frac{5}{2}\right\}e\,e_{\prime}\cos(2t-x+z-2y)$$
[90] [91]

$$+\left\{-1-\frac{5}{2}\right\}e\,e_{\prime}\cos(2t+x-z-2y)+\left\{-\frac{3}{8}-\frac{5}{2}+5\right\}e_{\prime}^2\cos(2z-2y)$$
[93] [95]

$$+\left\{-\frac{5}{8}-\frac{5}{2}+5\right\}e_{\prime}^2\cos(2z+2y)+\left\{\frac{5}{8}+\frac{5}{2}+5\right\}e_{\prime}^2\cos(2t-2z-2y)$$
[96] [97]

$$+\left\{\frac{19}{8}-\frac{15}{2}+5\right\}e_{\prime}^2\cos(2t+2z-2y)$$
[99]

Terms in R multiplied by $-\frac{3}{2}\cos^4\frac{\iota}{2}\frac{a^2}{a_{\prime}^3}$

$$=\frac{1}{2}+\frac{3}{4}e^2+\frac{3}{4}e_{\prime}^2+\frac{15}{16}e_{\prime}^4+\frac{9}{8}e^2e_{\prime}^2+\left\{\frac{1}{2}-\frac{5}{4}e^2-\frac{5}{4}e_{\prime}^2+\frac{23}{32}e^4+\frac{13}{32}e_{\prime}^4+\frac{25}{8}e^2e_{\prime}^2\right\}\cos 2t$$
[1]

$$+\left\{-1+\frac{e^2}{8}-\frac{3}{2}e_{\prime}^2\right\}e\cos x+\left\{-\frac{3}{2}+\frac{13}{16}e^2+\frac{15}{4}e_{\prime}^2\right\}e\cos(2t-x)$$
[2] [3]

$$+\left\{\frac{1}{2}-\frac{19}{16}e^2-\frac{5}{4}e_{\prime}^2\right\}e\cos(2t+x)+\left\{\frac{3}{2}+\frac{9}{4}e^2+\frac{27}{16}e_{\prime}^2\right\}e_{\prime}\cos z$$
[4] [5]

$$+\left\{\frac{7}{4}-\frac{35}{8}e^2-\frac{123}{32}e_{\prime}^2\right\}e_{\prime}\cos(2t-z)$$
[6]

$$+\left\{-\frac{1}{4}+\frac{5}{8}e^2+e_{\prime}^2\right\}e_{\prime}\cos(2t+z)+\left\{-\frac{1}{4}+\frac{e^2}{12}-\frac{3}{8}e_{\prime}^2\right\}e^2\cos 2x$$
[7] [8]

$$+\left\{\frac{5}{4}-\frac{25}{8}e_{\prime}^2\right\}e^2\cos(2t-2x)+\left\{\frac{1}{2}-\frac{5}{4}e^2-\frac{5}{4}e_{\prime}^2\right\}e^2\cos(2t+2x)$$
[9] [10]

$$+\left\{-\frac{3}{2}+\frac{3}{16}e^2-\frac{27}{16}e_{\prime}^2\right\}e\,e_{\prime}\cos(x+z)+\left\{-\frac{21}{4}+\frac{91}{32}e^2+\frac{369}{32}e_{\prime}^2\right\}e\,e_{\prime}\cos(2t-x-z)$$
[11] [12]

$$+\left\{-\frac{1}{4}+\frac{19}{32}e^2+\frac{e_{\prime}^2}{32}\right\}e\,e_{\prime}\cos(2t+x+z)+\left\{-\frac{3}{2}+\frac{3}{16}e^2-\frac{27}{16}e_{\prime}^2\right\}e\,e_{\prime}\cos(x-z)$$
[13] [14]

$$+\left\{\frac{3}{4}-\frac{13}{32}e^2-\frac{3}{32}e_{\prime}^2\right\}e\,e_{\prime}\cos(2t-x+z)+\left\{\frac{7}{4}-\frac{133}{32}e^2-\frac{123}{32}e_{\prime}^2\right\}e\,e_{\prime}\cos(2t+x-z)$$
[15] [16]

$$+\left\{\frac{9}{4}+\frac{27}{8}e^2+\frac{21}{12}e_{\prime}^2\right\}e_{\prime}^2\cos 2z+\left\{\frac{17}{4}-\frac{85}{8}e^2-\frac{115}{12}e_{\prime}^2\right\}e_{\prime}^2\cos(2t-2z)^*-\frac{e^3}{8}\cos 3x$$
[17] [18] [20]

$$-\frac{7}{48}e^3\cos(2t-3x)+\frac{25}{48}e^3\cos(2t+3x)-\frac{3}{8}e^2e_{\prime}\cos(2x+z)$$
[21] [22] [23]

$$+\frac{35}{8}e^2e_{\prime}\cos(2t-2x-z)-\frac{e^2e_{\prime}}{4}\cos(2t+2x+z)-\frac{3}{8}e^2e_{\prime}\cos(2x-z)$$
[24] [25] [26]

$$-\frac{5}{8}e^2e_{\prime}\cos(2t-2x+z)+\frac{7}{4}e^2e_{\prime}\cos(2t+2x-z)$$
[27] [28]

$$-\frac{9}{4}e\,e_{\prime}^2\cos(x+2z)-\frac{51}{4}e\,e_{\prime}^2\cos(2t-x-2z)$$
[29] [30]

$$-\frac{9}{4}e\,e_{\prime}^2\cos(x-2z)+\frac{17}{4}e\,e_{\prime}^2\cos(2t+x-2z)$$
[32] [34]

$$+\frac{53}{16}e_{\prime}^3\cos 3z+\frac{845}{96}e_{\prime}^3\cos(2t-3z)+\frac{e^3e_{\prime}}{96}\cos(2t+3z)-\frac{e^4}{12}\cos 4x-\frac{e^4}{32}\cos(2t-4x)$$
[35] [36] [37] [38] [39]

* It is remarkable that the coefficient of argument 19 equals zero.

$$+ \underset{[40]}{\frac{9}{16} e^4 \cos (2t + 4x)} - \underset{[41]}{\frac{3}{16} e^3 e_{\prime} \cos (3x + z)}$$

$$- \underset{[42]}{\frac{49}{96} e^3 e_{\prime} \cos (2t + 3x - z)} - \underset{[43]}{\frac{25}{96} e^3 e_{\prime} \cos (2t - 3x - z)} - \underset{[44]}{\frac{3}{16} e^3 e_{\prime} \cos (3x - z)}$$

$$+ \underset{[45]}{\frac{7}{96} e^3 e_{\prime} \cos (2t - 3x + z)} + \underset{[46]}{\frac{175}{96} e^3 e_{\prime} \cos (2t + 3x - z)} - \underset{[47]}{\frac{9}{16} e^2 e_{\prime}^2 \cos (2x + 2z)}$$

$$+ \underset{[48]}{\frac{85}{8} e^2 e_{\prime}^2 \cos (2t - 2x - 2z)} - \underset{[50]}{\frac{9}{16} e^2 e_{\prime}^2 \cos (2x - 2z)} + \underset{[52]}{\frac{17}{4} e^2 e_{\prime}^2 \cos (2t + 2x - 2z)}$$

$$- \underset{[53]}{\frac{53}{16} e e_{\prime}^3 \cos (x + 3z)} - \underset{[54]}{\frac{845}{32} e e_{\prime}^3 \cos (2t \quad x - 3z)} + \underset{[55]}{\frac{e e_{\prime}^3}{96} \cos (2t + x - 3z)}$$

$$- \underset{[56]}{\frac{53}{16} e e_{\prime}^3 \cos (x - 3z)} - \underset{[57]}{\frac{e e_{\prime}^3}{32} \cos (2t - x + 3z)} - \underset{[58]}{\frac{25}{96} e e_{\prime}^3 \cos (2t + x - 3z)}$$

$$- \underset{[59]}{\frac{283}{96} e_{\prime}^4 \cos 4z} + \underset{[60]}{\frac{2453}{192} e_{\prime}^4 \cos (2t - 4z)} - \underset{[61]}{\frac{741}{192} e_{\prime}^4 \cos (2t + 4z)}$$

Terms in R multiplied by $-\frac{3}{2} \sin^2 \frac{\iota}{2} \cos^2 \frac{\iota}{2} \frac{a^2}{a_{\prime}^3}$ or $-\frac{3}{8} \sin^2 \iota \frac{a^2}{a_{\prime}^3}$

$$= \underset{[62]}{\left\{ 1 - \frac{5}{2} e^2 + \frac{3}{2} e_{\prime}^2 \right\} \cos 2y} + \underset{[63]}{\left\{ 1 + \frac{3}{2} e^2 - \frac{5}{2} e_{\prime}^2 \right\} \cos (2t - 2y)} - \underset{[65]}{3 e \cos (x - 2y)}$$

$$+ \underset{[66]}{e \cos (x + 2y)} - \underset{[67]}{e \cos (2t - x - 2y)} - \underset{[69]}{e \cos (2t + x - 2y)} + \underset{[71]}{\frac{3}{2} e_{\prime} \cos (z - 2y)} + \underset{[72]}{\frac{3}{2} e_{\prime} \cos (z + 2y)}$$

$$+ \underset{[73]}{\frac{7}{2} e_{\prime} \cos (2t - z - 2y)} - \underset{[75]}{\frac{e_{\prime}}{2} \cos (2t + z - 2y)} + \underset{[77]}{\frac{5}{2} e^2 \cos (2x - 2y)} + \underset{[78]}{e^2 \cos (2x + 2y)}$$

$$- \underset{[79]}{\frac{e^2}{4} \cos (2t - 2x - 2y)} - \underset{[81]}{\frac{e^2}{4} \cos (2t + 2x - 2y)} - \underset{[83]}{\frac{9}{2} e e_{\prime} \cos (x + z - 2y)}$$

$$+ \underset{[84]}{\frac{3}{2} e e_{\prime} \cos (x + z + 2y)} - \underset{[85]}{\frac{7}{2} e e_{\prime} \cos (2t - x - z - 2y)} + \underset{[87]}{\frac{e e_{\prime}}{2} \cos (2t + x + z - 2y)}$$

$$-\frac{9}{2}ee_{\prime}\cos(x-z-2y)+\frac{3}{2}ee_{\prime}\cos(x-z+2y)+\frac{ee_{\prime}}{2}\cos(2t-x+z-2y)$$
[89] [90] [91]

$$-\frac{7}{2}ee_{\prime}\cos(2t+x-z-2y)+\frac{17}{8}e_{\prime}^{2}\cos(2z-2y)+\frac{15}{8}e_{\prime}^{2}\cos(2z+2y)$$
[93] [95] [96]

$$+\frac{65}{8}e_{\prime}^{2}\cos(2t-2z-2y)-\frac{e_{\prime}^{2}}{8}\cos(2t+2z-2y)$$
[97] [99]

$$R=m_{\prime}\left\{-\frac{1}{r_{\prime}}-\frac{1}{4}\left\{1+\frac{3}{2}e^{2}+\frac{3}{2}e_{\prime}^{2}+\frac{9}{4}e^{2}e_{\prime}^{2}+\frac{15}{8}e_{\prime}^{4}-\frac{3}{2}\gamma^{2}-\frac{9}{4}\gamma^{2}e^{2}-\frac{9}{4}\gamma^{2}e_{\prime}^{2}+\frac{39}{8}\gamma^{4}\right\}\frac{a^{2}}{a_{\prime}^{3}}\right.$$
[0]

$$-\frac{3}{4}\left\{1-\frac{5}{2}e^{2}-\frac{5}{2}e_{\prime}^{2}+\frac{23}{16}e^{4}+\frac{25}{4}e^{2}e_{\prime}^{2}+\frac{13}{16}e_{\prime}^{4}\right\}\cos^{4}\frac{\iota}{2}\frac{a^{2}}{a_{\prime}^{3}}\cos 2t$$
[1]

$$+\frac{1}{2}\left\{1-\frac{e^{2}}{8}-\frac{3}{2}e_{\prime}^{2}-\frac{3}{2}\gamma^{2}\right\}\frac{a^{2}}{a_{\prime}^{3}}e\cos x$$
[2]

$$+\frac{9}{4}\left\{1-\frac{13}{24}e^{2}-\frac{5}{2}e_{\prime}^{2}\right\}\cos^{4}\frac{\iota}{2}\frac{a^{2}}{a_{\prime}^{3}}e\cos(2t-x)$$
[3]

$$-\frac{3}{4}\left\{1-\frac{19}{8}e^{2}-\frac{5}{2}e_{\prime}^{2}\right\}\cos^{4}\frac{\iota}{2}\frac{a^{2}}{a_{\prime}^{3}}e\cos(2t+x)$$
[4]

$$-\frac{3}{4}\left\{1+\frac{3}{2}e^{2}+\frac{9}{8}e_{\prime}^{2}-\frac{3}{2}\gamma^{2}\right\}\frac{a^{2}}{a_{\prime}^{3}}e_{\prime}\cos z$$
[5]

$$-\frac{21}{8}\left\{1-\frac{5}{2}e^{2}-\frac{123}{56}e_{\prime}^{2}\right\}\cos^{4}\frac{\iota}{2}\frac{a^{2}}{a_{\prime}^{3}}e_{\prime}\cos(2t-z)$$
[6]

$$+\frac{3}{8}\left\{1-\frac{5}{2}e^{2}-4e_{\prime}^{2}\right\}\cos^{4}\frac{\iota}{2}\frac{a^{2}}{a_{\prime}^{3}}e_{\prime}\cos(2t+z)$$
[7]

$$+\frac{1}{8}\left\{1-\frac{e^{2}}{3}+\frac{3}{2}e_{\prime}^{2}-\frac{3}{2}\gamma^{2}\right\}\frac{a^{2}}{a_{\prime}^{3}}e^{2}\cos 2x$$
[8]

$$-\frac{15}{8}\left\{1-\frac{5}{2}e_{\prime}^{2}\right\}\cos^{4}\frac{\iota}{2}\frac{a^{2}}{a_{\prime}^{3}}e^{2}\cos(2t-2x)$$
[9]

$$-\frac{3}{4}\left\{1-\frac{5}{2}e^{2}-\frac{5}{2}e_{\prime}^{2}\right\}\cos^{4}\frac{\iota}{2}\frac{a^{2}}{a_{\prime}^{3}}e^{2}\cos(2t+2x)$$
[10]

Development of R.

$$+\frac{3}{4}\left\{1-\frac{e^2}{8}+\frac{9}{8}e_i^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_i^3}e\,e_i\cos(x+z)$$
[11]

$$+\frac{63}{8}\left\{1-\frac{91}{168}e^2-\frac{123}{56}e_i^2\right\}\cos^4\frac{\iota}{2}\frac{a^2}{a_i^3}e\,e_i\cos(2t-x-z)$$
[12]

$$+\frac{3}{8}\left\{1-\frac{19}{8}e^2-\frac{e_i^2}{8}\right\}\cos^4\frac{\iota}{2}\frac{a^2}{a_i^3}e\,e_i\cos(2t+x+z)$$
[13]

$$+\frac{3}{4}\left\{1-\frac{e^2}{8}+\frac{9}{8}e_i^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_i^2}e\,e_i\cos(x-z)$$
[14]

$$-\frac{9}{8}\left\{1-\frac{13}{24}e^2-\frac{e_i^2}{8}\right\}\cos^4\frac{\iota}{2}\frac{a^2}{a_i^3}e\,e_i\cos(2t-x+z)$$
[15]

$$-\frac{21}{8}\left\{1-\frac{19}{8}e^2-\frac{123}{56}e_i^2\right\}\cos^4\frac{\iota}{2}\frac{a^2}{a_i^3}e\,e_i\cos(2t+x-z)$$
[16]

$$-\frac{9}{8}\left\{1+\frac{3}{2}e^2+\frac{7}{9}e_i^2-\frac{3}{2}\gamma^2\right\}\frac{a^2}{a_i^3}e_i^2\cos 2z$$
[17]

$$-\frac{51}{8}\left\{1-\frac{5}{2}e^2-\frac{115}{51}e_i^2\right\}\cos^4\frac{\iota}{2}\frac{a^2}{a_i^3}e_i^2\cos(2t-2z)$$
[18]

$$+\frac{1}{16}\frac{a^2}{a_i^3}e^3\cos 3x+\frac{7a^2}{32a_i^3}e^3\cos(2t-3x)-\frac{25a^2}{32a_i^3}e^3\cos(2t+3x)$$
[20] [21] [22]

$$+\frac{3a^2}{16a_i^3}e^2e_i\cos(2x+z)-\frac{105}{16}\frac{a^2}{a_i^3}e^2e_i\cos(2t-2x-z)+\frac{3}{8}\frac{a^2}{a_i^3}e^2e_i\cos(2t+2x+z)$$
[23] [24] [25]

$$+\frac{3}{16}\frac{a^2}{a_i^3}e^2e_i\cos(2x-z)+\frac{15}{16}\frac{a^2}{a_i^3}e^2e_i\cos(2t-2x+z)-\frac{21}{8}\frac{a^2}{a_i^3}e^2e_i\cos(2t+2x-z)$$
[26] [27] [28]

$$+\frac{9}{8}\frac{a^2}{a_i^3}e\,e_i^2\cos(x+2z)+\frac{153}{8}\frac{a^2}{a_i^3}e\,e_i^2\cos(2t-x-2z)$$
[29] [30]

$$+\frac{9}{8}\frac{a^2}{a_i^3}e\,e_i^2\cos(x-2z)-\frac{51}{8}\frac{a^2}{a_i^3}e\,e_i^2\cos(2t+x-2z)-\frac{53}{32}\frac{a^2}{a_i^3}e_i^3\cos 3z$$
[32] [34] [35]

$$-\frac{845a^2}{64a_i^3}e_i^3\cos(2t-3z)-\frac{1}{64}\frac{a^2}{a_i^3}e_i^3\cos(2t+3z)+\frac{1}{24}\frac{a^2}{a_i^3}e^4\cos 4x$$
[36] [37] [38]

$$+\frac{3}{64}\frac{a^2}{a_i^3}e^4\cos(2t-4x)-\frac{27}{32}\frac{a^2}{a_i^3}e^4\cos(2t+4x)+\frac{3}{32}\frac{a^2}{a_i^3}e^3e_i\cos(3x+z)$$
[39] [40] [41]

Development of R.

$$+\frac{49}{64}\frac{a^2}{a_{\prime}^3}e^3 e_{\prime}\cos(2t-3x-z) + \frac{25}{64}\frac{a^2}{a_{\prime}^3}e^3 e_{\prime}\cos(2t-3x-z) + \frac{3}{32}\frac{a^2}{a_{\prime}^3}e^3 e_{\prime}\cos(3x-z)$$
[42] [44]

$$-\frac{7}{64}\frac{a^2}{a_{\prime}^3}e^3 e_{\prime}\cos(2t-3x+z) - \frac{175}{64}\frac{a^2}{a_{\prime}^3}e^3 e_{\prime}\cos(2t+3x-z)$$
[45] [46]

$$+\frac{9}{32}\frac{a^2}{a_{\prime}^3}e^2 e_{\prime}^2\cos(2x+2z) - \frac{255}{16}e^2 e_{\prime}^2\cos(2t-2x-2z)$$
[47] [48]

$$+\frac{9}{32}\frac{a^2}{a_{\prime}^3}e^2 e_{\prime}^2\cos(2x-2z) - \frac{51}{8}\frac{a^2}{a_{\prime}^3}e^2 e_{\prime}^2\cos(2t+2x-2z) + \frac{53}{32}\frac{a^2}{a_{\prime}^3}e e_{\prime}^3\cos(x+3z)$$
[50] [52] [53]

$$+\frac{2535}{64}\frac{a^2}{a_{\prime}^3}e e_{\prime}^3\cos(2t-x-3z) - \frac{1}{64}\frac{a^2}{a_{\prime}^3}e e_{\prime}^3\cos(2t+x+3z)$$
[54] [55]

$$+\frac{53}{32}\frac{a^2}{a_{\prime}^3}e e_{\prime}^3\cos(x-3z) + \frac{3}{64}\frac{a^2}{a_{\prime}^3}e e_{\prime}^3\cos(2t-x+3z)$$
[56] [57]

$$+\frac{45}{64}\frac{a^2}{a_{\prime}^3}e e_{\prime}^3\cos(2t+x-3z) + \frac{591}{64}\frac{a^2}{a_{\prime}^3}e_{\prime}^4\cos 4z$$
[58] [59]

$$-\frac{2453}{128}\frac{a^2}{a_{\prime}^3}e_{\prime}^4\cos(2t-4z) + \frac{741}{128}\frac{a^2}{a_{\prime}^3}e_{\prime}^4\cos(2t+4z)$$
[60] [61]

$$-\frac{3}{8}\left\{1-\frac{5}{2}e^2+\frac{3}{2}e_{\prime}^2\right\}\frac{a^2}{a_{\prime}^3}\gamma^2\cos 2y - \frac{3}{8}\left\{1+\frac{3}{2}e^2-\frac{5}{2}e_{\prime}^2+\frac{\gamma^2}{8}\right\}\frac{a^2}{a_{\prime}^3}\gamma^2\cos(2t-2y)$$
[62] [63]

$$+\frac{9}{8}\frac{a^2}{a_{\prime}^3}\gamma^2 e\cos(x-2y)$$
[65]

$$-\frac{3}{8}\frac{a^2}{a_{\prime}^3}\gamma^2 e\cos(x+2y) + \frac{3}{8}\frac{a^2}{a_{\prime}^3}\gamma^2 e\cos(2t-x-2y)$$
[66] [67]

$$+\frac{3}{8}\frac{a^2}{a_{\prime}^3}\gamma^2 e\cos(2t+x-2y) - \frac{9}{16}\frac{a^2}{a_{\prime}^3}\gamma^2 e_{\prime}\cos(z-2y)$$
[69] [71]

$$-\frac{9}{16}\frac{a^2}{a_{\prime}^3}\gamma^2 e_{\prime}\cos(z+2y) - \frac{21}{16}\frac{a^2}{a_{\prime}^3}\gamma^2 e_{\prime}\cos(2t-z-2y)$$
[72] [73]

$$+\frac{3}{16}\frac{a^2}{a_{\prime}^3}\gamma^2 e_{\prime}\cos(2t+z-2y) - \frac{15}{16}\frac{a^2}{a_{\prime}^3}\gamma^2 e^2\cos(2x-2y)$$
[75] [77]

Development of R.

$$- \frac{3}{8} \frac{a^2}{a_,^3} \gamma^2 e^2 \cos (2x + 2y) + \frac{3}{32} \frac{a^2}{a_,^3} \gamma^2 e^2 \cos (2t - 2x - 2y)$$
[78] [79]

$$+ \frac{3}{32} \frac{a^2}{a_,^3} \gamma^2 e^2 \cos (2t + 2x - 2y) + \frac{27}{16} \frac{a^2}{a_,^3} \gamma^2 e e_, \cos (x + z - 2y)$$
[81] [83]

$$- \frac{9}{16} \frac{a^2}{a_,^3} \gamma^2 e e_, \cos (x + z + 2y) + \frac{21}{16} \frac{a^2}{a_,^3} \gamma^2 e e_, \cos (2t - x - z - 2y)$$
[84] [85]

$$- \frac{3}{16} \frac{a^2}{a_,^3} \gamma^2 e e_, \cos (2t + x + z - 2y) + \frac{27}{16} \frac{a^2}{a_,^3} \gamma^2 e e_, \cos (x - z - 2y)$$
[87] [89]

$$- \frac{9}{16} \frac{a^2}{a_,^3} \gamma^2 e e_, \cos (x - z + 2y) - \frac{3}{16} \frac{a^2}{a_,^2} \gamma^2 e e_, \cos (2t - x + z - 2y)$$
[90] [91]

$$+ \frac{21}{16} \frac{a^2}{a_,^3} \gamma^2 e e_, \cos (2t + x - z - 2y)$$
[93]

$$- \frac{51}{64} \frac{a^2}{a_,^3} \gamma^2 e_,^2 \cos (2z - 2y) - \frac{45}{64} \frac{a^2}{a_,^3} \gamma^2 e_,^2 \cos (2z + 2y)$$
[95] [96]

$$- \frac{195}{64} \frac{a^2}{a_,^3} \gamma^2 e_,^2 \cos (2t - 2z - 2y) + \frac{3}{64} \frac{a^2}{a_,^3} \gamma^2 e_,^2 \cos (2t + 2z - 2y)$$
[97] [99]

$$- \frac{3}{8} \left\{ 1 + 3e^2 + 3e_,^2 - \frac{11}{4} \gamma^2 \right\} \frac{a^3 *}{a_,^4} \cos t + \frac{15}{16} \frac{a^3}{a_,^4} e \cos (t - x)$$
[101] [102]

$$+ \frac{3}{16} \frac{a^3}{a_,^4} e \cos (t + x) - \frac{9}{8} \frac{a^3}{a_,^4} e_, \cos (t - z) - \frac{3}{8} \frac{a^3}{a_,^4} e_, \cos (t + x)$$
[103] [104] [105]

$$- \frac{33}{64} \frac{a^3}{a_,^4} e^2 \cos (t - 2x) + \frac{9}{64} \frac{a^3}{a_,^4} e^2 \cos (t + 2x) + \frac{45}{16} \frac{a^3}{a_,^4} e e_, \cos (t - x - z)$$
[106] [107] [108]

$$+ \frac{3}{16} \frac{a^3}{a_,^4} e e_, \cos (t + x + z) + \frac{15}{16} \frac{a^3}{a_,^4} e e_, \cos (t - x + z) + \frac{9}{16} \frac{a^3}{a_,^4} e e_, \cos (t + x - z)$$
[109] [110] [111]

$$- \frac{159}{64} \frac{a^3}{a_,^4} e_,^2 \cos (t - 2z) - \frac{33}{64} \frac{a^3}{a_,^4} e_,^2 \cos (t + 2z) - \frac{9}{16} \frac{a^3}{a_,^4} \gamma^2 \cos (t - 2y)$$
[112] [113] [114]

$$- \frac{15}{32} \frac{a^3}{a_,^4} \sin^2 \frac{\iota}{2} \cos (t + 2y) - \frac{5}{8} \left\{ 1 - 6e^2 - 6e_,^2 - \frac{3}{4} \gamma^2 \right\} \frac{a^3}{a_,^4} \cos 3t$$
[115] [116]

* For the coefficients of the terms multiplied by $\frac{a^3}{a_,^4}$ see p. 39.

$$+\frac{45}{16}\frac{a^3}{a_{\prime}^4}e\cos(3t-x) - \frac{15}{16}\frac{a^3}{a_{\prime}^4}e\cos(3t+x) - \frac{25}{8}\frac{a^3}{a_{\prime}^4}e_{\prime}\cos(3t-z)$$
[117] [118] [119]

$$+\frac{5}{8}\frac{a^3}{a_{\prime}^4}e_{\prime}\cos(3t+z) - \frac{285}{64}\frac{a^3}{a_{\prime}^4}e^2\cos(3t-2x) - \frac{75}{64}\frac{a^3}{a_{\prime}^4}e^2\cos(3t+2x)$$
[120] [121] [122]

$$-\frac{225}{16}\frac{a^3}{a_{\prime}^4}e e_{\prime}\cos(3t-x-z) + \frac{15}{16}\frac{a^3}{a_{\prime}^4}e e_{\prime}\cos(3t+x+z)$$
[123] [124]

$$-\frac{45}{16}\frac{a^3}{a_{\prime}^4}e e_{\prime}\cos(3t-x+z) - \frac{75}{16}\frac{a^3}{a_{\prime}^4}e e_{\prime}\cos(3t+x-z)$$
[125] [126]

$$-\frac{635}{64}\frac{a^3}{a_{\prime}^4}e_{\prime}^2\cos(3t-2z) - \frac{5}{64}\frac{a^3}{a_{\prime}^4}e_{\prime}^2\cos(3t+2z) - \frac{15}{32}\frac{a^3}{a_{\prime}^4}\gamma^2\cos(3t-2y)$$
[127] [128] [129]

In the elliptic movement ;

$$s = \gamma\sin(g\lambda - \nu)$$

$$\lambda = nt + 2e\sin x + \frac{5}{4}e^2\sin 2x$$

$$s = \gamma(1-e^2)\sin y + \gamma e\sin(x-y) + \gamma e\sin(x+y) + \gamma\frac{e^2}{8}\sin(2x-y) + \frac{9}{8}\gamma e^2\sin(2x+y)$$
[146] [149] [150] [161] [162]

$$s^2 = \frac{\gamma^2}{2} - \frac{\gamma^2}{2}(1-4e^2)\cos 2y + \gamma^2 e\cos(x-2y) - \gamma^2 e\cos(x+2y)$$
[62] [65] [66]

$$+\frac{5}{8}\gamma^2 e^2\cos(2x-2y) - \frac{5}{8}\gamma^2 e^2\cos(2x+2y)$$
[77] [78]

$$z^* = a\gamma\left(1-\frac{e^2}{2}\right)\sin y + \frac{3a\gamma e}{2}\sin(x-y) + \frac{a\gamma e}{2}\sin(x+y)$$
[146] [149] [150]

$$-\frac{a\gamma e^2}{8}\sin(2x-y) + \frac{3a\gamma e^2}{8}\sin(2x+y)$$
[161] [162]

$$\frac{s}{r} = \frac{\gamma}{a}(1-e^2)\sin y + \frac{\gamma e}{2a}\sin(x-y) + \frac{3\gamma e}{2a}\sin(x+y)$$
[146] [149] [150]

* This quantity z, which is one of the rectangular coordinates of the moon, must not be confounded with $z = n_{\prime}t - \varpi_{\prime}$; this latter quantity should rather be $x_{\prime}$, but I think it better to conform as far as possible to the notation of M. Damoiseau.

$$+\frac{\gamma e^2}{8a}\sin(2x-y)+\frac{17}{8}\frac{\gamma e^2}{a}\sin(2x+y)$$
[161] [162]

$$\frac{s}{r}\delta\cdot\frac{1}{r}=\left\{(1-e^2)r_0+\frac{e^2}{2}r_2\right\}\frac{\gamma}{a^2}\sin y-\left\{(1-e^2)\frac{r_1}{2}+\frac{e^2}{4}r_3-\frac{3e^2}{4}r_4\right\}\frac{\gamma}{a^2}\sin(2t-y)$$
[146] [147]

$$+\left\{(1-e^2)\frac{r_1}{2}-\frac{e^2}{4}r_4+\frac{3e^2}{4}r_3\right\}\frac{e\gamma}{a^2}\sin(2t+y)+\frac{e\gamma r_0}{2a^2}\sin(x-y)$$
[148] [149]

$$+\frac{3r_0}{2a^2}e\gamma\sin(x+y)+\left\{-\frac{r_3}{2}-\frac{3r_1}{4}\right\}\frac{e\gamma}{a^2}\sin(2t-x-y)$$
[150] [151]

$$+\left\{\frac{r_3}{2}-\frac{r_1}{4}\right\}\frac{e\gamma}{a^2}\sin(2t-x+y)+\left\{-\frac{r_4}{2}-\frac{r_1}{4}\right\}\frac{e\gamma}{a^2}\sin(2t+x-y)$$
[152] [153]

$$+\left\{\frac{r_4}{2}+\frac{3}{4}r_1\right\}\frac{e\gamma}{a^2}\sin(2t+x+y)+\frac{r_5e_{\prime}\gamma}{2a^2}\sin(z-y)+\frac{r_5e_{\prime}\gamma}{2a^2}\sin(z+y)$$
[154] [155] [156]

$$-\frac{r_6e_{\prime}\gamma}{2a^2}\sin(2t-z-y)+\frac{r_6e_{\prime}\gamma}{2a^2}\sin(2t-z+y)-\frac{r_7e_{\prime}\gamma}{2a^2}\sin(2t+z-y)$$
[157] [158] [159]

$$+\left\{-\frac{r_9}{2}-\frac{3}{4}r_3-\frac{17}{16}r_1\right\}\frac{e^2\gamma}{a^2}\sin(2t-2x-y)+\left\{\frac{r_9}{2}-\frac{r_3}{4}-\frac{r_1}{16}\right\}\frac{e^2\gamma}{a^2}\sin(2t-2x+y)$$
[163] [164)

$$+\left\{-\frac{r_{10}}{2}+\frac{r_4}{4}+\frac{r_1}{16}\right\}\frac{e^2\gamma}{a^2}\sin(2t+2x-y)+\left\{\frac{r_{10}}{2}+\frac{3r_4}{4}+\frac{17}{16}r_1\right\}\frac{e^2\gamma}{a^2}\sin(2t+2x+y)$$
[165] [166]

$$+\left\{-\frac{r_{11}}{2}+\frac{r_5}{4}\right\}\frac{ee_{\prime}\gamma}{a^2}\sin(x+z-y)+\left\{\frac{r_{11}}{2}+\frac{3r_5}{4}\right\}\frac{ee_{\prime}\gamma}{a^2}\sin(x+z+y)$$
[167] [168]

$$+\left\{-\frac{r_{12}}{2}+\frac{r_6}{4}-\frac{3}{4}r_6\right\}\frac{ee_{\prime}\gamma}{a^2}\sin(2t-x-z-y)+\left\{\frac{r_{12}}{2}+\frac{r_6}{4}\right\}\frac{ee_{\prime}\gamma}{a^2}\sin(2t-x-z+y)$$
[169] [170]

$$+\left\{-\frac{r_{13}}{2}+\frac{r_7}{4}\right\}\frac{ee_{\prime}\gamma}{a^2}\sin(2t+x+z-y)+\left\{\frac{r_{13}}{2}+\frac{3}{4}r_7\right\}\frac{ee_{\prime}\gamma}{a^2}\sin(2t+x+z+y)$$
(171) (172)

$$+\left\{-\frac{r_{14}}{2}+\frac{r_5}{2}\right\}\frac{ee_{\prime}\gamma}{a^2}\sin(x-z-y)+\left\{\frac{r_{14}}{2}+\frac{3}{4}r_5\right\}\frac{ee_{\prime}\gamma}{a^2}\sin(x-z+y)$$
(173) (174)

$$+\left\{-\frac{r_{15}}{2}-\frac{3}{4}r_7\right\}\frac{ee_{\prime}\gamma}{a^2}\sin(2t-x+z-y)+\left\{\frac{r_{15}}{2}-\frac{r_7}{2}\right\}\frac{ee_{\prime}\gamma}{a^2}\sin(2t-x+z+y)$$
[175] [176]

$$- \frac{r_{16} e e_{,} \gamma}{2 a^2} \sin (2t + x - z - y) + \left\{ \frac{r_{16}}{2} + \frac{3}{4} r_6 \right\} \frac{e e_{,} \gamma}{a^2} \sin (2t + x - z + y)$$
[177] [178]

$$- \frac{r_{17}}{2} \frac{e_{,}^2 \gamma}{a^2} \sin (2z - y) + \frac{r_{17}}{2} \frac{e_{,}^2 \gamma}{a^2} \sin (2z + y) - \frac{r_{18}}{2} \frac{e_{,}^2 \gamma}{a^2} \sin (2t - 2z - y)$$
[179] [180] [181]

$$+ \frac{r_{18}}{2} \frac{e_{,}^2 \gamma}{a^2} \sin (2t - 2z + y) - \frac{r_{19}}{2} \frac{e_{,}^2 \gamma}{a^2} \sin (2t + 2z - y) + \frac{r_{19}}{2} \frac{e_{,}^2 \gamma}{a^2} \sin (2t + 2z + y)$$
[182] [183] [184]

$$\frac{m_{,} z}{r^3} = \frac{m_{,} a \gamma}{a_{,}^3} \left(1 + \frac{3}{2} e_{,}^2 - \frac{e^2}{2}\right) \sin y + \frac{3 m_{,} a \gamma e}{2 a_{,}^3} \sin (x - y) + \frac{m_{,} a \gamma}{2 a_{,}^3} \sin (x + y)$$
[146] [149] [150]

$$- \frac{3 m_{,} a \gamma e_{,}}{2 a_{,}^3} \sin (z - y) + \frac{3 m_{,} a \gamma e_{,}}{2 a_{,}^3} \sin (z + y) - \frac{m_{,} a \gamma e^2}{8 a_{,}^3} \sin (2x - y)$$
[155] [156] [161]

$$+ \frac{3 m_{,} a \gamma e^2}{8 a_{,}^3} \sin (2x + y) + \frac{9 m_{,} a \gamma e e_{,}}{4 a_{,}^3} \sin (x + z - y) + \frac{3 m_{,} a \gamma e e_{,}}{4 a_{,}^3} \sin (x + z + y)$$
[162] [167] [168]

$$+ \frac{9 m_{,} a \gamma e e_{,}}{4 a_{,}^3} \sin (x - z - y) + \frac{3 m_{,} a \gamma e e_{,}}{4 a_{,}^3} \sin (x - z + y) - \frac{9 m_{,} a \gamma e_{,}^2}{4 a_{,}^3} \sin (2z - y)$$
[173] [174] [179]

$$+ \frac{9 m_{,} a \gamma e_{,}^2}{4 a_{,}^3} \sin (2z + y)$$
[180]

$$\frac{a^3}{\mathrm{r}^3} = 1 + \frac{3}{2} e^2 + 3 e \cos x + \frac{9}{2} e^2 \cos 2x$$

r being the elliptic value of r.

If $z = a \gamma z_{146} \sin y + a \gamma z_{147} \sin (2t - y) + a \gamma z_{148} \sin (2t + y)$ &c.

$$\frac{z}{r^3}{}^* = \left\{ \left(1 + \frac{3e^2}{2}\right) z_{146} + \frac{3}{2} e^2 z_{150} - \frac{3e^2}{2} z_{149} \right\} \frac{\gamma}{a^2} \sin y$$
[146]

$$+ \left\{ \left(1 + \frac{3e^2}{2}\right) z_{147} + \frac{3e^2}{2} z_{151} + \frac{3e^2}{2} z_{153} \right\} \frac{\gamma}{a^2} \sin (2t - y)$$
[147]

$$+ \left\{ \left(1 + \frac{3e^2}{2}\right) z_{148} + \frac{3e^2}{2} z_{152} + \frac{3e^2}{2} z_{154} \right\} \frac{\gamma}{a^2} \sin (2t + y)$$
[148]

* This multiplication of z by r^{-3} may be effected at once by means of Table II.

$$+ \left\{ z_{149} - \frac{3}{2} z_{146} \right\} \frac{e\gamma}{a^2} \sin(x-y) + \left\{ z_{150} + \frac{3}{2} z_{146} \right\} \frac{e\gamma}{a^2} \sin(x+y)$$
[149] [150]

$$+ \left\{ z_{151} + \frac{3}{2} z_{147} \right\} \frac{e\gamma}{a^2} \sin(2t-x-y) + \left\{ z_{152} + \frac{3}{2} z_{148} \right\} \frac{e\gamma}{a^2} \sin(2t-x+y)$$
[151] [152]

$$+ \left\{ z_{153} + \frac{3}{2} z_{147} \right\} \frac{e\gamma}{a^2} \sin(2t+x-y)$$
[153]

$$+ \left\{ z_{154} + \frac{3}{2} z_{148} \right\} \frac{e\gamma}{a^2} \sin(2t+x+y) + z_{155} \frac{e_{\prime}\gamma}{a^2} \sin(z-y) + z_{156} \frac{e_{\prime}\gamma}{a^2} \sin(z+y)$$
[154] [155] [156]

$$+ \left\{ z_{161} + \frac{3}{2} z_{149} - \frac{9}{4} z_{146} \right\} \frac{e^2\gamma}{a^2} \sin(2x-y) + \left\{ z_{162} + \frac{3}{2} z_{150} + \frac{9}{4} z_{146} \right\} \frac{e^2\gamma}{a^2} \sin(2x+y)$$
[161] [162]

$$+ \left\{ z_{163} + \frac{3}{2} z_{151} + \frac{9}{4} z_{147} \right\} \frac{e^2\gamma}{a^2} \sin(2t-2x-y)$$
[163]

$$+ \left\{ z_{164} + \frac{3}{2} z_{152} + \frac{9}{4} z_{148} \right\} \frac{e^2\gamma}{a^2} \sin(2t-2x+y)$$
[164]

$$+ \left\{ z_{165} + \frac{3}{2} z_{153} + \frac{9}{4} z_{147} \right\} \frac{e^2\gamma}{a^2} \sin(2t+2x-y)$$
[165]

$$+ \left\{ z_{166} + \frac{3}{2} z_{154} + \frac{9}{4} z_{148} \right\} \frac{e^2\gamma}{a^2} \sin(2t+2x+y) + \&c.$$
[166]

$$+ \left\{ z_{167} + \frac{3}{2} z_{155} \right\} \frac{e e_{\prime}\gamma}{a^2} \sin(x+z-y) + \left\{ z_{168} + \frac{3}{2} z_{156} \right\} \frac{e e_{\prime}\gamma}{a^2} \sin(x+z+y)$$
[167] [168]

$$+ \left\{ z_{169} + \frac{3}{2} z_{157} \right\} \frac{e e_{\prime}\gamma}{a^2} \sin(2t-x-z-y) + \left\{ z_{170} + \frac{3}{2} z_{158} \right\} \frac{e e_{\prime}\gamma}{a^2} \sin(2t-x-z+y)$$
[169] [170]

$$+ \left\{ z_{171} + \frac{3}{2} z_{159} \right\} \frac{e e_{\prime}\gamma}{a^2} \sin(2t+x+z-y) + \left\{ z_{172} + \frac{3}{2} z_{160} \right\} \frac{e e_{\prime}\gamma}{a^2} \sin(2t+x+z+y)$$
[171] [172]

$$+ \left\{ z_{173} - \frac{3}{2} z_{156} \right\} \frac{e e_{\prime}\gamma}{a^2} \sin(x-z-y) + \left\{ z_{174} - \frac{3}{2} z_{155} \right\} \frac{e e_{\prime}\gamma}{a^2} \sin(x-z+y)$$
[173] [174]

$$+ \left\{ z_{175} + \frac{3}{2} z_{159} \right\} \frac{e e_{\prime}\gamma}{a^2} \sin(2t-x+z-y) + \left\{ z_{176} + \frac{3}{2} z_{160} \right\} \frac{e e_{\prime}\gamma}{a^2} \sin(2t-x+z+y)$$
[175] [176]

$$+ \left\{ z_{177} + \frac{3}{2} z_{157} \right\} \frac{e e_{\prime}\gamma}{a^2} \sin(2t+x-z-y) + \left\{ z_{178} + \frac{3}{2} z_{158} \right\} \frac{e e_{\prime}\gamma}{a^2} \sin(2t+x-z+y)$$
[177] [178]

$$s = \frac{z}{r} \text{ nearly,}$$

$$= \left\{z_{146} + \frac{e^2}{2} z_{150} + \frac{e^2}{2} z_{149}\right\} \gamma \sin y$$

$$+ \left\{z_{147} + \frac{e^2}{2} z_{151} + \frac{e^2}{2} z_{153}\right\} \gamma \sin (2\,t - y)$$

$$+ \left\{z_{148} + \frac{e^2}{2} z_{152} + \frac{e^2}{2} z_{154}\right\} \gamma \sin (2\,t + y) + \&c.$$

$$\frac{\mathrm{d}^2 . r^2}{2 . \mathrm{d}\, t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2 \int \mathrm{d}\, R + r \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, r}\right) = 0$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\, t^2} + \frac{\mu\, z}{r^3} + \frac{m_{\prime}\, z}{\{r^2 - 2\, r\, r^{\backprime} \cos (\lambda - \lambda) + r_{\prime}^2\}^{\frac{3}{2}}}$$

$$r^{\backprime 4} . \frac{\mathrm{d}\, \lambda^{\backprime 2}}{\mathrm{d}\, t^2} = h^2 - 2 \int r^{\backprime 2} \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, \lambda^{\backprime}}\right) \mathrm{d}\, \lambda^{\backprime}$$

Neglecting the square of the disturbing force

$$\frac{- \mathrm{d}^2 . \mathrm{r}^3 \delta . \frac{1}{r}}{\mathrm{d}\, t^2} - \mu\, \delta . \frac{1}{r} + 2 \int \mathrm{d}\, R + r \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, r}\right) = 0$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\, t^2} + \frac{\mu\, z}{r^3} + \frac{m_{\prime}\, z}{r_{\prime}^3} + \frac{3\, m_{\prime}\, z\, r^{\backprime}\, r \cos (\lambda^{\backprime} - \lambda)}{r_{\prime}^5} = 0$$

$$\frac{\mathrm{d}^2 . \delta\, z}{\mathrm{d}\, t^2} + \frac{3\, \mu\, s\, \delta . \frac{1}{r}}{\mathrm{r}} + \frac{\mu\, \delta . z}{\mathrm{r}^3} + \frac{m_{\prime}\, z}{r_{\prime}^3} + \frac{3\, \mu_{\prime}\, z\, r^{\backprime}\, r \cos (\lambda^{\backprime} - \lambda)}{r_{\prime}^5} = 0$$

$$\frac{\mathrm{d}\, \lambda^{\backprime}}{\mathrm{d}\, t} = \frac{h (1 + s^2)}{r^2} - \frac{(1 + \mathrm{s}^2)}{\mathrm{r}^2} \int \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, \lambda^{\backprime}}\right) \mathrm{d}\, t$$

$$r \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, r}\right) = a \left(\frac{\mathrm{d}\, R}{\mathrm{d}\, a}\right), \; \frac{\mathrm{d}\, R}{\mathrm{d}\, \lambda^{\backprime}} = \frac{\mathrm{d}\, R}{\mathrm{d}\, t}, \; (t \text{ being used for } n\, t - n_{\prime}\, t).$$

Integrating the equation of p. 270, line 9, by the method of indeterminate coefficients, neglecting the cubes and higher powers of e in order to obtain a first approximation, m being equal to $\frac{n_{\prime}}{n}$ as in the notation of M. DAMOISEAU;

$$- r_0 - \frac{m_{\prime}\, a^3}{2\, \mu\, a_{\prime}^3} \left\{1 + \frac{3}{2} e^2 + \frac{3}{2} e_{\prime}^2 - \frac{3}{2} \gamma^2\right\} = 0$$

$$4 (1 - m)^2 \left\{ (1 + 3\, e^2)\, r_1 - \frac{3\, e^2}{2} \{r_3 + r_4\} \right\} - r_1$$

$$-\frac{3\,m_{\prime}\,a^3}{2\,\mu\,a_{\prime}^3}\left\{1-\frac{5}{2}\,e^2-\frac{5}{2}\,e_{\prime}^2-\frac{\gamma^2}{2}\right\}\left\{\frac{1}{1-m}+1\right\}=0$$

$$c^2 * \{1-3\,r_0\}-1+\frac{2\,m_{\prime}\,a^3}{\mu\,a^3}=0$$

$$(2-2\,m-c)^2\left\{r_3-\frac{3}{2}\,r_1\right\}-r_3+\frac{9}{2}\,\frac{m_{\prime}}{\mu}\,\frac{a^3}{a_{\prime}^3}\left\{\frac{2-c}{2-2\,m-c}+1\right\}=0$$

$$(2-2\,m+c)^2\left\{r_4-\frac{3}{2}\,r_1\right\}-r_4-\frac{3}{2}\,\frac{m_{\prime}}{\mu}\,\frac{a^3}{a_{\prime}^3}\left\{\frac{2+c}{2-2\,m+c}+1\right\}=0$$

$$m^2\,r_5-r_5-\frac{3}{2}\,\frac{m_{\prime}}{\mu}\,\frac{a^3}{a_{\prime}^3}=0$$

$$(2-3\,m)^2\,r_6-r_6-\frac{21}{4}\,\frac{m_{\prime}}{\mu}\,\frac{a^3}{a_{\prime}^3}\left\{\frac{2}{2-3\,m}+1\right\}=0$$

$$(2-m)^2\,r_7-r_7+\frac{3}{4}\,\frac{m_{\prime}}{\mu}\,\frac{a^3}{a_{\prime}^3}\left\{\frac{2}{2-m}+1\right\}=0$$

$$4\,c^2\left\{(1-3\,r_0)\,r_8-\frac{3}{4}+3\,r_0\right\}-r_8+\frac{m_{\prime}}{2\,\mu}\,\frac{a^3}{a_{\prime}^3}=0$$

$$(2-2\,m-2\,c)^2\left\{r_9-\frac{3}{2}\,r_3\right\}-r_9-\frac{15}{4}\,\frac{m_{\prime}}{\mu}\,\frac{a^3}{a_{\prime}^3}\left\{\frac{2-2\,c}{2-2\,m-2\,c}+1\right\}=0$$

$$(2-2\,m+2\,c)^2\left\{r_{10}-\frac{3}{2}\,r_4\right\}-r_{10}-\frac{3}{2}\,\frac{m_{\prime}}{\mu}\,\frac{a^3}{a_{\prime}^3}\left\{\frac{2+2\,c}{2-2\,m+2\,c}+1\right\}=0$$

$$(c+m)^2\left\{r_{11}-\frac{3}{2}\,r_5\right\}-r_{11}+\frac{3}{2}\,\frac{m_{\prime}}{\mu}\,\frac{a^3}{a_{\prime}^3}\left\{\frac{c}{c+m}+1\right\}=0$$

$$(2-3\,m-c)^2\left\{r_{12}-\frac{3}{2}\,r_6\right\}-r_{12}+\frac{63}{4}\,\frac{m_{\prime}}{\mu}\,\frac{a^3}{a_{\prime}^3}\left\{\frac{c}{2-3\,m-c}+1\right\}=0$$

$$(2-m+c)^2\left\{r_{13}-\frac{3}{2}\,r_7\right\}-r_{13}+\frac{3}{4}\,\frac{m_{\prime}}{\mu}\,\frac{a^2}{a_{\prime}^3}\left\{\frac{2+c}{2-m+c}+1\right\}=0$$

$$(c-m)^2\left\{r_{14}-\frac{3}{2}\,r_5\right\}-r_{14}+\frac{3}{2}\,\frac{m_{\prime}}{\mu}\,\frac{a^3}{a_{\prime}^3}\left\{\frac{c}{c-m}+1\right\}=0$$

$$(2-m-c)^2\left\{r_{15}-\frac{3}{2}\,r_7\right\}-r_{15}-\frac{9}{4}\,\frac{m_{\prime}}{\mu}\,\frac{a^3}{a_{\prime}^3}\left\{\frac{2-c}{2-m-c}+1\right\}=0$$

* The letter c does not strictly denote the same quantity as in the notation of M. Damoiseau, or in that of the Mathematical Tracts, p. 33.

$$(2-3m+c)^2\left\{r_{16}-\frac{3}{2}r_6\right\}-r_{16}-\frac{21}{4}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\left\{\frac{2+c}{2-3m+c}+1\right\}=0$$

$$4m^2r_{17}-r_{17}-\frac{9}{4}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}=0$$

$$(2-4m)^2r_{18}-r_{18}-\frac{51}{4}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\left\{\frac{2}{2-4m}+1\right\}$$

$$4r_{19}-r_{19}=0$$

The equation for determining z may be integrated in the same way.

$$-g^2z_{146}+3r_0+z_{146}+\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}=0$$

$$-\left\{2(1-m)-g\right\}^2z_{147}-\frac{3r_1}{2}+z_{147}=0$$

$$-\left\{2(1-m)+g\right\}^2z_{148}+\frac{3r_1}{2}+z_{148}=0$$

$$-\left\{c-g\right\}^2z_{149}+\frac{3}{2}r_6+z_{149}-\frac{3}{2}z_{146}+\frac{3m_{\prime}}{2\mu}\frac{a^3}{a_{\prime}^3}=0$$

$$-\left\{c+g\right\}^2z_{150}+\frac{9}{2}r_6+z_{150}+\frac{3}{2}z_{146}+\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}=0$$

$$-\left\{2(1-m)-c-g\right\}^2z_{147}+3\left\{-\frac{3r_1}{4}-\frac{r_3}{2}\right\}+z_{151}+\frac{3}{2}z_{147}=0$$

$$-\left\{2(1-m)-c+g\right\}^2z_{148}+3\left\{-\frac{r_1}{4}+\frac{r_3}{2}\right\}+z_{152}+\frac{3}{2}z_{148}=0$$

$$-\left\{2(1-m)+c-g\right\}^2z_{149}+3\left\{\frac{r_1}{4}-\frac{r_4}{2}\right\}+z_{153}+\frac{3}{2}z_{147}=0$$

$$-\left\{2(1-m)+c+g\right\}^2z_{150}+3\left\{\frac{3}{4}r_1+\frac{r_4}{2}\right\}+z_{154}+\frac{3}{2}z_{148}=0$$

$$-\left\{m-g\right\}^2z_{151}+\frac{3}{2}r_5+z_{155}-\frac{3m_{\prime}}{2\mu}\frac{a^3}{a_{\prime}^3}=0$$

$$-\left\{m+g\right\}^2z_{152}+\frac{3}{2}r_5+z_{156}+\frac{3m_{\prime}}{2\mu}\frac{a^3}{a_{\prime}^3}=0$$

$$-\left\{2(1-m)-m-g\right\}^2z_{153}-\frac{3}{2}r_6+z_{157}=0$$

$$-\left\{2(1-m)-m+g\right\}^2z_{154}+\frac{3}{2}r_6+z_{158}=0$$

$$-\left\{2(1-m)+m-g\right\}^2z_{155}-\frac{3}{2}r_7+z_{159}=0$$

$$-\left\{2(1-m)+m+g\right\}^2 z_{156}+\frac{3}{2}r_7+z_{160}=0$$

$$\frac{d\lambda'}{dt}=\frac{h}{r^2}+\frac{2h}{r'}\delta.\ \frac{1}{r}+\frac{hz^2}{r^4}-\frac{(1+s^2)}{r^2}\int\left(\frac{dR}{d\lambda'}\right)dt$$

$$\lambda'=\frac{h}{a^2}\left\{1+\frac{e^2}{2}+\frac{\gamma^2}{2}+2r_0\right\}t+\frac{2e(1+r_0)}{c}\sin x+\frac{5e^2(1+r_0)}{4c}\sin 2x$$

$$+\left\{2r_1+e^2(r_3+r_4)-\left\{-(1-\frac{5}{2}e^2-\frac{5}{2}e_{\prime}^2-\frac{\gamma^2}{2})\frac{3}{4(1-m)}+\frac{9e^2}{2(2-2m-c)}\right.\right.$$

$$\left.\left.-\frac{3e^2}{2(2-2m+c)}\right\}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\right\}\frac{1}{2(1-m)}\sin 2t$$

$$+\left\{2r_3+e^2r_1-\left\{\frac{9}{2(2-m-c)}-\frac{3}{2(2-m)}\right\}\frac{m_{\prime}a^3}{\mu a_{\prime}^3}\right\}\frac{e}{(2-2m-c)}\sin(2t-x)$$

$$+\left\{2r_4+e^2r_1-\left\{-\frac{3}{2(2-m+c)}-\frac{3}{2(2-m)}\right\}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\right\}\frac{e}{(2-m+c)}\sin(2t+x)$$

$$+\frac{2r_5}{m}\sin z$$

$$+\left\{2r_6+\frac{21}{4(2-3m)}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\right\}\frac{e_{\prime}}{(2-3m)}\sin(2t-z)$$

$$+\left\{2r_7-\frac{3}{4(2-m)}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\right\}\frac{e_{\prime}}{(2-m)}\sin(2t+z)$$

$$+\left\{2r_9+r_3-\left\{-\frac{15}{4(2-2m-2c}+\frac{9}{2(2-2m-c}\right\}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\right\}\frac{e^2}{2(1-m-c)}\sin(2t-2x)$$

$$+\left\{2r_{10}+r_4-\left\{-\frac{3}{2(2-2m+2c)}-\frac{3}{2(2-m+c)}\right\}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\right\}\frac{e^2}{2(1-m+c)}\sin(2t+2x)$$

$$+\left\{2r_{11}+r_5\right\}\frac{ee_{\prime}}{(c+m)}\sin(x+z)$$

$$+\left\{2r_{12}+r_6-\left\{\frac{63}{4(2-3m-c)}-\frac{21}{4(2-3m)}\right\}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\right\}\frac{ee_{\prime}}{(2-3m-c)}\sin(2t-x-z)$$

$$+\left\{2r_{13}+r_7-\left\{\frac{3}{4(2-m+c)}+\frac{3}{4(2-m)}\right\}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\right\}\frac{ee_{\prime}}{(2-m+c)}\sin(2t+x+z)$$

Considering the terms which depend on the square of the disturbing force

$$\frac{d^2.r^2}{2dt^2}-\frac{\mu}{r}+\frac{\mu}{a}+2\int dR+r\left(\frac{dR}{dr}\right)=0$$

$$\frac{d^2 . r^2}{2\,d\,t^2} - \frac{d^2 . r^3 \delta . \frac{1}{r}}{d\,t^2} + \frac{3\,d^2 . r^4 \left(\delta . \frac{1}{r}\right)^2}{2\,d\,t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int d\,R + r\left(\frac{d\,R}{d\,r}\right) = 0$$

$$\frac{d^2 z}{d\,t^2} + \frac{\mu z}{r^3} + \frac{m_{,} z}{r_{,}^3} - \frac{3\,m_{,} z\, r\, r \cos(\lambda' - \lambda)}{r_{,}^5} = 0.$$

$$\frac{d\,\lambda'}{d\,t} = \frac{h}{r'^2}\left\{1 - \frac{1}{h}\int\left(\frac{d\,R}{d\,\lambda'}\right)d\,t\left\{1 - \frac{1}{h^2}\int\left(\frac{d\,R}{d\,\lambda'}\right)d\,t\right\} - \frac{1}{2\,h^2}\left\{\int\left(\frac{d\,R}{d\,\lambda'}\right)d\,t\right\}^2\right.$$

$$= \frac{h(1+s^2)}{r^2} - \frac{(1+s^2)}{r^2}\int\left(\frac{d\,R}{d\,\lambda'}\right)d\,t + \frac{(1+s^2)}{2\,r^2\,h}\left\{\int\left(\frac{d\,R}{d\,\lambda'}\right)d\,t\right\}^2$$

d R = the differential of R, supposing $n\,t$ only variable + the differential of R, with regard to $n_{,} t$ only in as much as it is contained in the terms in r, λ and s due to the perturbations ; hence

d R = the differential of R, supposing only $n\,t$ variable + $\frac{d\,R}{d\,r'}$. d . $\delta\,r'$ + $\frac{d\,R}{d\,r}$ d . $\delta\,\lambda'$ + $\frac{d\,R}{d\,r}$ d . $\delta\,s$; d . $\delta\,r'$, d . $\delta\,\lambda$, and d . $\delta\,s$, being restrained to mean the differentials of those quantities with regard to $n_{,} t$ only.

$$\delta\,R = \left(\frac{d\,R}{d\,r'}\right)\delta\,r' + \left(\frac{d\,R}{d\,\lambda'}\right)\delta\,\lambda' + \left(\frac{d\,R}{d\,s}\right)\delta\,s = -\,a\left(\frac{d\,R}{d\,a}\right)r'\,\delta . \frac{1}{r'} + \left(\frac{d\,R}{d\,t}\right)\delta\,\lambda' + \left(\frac{d\,R}{d\,s}\right)\delta\,s,$$

(t being used in the sense $n\,t - n_{,} t$.) $\left(\frac{d\,R}{d\,s}\right)\delta\,s = \frac{r^2}{2\,r_{,}^3}\,s\,\delta\,s$ nearly.

$$\left(\frac{d\,R}{d\,r}\right)d . \delta\,r + \left(\frac{d\,R}{d\,\lambda'}\right)d . \delta\,\lambda' + \left(\frac{d\,R}{d\,s}\right)d . \delta\,s = -\,a\left(\frac{d\,R}{d\,a}\right)d . r\,\delta . \frac{1}{r} + \left(\frac{d\,R}{d\,t}\right)d . \delta\,\lambda' + \left(\frac{d\,R}{d\,s}\right)d . \delta\,s$$

d . $r'\delta\frac{1}{r'}$, d . $\delta\,\lambda'$ and d . $\delta\,s$ being restrained to mean the differentials of those quantities with regard to $n_{,}t$ only.

$$\delta . r\left(\frac{d\,R}{d\,r}\right) = d . \frac{r\left(\frac{d\,R}{d\,r}\right)}{d\,r'} . \delta\,r' + d . \frac{r\left(\frac{d\,R}{d\,r}\right)}{d\,\lambda'}\,\delta\,\lambda' + d . \frac{r\left(\frac{d\,R}{d\,r}\right)}{d\,s}\,\delta\,s$$

$$= -\,a\,d . \frac{r\left(\frac{d\,R}{d\,r}\right)}{d\,a}\,r' . \delta . \frac{1}{r'} + d . \frac{r\left(\frac{d\,R}{d\,r}\right)}{d\,t} . \delta\,\lambda' + d . \frac{r\left(\frac{d\,R}{d\,r}\right)}{d\,s}\,\delta\,s$$

$$\delta . \left(\frac{d\,R}{d\,\lambda'}\right) = -\,a . d . \frac{\left(\frac{d\,R}{d\,\lambda'}\right)}{d\,a}\,r'\,\delta . \frac{1}{r'} + d . \frac{\left(\frac{d\,R}{d\,\lambda}\right)}{d\,t}\,\delta\,\lambda' + \frac{\left(\frac{d\,R}{d\,\lambda'}\right)}{d\,s}\,\delta\,s$$

A similar theorem exists with the quantity $\delta . \frac{\mathrm{d} R}{\mathrm{d} z}$, and it will readily be seen that all the developments δR, $\delta . r \left(\frac{\mathrm{d} R}{\mathrm{d} r}\right)$, $\delta . \left(\frac{\mathrm{d} R}{\mathrm{d} \lambda}\right)$ and $\delta . \left(\frac{\mathrm{d} R}{\mathrm{d} z}\right)$ may be effected very easily by means of Table II.

Similarly, if δ' denote the variation due to the disturbance of the earth by the moon,

$$\delta' R = - a_{,} \mathrm{d} . \left(\frac{\mathrm{d} R}{\mathrm{d} a_{,}}\right) r_{,} \delta . \frac{1}{r_{,}} - \mathrm{d} . \left(\frac{\mathrm{d} R}{\mathrm{d} t}\right) \delta \lambda_{,}$$

In d R the terms which arise from

$$- a \left(\frac{\mathrm{d} R}{\mathrm{d} a}\right) \mathrm{d} . r' \delta . \frac{1}{r'} + \left(\frac{\mathrm{d} R}{\mathrm{d} t}\right) \mathrm{d} . \delta \lambda' + \left(\frac{\mathrm{d} R}{\mathrm{d} s}\right) \mathrm{d} . \delta s$$

are multiplied by the small quantity m.

Considering in $r' \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right)$ and R the terms multiplied by $\frac{a^2}{a_{,}^3}$,

$$r' \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right) = 2 R, \qquad \delta . r' \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right) = 2 \delta R ;$$

considering the terms multiplied by $\frac{a^3}{a_{,}^4}$,

$$r' \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right) = 3 R, \qquad \delta . r' \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right) = 3 \delta R$$

Hence the value of $r' \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right)$ and $\delta . r' \left(\frac{\mathrm{d} R}{\mathrm{d} r'}\right)$ may at once be inferred from R and δR.

I reserve the formation of these developments and of the final equations for determining the coefficients of the different inequalities to another opportunity. These equations are voluminous when all sensible quantities are taken into account; but they are formed with so much facility by means of Table II., that error is not likely to arise in this part of the process. Error is more, I think, to be apprehended in the terms of R multiplied by the cubes and fourth powers of the eccentricities; the rest have been verified by an independent method. See p. 39.

Addition to Table I.

	146	149	150	
1	148 147	153 152	154 151	1
2	150 149	161 162	162 −146	2
3	152 151	147 164	148 163	3
4	154 153	165 148	166 147	4
5	156 155	167 −173	168 −174	5
6	158 157	169 170	178 169	6
7	160 159	171 176	172 175	7
146	62 0	2 − 65	66 − 2	146

	146	149	150	
147	1 63	69 3	4 67	147
148	64 1	4 68	70 3	148
149	2 65	77 0	8 − 62	149
150	66 2	8 62	78 0	150
151	3 67	63 9	1 79	151
152	68 3	1 80	64 9	152
153	4 69	81 1	10 63	153
154	70 4	10 64	82 1	154

	146	149	150	
155	5 71	83 − 14	11 − 90	155
156	72 5	11 − 89	84 − 14	156
157	6 73	93 12	16 85	157
158	74 6	16 86	94 12	158
159	7 75	87 15	13 91	159
160	76 7	13 92	88 15	160

	161	162	
	165 164	166 163	1
146	8 − 77	78 − 8	146

	161	162	
147	81 9	10 79	147
148	10 80	82 9	148

Addition to Table II.

	146	149	150	
1	147 148	152 153	151 154	1
2	149 150	146	...... −146	2
3	151 152	 147	 148	3
4	153 154	148	147	4
5	155 156	....	....	5
6	157 158	....	....	6
7	159 160	....	....	7
8	161 162	150	149	8
9	163 164	 151	 152	9

	146	149	150	
10	165 166	154	153	10
11	167 168	156	155	11
12	169 170	 157	 158	12
13	171 172	160	159	13
14	173 174	...... −155	...... −156	14
15	175 176	 159	 160	15
16	177 178	158	157	16
62	146	 150	...... −149	62
63	 147	151	 153	63

	146	149	150	
64	148	 154	152	64
65	 149	...... −146	....	65
66	150	....	146	66
67	 151	....	 147	67
68	152	 148	....	68
69	 153	147	....	69
70	154	....	148	70
71	 155	....	....	71
72	156	....	....	72

	146	149	150	
73	 157	....	....	73
74	158	....	....	74
75	 159	....	....	75
76	160	....	....	76
77	 161	149	....	77
78	162	....	150	78
79	 163	....	 151	79
80	164	 152	....	80
81	 165	153	....	81
82	166	....	154	82
83	 167	155	....	83
84	168	....	156	84
85	 169	....	 157	85
86	170	 158	....	86
87	 171	159	....	87
88	172	....	160	88
89	 173	...... −156	....	89
90	174	....	...... −155	90
91	 175	....	 159	91
92	176	 160	....	92
93	 177	157	....	93

	146	149	150	
94	178	....	158	94
146	 62	....	...... − 2	146
147	63 1	3	 4	147
148	1 64	 4	3	148
149	 2	....	....	149
150	2	....	....	150
151	 3	....	 1	151
152	3	 1	....	152
153	 4	1	....	153
154	4	....	1	154
155	 5	....	....	155
156	5	....	....	156
157	 6	....	....	157
158	6	....	....	158
159	 7	....	....	159
160	7	....	....	160
161	 8	2	....	161
162	8	 2	2	162
163	 9	....	 3	163
164	9	 3	....	164
165	 10	4	....	165

	146	149	150	
166	10	....	4	166
167	 11	5	....	167
168	11	....	5	168
169	 12	6	 6	169
170	12	 6	....	170
171	 13	7	....	171
172	13	....	7	172
173	 14	...... − 5	....	173
174	14	....	...... − 5	174
175	 15	....	 7	175
176	15	 7	....	176
177	 16	....	....	177
178	16	....	6	178
179	 17	....	....	179
180	17	....	....	180
181	 18	....	....	181
182	18	....	....	182
183	 19	....	....	183
184	19	....	....	184

	161	162	
8	146	...... −146	8
9	 147	 148	9
10	148	147	10
77	...... −146	....	77
78	....	146	78
79	....	 147	79
80	 148	....	80
81	147	....	81
82	....	148	82
163	....	 1	163
164	 1	....	164
165	1	....	165
166	....	1	166

On the Precession of the Equinoxes, supposing the Earth to revolve in a resisting medium.

In my last paper on Physical Astronomy, I gave expressions for the variations of the six constants which enter into the solution of this problem, upon the hypothesis that the body revolves in a medium devoid of resistance.

If we suppose a plane to revolve in a resisting medium, about an axis perpendicular to itself, the resistance of the medium can produce no effect, and the phenomena will only be modified in a slight degree by the friction of the plane surface against the medium. If, however, the inclination of the plane on the axis of rotation differs from 90°, the effect of the resistance of the medium becomes sensible, tending to retard the motion of the plane; the effect being greatest when the axis of rotation is parallel to the plane.

This principle is used in some machines, as in self-playing organs, to regulate the motion by means of a vane, of which the inclination to its axis of rotation can be varied at pleasure.

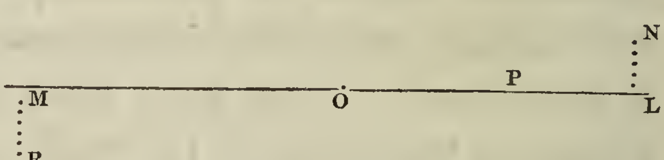
In the case of a sphere, whatever be the direction of the axis of rotation, this effect of the resistance is insensible; and also in the case of a solid of revolution when the axis of rotation coincides with the axis of the figure, but not otherwise. If the difference of the latitude of the axis of rotation from 90° (supposing the equator from which the latitudes are reckoned to coincide with the equator of the figure) be at any time small, the mathematical investigation appears to show, that the effect of the resistance of the medium is to diminish continually this difference. In the case of the earth, this quantity is now insensible; but as the probability is small that this was the case in the first instance, may this circumstance arise from the resistance of a medium of small density acting for a great length of time? and can the change of climate on the surface of the earth, a change of which the probability is indicated by many geological phenomena, be explained in the same manner? It may be remarked, however, that the effect of a resisting medium in diminishing the eccentricities of the orbits of the planets is of the same order, and that these, although for the most part small, are far from having reached zero. The tendency of a resisting medium is also to diminish the major axes of the orbits of the planets; these effects, if they exist, will probably be most sensible

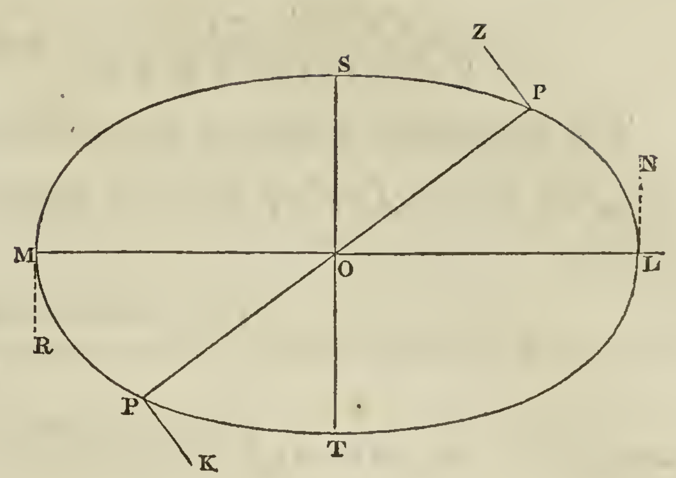
in the case of comets, not only on account of their great eccentricity, but also on account of their small density, in the same manner as a flock of any light substance is wafted by the gentlest wind and prevented from reaching the ground. The eccentricity of the orbit of the comet of Halley in 1759 is known with great accuracy, and as its perturbations have been calculated with great care by MM. Damoiseau and de Pontécoulant, the eccentricity which it should have in 1835, when it will again visit this part of space, unless it be affected by a resisting medium, is also known with great precision. It is scarcely probable, however, that any change will be perceptible in one revolution, even if the cause exists; but the succeeding revolutions of this body will no doubt throw light upon this question. The ratio of the change of the semi-major axis to the change of the eccentricity, due to the action of the resisting medium, is known, being a function of the eccentricity, and independent of the constant, which depends upon the density of the medium; this ratio therefore may also tend to elucidate the question, if it can be determined by observation with sufficient accuracy.

Let x', y', z' be the co-ordinates of any point P corresponding to the elementary portion of the surface d s, and referred to axes passing through the centre of gravity and revolving with the body in motion.

Let P be the point of which the co-ordinates are x', y', z', A P the direction of the normal at the point P, B P perpendicular to the axis of instantaneous rotation, and cutting it in B, and C P the direction of motion of the point P. I suppose the resistance of the medium to create a force proportional to v^2 cos A P C d s, acting in the direction of the normal A P upon the point P, v being the velocity of the point P.

Suppose the straight line M O P L to revolve about an axis passing through O, and perpendicular to it, and in the direction L N, the action of the resisting medium will be in the direction N L, on one side only of the line O L, upon all the points P between O and L, and upon all the points between M P it will be in the contrary direction R M, and on the other side of the line.





Now, let L S M T be the section of a cylinder revolving about an axis, passing through O perpendicular to the plane L S M T, and let the cylinder revolve in the direction L N. The action of the resisting medium will be in the direction Z P, perpendicular to O P upon all the points P between L S; and in the contrary direction K P upon all the points, P between T M. These remarks show that in what follows, the integrations must not be made throughout the whole surface of the body revolving: this consideration however does not affect the nature of the results.

The equation to a plane perpendicular to the axis of rotation, and passing through the centre of gravity of the body, is $p x + q y + r z = 0$.

Let the body revolving be a spheroid of which the equation is

$$x^2 + y^2 + z^2 (1 + e^2) = a^2 (1 + e^2)$$

The equation to the tangent plane to the spheroid at the point x, y, z is

$$x x' + y y' + z z' (1 + e^2) = a^2 (1 + e^2)$$

The equations to the planes from whose intersection the line P B results, are

$$*\, x (q z' - r y') + y (r x' - p z') + z (p y' - q x') = 0$$
$$p x + q y + r z = D$$

D being a constant. The equations to the line P C are

$$x\{r (q z' - r y') - p (p y' - q x')\} + y \{r (r x' - p z') - q (p y' - q x')\} = 0$$
$$x\{q (q z' - r y') - p (r x' - p z')\} + z \{q (p y' - q x') - r (r x' - p z')\} = 0$$

and neglecting p^2, q^2, $p q$,

$$x (q z' - r y') = y (p z' - r x')$$
$$x (q y' + p x') = z (p z' - r x')$$

The equations to the direction of motion of the point P are

$$x (p z' - r x') = y (r y' - q r')$$
$$x (q x' - p y') = z (r y' - q z')$$

Cos. angle, which the direction of motion of P makes with the normal to the surface or cos A P C

$$= \frac{x' (r y' - q z') + y' (p z' - r x') + z' (1 + e^2) (q x' - p y')}{\sqrt{\{(r y' - q z')^2 + (p z' - r x')^2 + (q y - p x')^2\}} \sqrt{\{x'^2 + y'^2 + z'^2 (1 + 2 e^2)\}}}$$

* The notation is the same as p. 20, except that the accents at foot of $x_{,}$, $y_{,}$, $z_{,}$ are omitted.

$$= \frac{e^2 z' (q x' - p y')}{r \sqrt{x'^2 + y'^2} \sqrt{x'^2 + y'^2 + z'^2}} \quad \text{nearly.}$$

The resistance acting in the direction of the normal, and since the velocity $= \sqrt{x'^2 + y'^2} \sqrt{(p^2 + q^2 + r^2)}$ nearly;

$$C \, \mathrm{d} r = 0$$

$$B \, \mathrm{d} q + (A - C) \, r p \, \mathrm{d} t = \mathrm{d} t \int \frac{\{z' x' - x' z' (1 + e^2)\} e^2 z' (q x' - p y') \sqrt{x'^2 + y'^2} \, \mathrm{d} s \, (p^2 + q^2 + r^2)}{r \{x'^2 + y'^2 + z'^2\}}$$

$$A \, \mathrm{d} p + (C - B) \, q r \, \mathrm{d} t = \mathrm{d} t \int \frac{\{y' z' (1 + e^2) - z' y'\} e^2 z' (q x' - p y') \sqrt{x'^2 + y'^2} \, \mathrm{d} s \, (p^2 + q^2 + r^2)}{r \{x'^2 + y'^2 + z'^2\}}$$

$$\sin \frac{C - A}{A} (n t + \gamma) \, \mathrm{d} c + c \frac{(C - A)}{A} \cos \frac{C - A}{A} (n t + \gamma) \, \mathrm{d} \gamma$$

$$= - \frac{n \, \mathrm{d} t \, e^4}{A} \int \frac{x' z'^2 (q x' - p y') \sqrt{x'^2 + y'^2} \, \mathrm{d} s}{\{x'^2 + y'^2 + z'^2\}}$$

$$\cos \frac{C - A}{A} (n t + \gamma) \, \mathrm{d} c - c \frac{(C - A)}{A} \sin \frac{C - A}{A} (n t + \gamma) \, \mathrm{d} \gamma$$

$$= \frac{n \, \mathrm{d} t \, e^4}{A} \int \frac{y' z'^2 (q x' - p y') \sqrt{x'^2 + y'^2} \, \mathrm{d} s}{\{x'^2 + y'^2 + z'^2\}}$$

since $\int x'^2 z'^2 \, \mathrm{d} s = \int y'^2 z'^2 \, \mathrm{d} s$

$$\mathrm{d} c = - \frac{n \, \mathrm{d} t \, e^4 c}{A} \int \frac{x'^2 z'^2 \sqrt{x'^2 + y'^2} \, \mathrm{d} s}{\{x'^2 + y'^2 + z'^2\}} + \frac{n \, \mathrm{d} t \, e^4}{2 A} \sin 2 \frac{(C - A)}{A} (n t + \gamma) \int \frac{x' y' z'^2 \sqrt{x'^2 + y'^2} \, \mathrm{d} s}{\{x'^2 + y'^2 + z'^2\}}$$

neglecting the term which is periodic,

$$\mathrm{d} c = - n c \frac{e^4 \, \mathrm{d} t}{A} \int \frac{x'^2 z'^2 \sqrt{x'^2 + y'^2} \, \mathrm{d} s}{\{x'^2 + y'^2 + z'^2\}}$$

Let $\int \frac{x'^2 z'^2 \sqrt{x'^2 + y'^2} \, \mathrm{d} s}{\{x'^2 + y'^2 + z'^2\}} = D$

D being a positive quantity.

$$\mathrm{d} c = - \frac{n D c e^4 \, \mathrm{d} t}{A} \qquad e^{\frac{1}{c}} = \frac{n D e^4 t}{A},$$ e being the base of Naperian logarithms.

When t is infinite $c = 0$; hence the latitude of the axis of instantaneous rotation increases until it reaches 90°, which is its limit.

Having determined the variations of c, γ and n by means of the above equations, the variations of the other constants ω, ψ_0 and φ_0 may be determined from the equations

$$p \, \mathrm{d} t = \sin \varphi \sin \theta \, \mathrm{d} \psi - \cos \varphi \, \mathrm{d} \theta$$
$$q \, \mathrm{d} t = \cos \varphi \sin \theta \, \mathrm{d} \psi + \sin \varphi \, \mathrm{d} \theta$$
$$r \, \mathrm{d} t = \mathrm{d} \varphi - \cos \theta \, \mathrm{d} \psi$$

XVI. *Researches in Physical Astronomy.* *By* JOHN WILLIAM LUBBOCK, *Esq. V.P. and Treas. R.S.*

Read June 9, 1831.

I PROPOSE in this paper to extend the equations I have already given for determining the planetary inequalities, as far as the terms depending on the squares and products of the eccentricities, to the terms depending on the cubes of the eccentricities and quantities of that order, which is done very easily by a Table similar to Table II. in my Lunar Theory; and particularly to the determination of the great inequality of Jupiter, or at least such part of it as depends on the first power of the disturbing force. That part which depends on the square of the disturbing force may I think be most easily calculated by the methods given in my Lunar Theory; but not without great care and attention can accurate numerical results be expected. I have however given the analytical form of the coefficients of the arguments in the development of R, upon which that inequality principally depends.

It is I think particularly convenient to designate the arguments of the planetary disturbances by indices. The system of indices adopted in this paper is given as appearing better adapted for the purpose than that used in my former paper on the Planetary Theory; but it is not advisable to make use of the same indices in this as in the Lunar Theory.

I have also given analytical expressions for the development of R to the terms multiplied by the squares and products of the eccentricities inclusive, and for the terms in $r\left(\frac{\mathrm{d} R}{\mathrm{d} r}\right)$ multiplied by the first power of the eccentricities, which are I believe the simplest that can be proposed.

The following are the arguments which occur in the Planetary Theory.

Column 1 contains the index.

—— 2 contains the index of the argument, which is symmetrical.

—— 3 contains the index used Phil. Trans. Part II. 1830, p. 349.

0	..	..	0	104	..	39	$4t + x - z = 5nt - 5n_{,}t - \varpi + \varpi_{,}$
1	..	..	$t = nt - n_{,}t$	110	50	57	$2z = 2n_{,}t - 2\varpi_{,}$
2	..	..	$2t = 2nt - 2n_{,}t$	111	61	63	$t - 2z = nt - 3n_{,}t + 2\varpi_{,}$
3	..	..	$3t = 3nt - 3n_{,}t$	112	62	64	$2t - 2z = 2nt - 4n_{,}t + 2\varpi_{,}$
4	..	..	$4t = 4nt - 4n_{,}t$	113	63	65	$3t - 2z = 3nt - 5n_{,}t + 2\varpi_{,}$
10	30	7	$x = nt - \varpi$	114	64	66	$4t - 2z = 4nt - 6n_{,}t + 2\varpi_{,}$
11	41	6	$t - x = -n_{,}t + \varpi$	121	51	58	$t + 2z = nt + n_{,}t - 2\varpi_{,}$
12	42	12	$2t - x = nt - 2n_{,}t + \varpi$	122	52	59	$2t + 2z = 2nt - 2\varpi_{,}$
13	43	13	$3t - x = 2nt - 3n_{,}t + \varpi$	123	52	60	$3t + 2z = 3nt - n_{,}t - 2\varpi_{,}$
14	44	14	$4t - x = 3nt - 4n_{,}t + \varpi$	124	54	61	$4t + 2z = 4nt - 2n_{,}t - 2\varpi_{,}$
21	31	8	$t + x = 2nt - n_{,}t - \varpi$	130	..	69	$2y_{,} = 2n_{,}t - 2\nu_{,}$
22	32	9	$2t + x = 3nt - 2n_{,}t - \varpi$	131	..	71	$t - 2y = nt - 3n_{,}t + 2\nu_{,}$
23	33	10	$3t + x = 4nt - 3n_{,}t - \varpi$	132	..	73	$2t - 2y = 2nt - 4n_{,}t + 2\nu_{,}$
24	34	11	$4t + x = 5nt - 4n_{,}t - \varpi$	133	..	..	$3t - 2y = 3nt - 5n_{,}t + 2\nu_{,}$
30	10	15	$z = n_{,}t - \varpi_{,}$	134	..	..	$4t - 2y = 4nt - 6n_{,}t + 2\nu_{,}$
31	21	20	$t - z = nt - 2n_{,}t + \varpi_{,}$	141	..	68	$t + 2y = nt + n_{,}t - 2\nu_{,}$
32	22	21	$2t - z = 2nt - 3n_{,}t + \varpi_{,}$	142	..	70	$2t + 2y = 2nt - 2\nu_{,}$
33	23	22	$3t - z = 3nt - 4n_{,}t + \varpi_{,}$	143	..	72	$3t + 2y = 3nt - n_{,}t - 2\nu_{,}$
34	24	23	$4t - z = 4nt - 5n_{,}t + \varpi_{,}$	144	..	..	$4t + 2y = 4nt - 2n_{,}t - 2\nu_{,}$
41	11	16	$t + z = nt - \varpi_{,}$	150	250	..	$3x = 3nt - 3\varpi$
42	12	17	$2t + z = 2nt - n_{,}t - \varpi_{,}$	151	261	..	$t - 3x = -2nt - n_{,}t + 3\varpi$
43	13	18	$3t + z = 3nt - 2n_{,}t - \varpi_{,}$	152	262	..	$2t - 3x = -nt - 2n_{,}t + 3\varpi$
44	14	19	$4t + z = 4nt - 3n_{,}t - \varpi_{,}$	153	263	..	$3t - 3x = -3n_{,}t + 3\varpi$
50	110	26	$2x = 2nt - 2\varpi$	154	264	..	$4t - 3x = nt - 4n_{,}t + 3\varpi$
51	121	25	$t - 2x = -nt - n_{,}t + 2\varpi$	161	251	..	$t + 3x = 4nt - n_{,}t - 3\varpi$
52	122	24	$2t - 2x = -2n_{,}t + 2\varpi$	162	252	..	$2t + 3x = 5nt - 2n_{,}t - 3\varpi$
53	123	32	$3t - 2x = nt - 3n_{,}t - 2\varpi$	163	253	..	$3t + 3x = 6nt - 3n_{,}t - 3\varpi$
54	124	33	$4t - 2x = 2nt - 4n_{,}t + 2\varpi$	164	254	..	$4t + 3x = 7nt - 4n_{,}t - 3\varpi$
61	111	27	$t + 2x = 3nt - n_{,}t - 2\varpi$	170	210	..	$2x + z = 2nt + n_{,}t - 2\varpi - \varpi_{,}$
62	112	28	$2t + 2x = 4nt - 2n_{,}t - 2\varpi$	171	221	..	$t - 2x - z = -nt - 2n_{,}t + 2\varpi + \varpi_{,}$
63	113	29	$3t + 2x = 5nt - 3n_{,}t - 2\varpi$	172	222	..	$2t - 2x - z = -3n_{,}t + 2\varpi + \varpi_{,}$
64	114	30	$4t + 2x = 6nt - 4n_{,}t - 2\varpi$	173	223	..	$3t - 2x - z = nt - 4n_{,}t + 2\varpi + \varpi_{,}$
70	..	47	$x + z = nt + n_{,}t - \varpi - \varpi_{,}$	174	224	..	$4t - 2x - z = 2nt - 5n_{,}t + 2\varpi + \varpi_{,}$
71	81	46	$t - x - z = -2n_{,}t + \varpi + \varpi_{,}$	181	211	..	$t + 2x + z = 3nt - 2\varpi - \varpi_{,}$
72	82	53	$2t - x - z = nt - 3n_{,}t + \varpi + \varpi_{,}$	182	212	..	$2t + 2x + z = 4nt - n_{,}t - 2\varpi - \varpi_{,}$
73	83	54	$3t - x - z = 2nt - 4n_{,}t + \varpi + \varpi_{,}$	183	213	..	$3t + 2x + z = 5nt - 2n_{,}t - 2\varpi - \varpi_{,}$
74	84	55	$4t - x - z = 3nt - 5n_{,}t + \varpi + \varpi_{,}$	184	214	..	$4t + 2x + z = 6nt - 3n_{,}t - 2\varpi - \varpi_{,}$
81	71	48	$t + x + z = 2nt - \varpi - \varpi_{,}$	190	230	..	$2x - z = 2nt - n_{,}t - 2\varpi + \varpi_{,}$
82	72	49	$2t + x + z = 3nt - n_{,}t - \varpi - \varpi_{,}$	191	231	..	$t - 2x + z = -nt + 2\varpi - \varpi_{,}$
83	73	50	$3t + x + z = 4nt - 2n_{,}t - \varpi - \varpi_{,}$	192	232	..	$2t - 2x + z - n_{,}t + 2\varpi - \varpi_{,}$
84	74	51	$4t + x + z = 5nt - 3n_{,}t - \varpi - \varpi_{,}$	193	233	..	$3t - 2x + z = nt - 2n_{,}t + 2\varpi - \varpi_{,}$
90	..	35	$x - z = nt - n_{,}t - \varpi + \varpi_{,}$	194	234	..	$4t - 2x + z = 2nt - 3n_{,}t + 2\varpi - \varpi_{,}$
91	..	41	$t - x + z = \varpi - \varpi_{,}$	201	241	..	$t + 2x - z = 3nt - 2n_{,}t - 2\varpi + \varpi_{,}$
92	..	42	$2t - x + z = nt - n_{,}t + \varpi - \varpi_{,}$	202	242	..	$2t + 2x - z = 4nt - 3n_{,}t - 2\varpi + \varpi_{,}$
93	..	43	$3t - x + z = 2nt - 2n_{,}t + \varpi + \varpi_{,}$	203	243	..	$3t + 2x - z = 5nt - 4n_{,}t - 2\varpi + \varpi_{,}$
94	..	44	$4t - x + z = 3nt - 3n_{,}t + \varpi + \varpi_{,}$	204	244	..	$4t + 2x - z = 6nt - 5n_{,}t - 2\varpi + \varpi_{,}$
101	..	36	$t + x - z = 2nt - 2n_{,}t - \varpi + \varpi_{,}$	210	170	..	$x + 2z = nt + 2n_{,}t - \varpi - 2\varpi_{,}$
102	..	37	$2t + x - z = 3nt - 3n_{,}t - \varpi + \varpi_{,}$	211	222	..	$t - x - 2z = -3n_{,}t + \varpi + 2\varpi_{,}$
103	..	38	$3t + x - z = 4nt - 4n_{,}t - \varpi + \varpi_{,}$	212	223	..	$2t - x - 2z = nt - 4n_{,}t + \varpi + 2\varpi_{,}$

213	224	$3t-x-2z=2nt-5n_{,}t+\varpi+2\varpi_{,}$	282	..	$2t+x+2y=3nt-\varpi-2\nu_{,}$
214	225	$4t-x-2z=3nt-6n_{,}t+\varpi+2\varpi_{,}$	283	..	$3t+x+2y=4nt-n_{,}t-\varpi-2\nu_{,}$
221	171	$t+x+2z=2nt+n_{,}t-\varpi-2\varpi_{,}$	284	..	$4t+x+2y=5nt-2n_{,}t-\varpi-2\nu_{,}$
222	172	$2t+x+2z=3nt-\varpi-2\varpi_{,}$	290	..	$x-2y=nt-2n_{,}t-\varpi+2\nu_{,}$
223	173	$3t+x+2z=4nt-n_{,}t-\varpi-2\varpi_{,}$	291	..	$t-x+2y=n_{,}t+\varpi-2\nu_{,}$
224	174	$4t+x+2z=5nt-n_{,}t-\varpi-2\varpi_{,}$	292	..	$2t-x+2y=nt+\varpi-2\nu_{,}$
230	190	$x-2z=nt-2n_{,}t-\varpi+2\varpi_{,}$	293	..	$3t-x+2y=2nt-n_{,}t+\varpi-2\nu_{,}$
231	191	$t-x+2z=n_{,}t+\varpi-2\varpi_{,}$	294	..	$4t-x+2y=3nt-2n_{,}t+\varpi-2\nu_{,}$
232	192	$2t-x+2z=nt+\varpi-2\varpi_{,}$	301	..	$t+x-2y=2nt-3n_{,}t-\varpi+2\nu_{,}$
233	193	$3t-x+2z=2nt-n_{,}t+\varpi-2\varpi_{,}$	302	..	$2t+x-2y=3nt-4n_{,}t-\varpi+2\nu_{,}$
234	194	$4t-x+2z=3nt-2n_{,}t+\varpi-2\varpi_{,}$	303	..	$3t+x-2y=4nt-5n_{,}t-\varpi+2\nu_{,}$
241	201	$t+x-2z=2nt-3n_{,}t-\varpi+2\varpi_{,}$	304	..	$4t+x-2y=5nt-6n_{,}t-\varpi+2\nu_{,}$
242	202	$2t+x-2z=3nt-4n_{,}t-\varpi+2\varpi_{,}$	310	..	$z+2y=3n_{,}t-\varpi_{,}-2\nu_{,}$
243	203	$3t+x-2z=4nt-5n_{,}t-\varpi+2\varpi_{,}$	311	..	$t-z-2y=nt-4n_{,}t+\varpi_{,}+2\nu_{,}$
244	204	$4t+x-2z=5nt-6n_{,}t-\varpi+2\varpi_{,}$	312	..	$2t-z-2y=2nt-5n_{,}t+\varpi_{,}+2\nu_{,}$
250	150	$3z=3n_{,}t-3\varpi_{,}$	313	..	$3t-z-2y=3nt-6n_{,}t+\varpi_{,}+2\nu_{,}$
251	161	$t-3z=nt-4n_{,}t+3\varpi_{,}$	314	..	$4t-z-2y=4n_{,}t-7n_{,}t+\varpi_{,}+2\nu_{,}$
252	162	$2t-3z=2nt-5n_{,}t+3\varpi_{,}$	321	..	$t+z+2y=nt+2n_{,}t-\varpi_{,}-2\nu_{,}$
253	163	$3t-3z=3nt-6n_{,}t+3\varpi_{,}$	322	..	$2t+z+2y=2nt+n_{,}t-\varpi_{,}-2\nu_{,}$
254	164	$4t-3z=4nt-7n_{,}t+3\varpi_{,}$	323	..	$3t+z+2y=3nt-\varpi_{,}-2\nu_{,}$
261	151	$t+3z=nt+2n_{,}t-3\varpi_{,}$	324	..	$4t+z+2y=4nt-n_{,}t-\varpi_{,}-2\nu_{,}$
262	152	$2t+3z=2n+n_{,}t-3\varpi_{,}$	330	..	$z-2y=-n_{,}t-\varpi_{,}+2\nu_{,}$
263	153	$3t+3z=3nt-3\varpi_{,}$	331	..	$t-z+2y=nt+\varpi_{,}-2\nu_{,}$
264	154	$4t+3z=4nt-n_{,}t-3\varpi_{,}$	332	..	$2t-z+2y=2nt-n_{,}t+\varpi_{,}-2\nu_{,}$
270	..	$x+2y=nt+2n_{,}t-\varpi-2\nu_{,}$	333	..	$3t-z+2y=3nt-2n_{,}t+\varpi_{,}+2\nu_{,}$
271	..	$t-x-2y=-3n_{,}t+\varpi+2\nu_{,}$	334	..	$4t-z+2y=4nt-3n_{,}t+\varpi_{,}-2\nu_{,}$
272	..	$2t-x-2y=nt-4n_{,}t+\varpi+2\nu_{,}$	341	..	$t+z-2y=nt-2n_{,}t-\varpi_{,}+2\nu_{,}$
273	..	$3t-x-2y=2nt-5n_{,}t+\varpi+2\nu_{,}$	342	..	$2t+z-2y=2nt-3n_{,}t-\varpi_{,}+2\nu_{,}$
274	..	$4t-x-2y=3nt-6n_{,}t+\varpi+2\nu_{,}$	343	..	$3t+z-2y=3nt-4n_{,}t-\varpi_{,}+2\nu_{,}$
281	..	$t+x+2y=2nt+n_{,}t-\varpi-2\nu_{,}$	344	..	$4t+z-2y=4nt-5n_{,}t-\varpi_{,}+2\nu_{,}$

Table I.

Showing the arguments which result from the combination of the arguments 10, 50 and 150 with the arguments in the first or left-hand column, by addition and subtraction.

	10	50	150			10	50	150			10	50	150	
1	21 11	61 51	161 151	1	51	11 151	....	....	51	102	202 32	....	....	102
2	22 12	62 52	162 152	2	52	12 152	....	....	52	103	203 33	....	....	103
3	23 13	63 53	163 153	3	53	13 153	....	....	53	104	204 34	....	....	104
4	24 14	64 54	164 154	4	54	14 154	....	....	54	110	210 –230	....	....	110
10	50 0	150 – 10	....	10	61	161 21	....	....	61	111	241 211	....	....	111
11	1 51	21 151	....	11	62	162 22	....	....	62	112	242 212	....	....	112
12	2 52	22 152	....	12	63	163 23	....	....	63	113	243 213	....	....	113
13	3 53	23 153	....	13	64	164 24	....	....	64	114	244 214	....	....	114
14	4 54	24 154	....	14	70	170 30	....	....	70	121	221 231	....	....	121
21	61 1	161 11	....	21	71	31 171	....	....	71	122	222 232	....	....	122
22	62 2	162 12	....	22	72	32 172	....	....	72	123	223 233	....	....	123
23	63 3	163 13	....	23	73	33 173	....	....	73	124	224 234	....	....	124
24	64 4	164 14	....	24	74	34 174	....	....	74	130	270 –290	....	....	130
30	70 – 90	170 –190	....	30	81	181 41	....	....	81	131	301 271	....	....	131
31	101 71	201 171	....	31	82	182 42	....	....	82	132	302 272	....	....	132
32	102 72	202 172	....	32	83	183 43	....	....	83	133	303 273	....	....	133
33	103 73	203 173	....	33	84	184 44	....	....	84	134	304 274	....	....	134
34	104 74	204 174	....	34	90	190 – 30	....	....	90	141	281 291	....	....	141
41	81 91	181 191	....	41	91	41 191	....	....	91	142	282 292	....	....	142
42	82 92	182 192	....	42	92	42 192	....	....	92	143	283 293	....	....	143
43	83 93	183 193	....	43	93	43 193	....	....	93	144	284 294	....	....	144
44	84 94	184 194	....	44	94	44 194	....	....	94					
50	150 10	....	....	50	101	201 31	....	....	101					

Table II.

Showing the arguments which, by their combination with the arguments 10, 50, and 150, by addition and subtraction, produce the arguments in the first or left-hand column.

	10	50	150			10	50	150			10	50	150	
1	11 21	....	....	1	43	93 83	....	....	43	90	...... – 30	....	....	90
2	12 22	....	....	2	44	94 84	....	....	44	91	 41	....	....	91
3	13 23	....	....	3	50	10	0	....	50	92	 42	....	....	92
4	14 24	....	....	4	51	 11	 1	....	51	93	 43	....	....	93
10	0 50	...... – 10	....	10	52	 12	 2	....	52	94	 44	....	....	94
11	51 1	 21	....	11	53	 13	 3	....	53	101	31	....	....	101
12	52 2	 22	....	12	54	 14	 4	....	54	102	32	....	....	102
13	53 3	 23	....	13	61	21	1	....	61	103	33	....	....	103
14	54 4	 24	....	14	62	22	2	....	62	104	34	....	....	104
21	1 61	11	....	21	63	23	3	....	63	150	50	10	0	150
22	2 62	12	....	22	64	24	4	....	64	151	 51	 11	 1	151
23	3 63	13	....	23	70	30	....	....	70	152	 52	 12	 2	152
24	4 64	14	....	24	71	 31	....	....	71	153	 53	 13	 3	153
30	70. – 90	....	....	30	72	 32	....	....	72	154	 54	 14	 4	154
31	71 101	....	....	31	73	 33	....	....	73	161	61	21	1	161
32	72 102	....	....	32	74	 34	....	....	74	162	62	22	2	162
33	73 103	....	....	33	81	41	....	....	81	163	63	23	3	163
34	74 104	....	....	34	82	42	....	....	82	164	64	24	4	164
41	91 81	....	....	41	83	43	....	....	83	170	70	30	....	170
42	92 82	....	....	42	84	44	....	....	84	171	 71	 31	....	171

TABLE II. (Continued.)

	10	50	150			10	50	150			10	50	150	
172	 72	 32	....	172	212	 112	....	....	212	272	 132	....	....	272
173	 73	 33	....	173	213	 113	....	....	213	273	 133	....	....	273
174	 74	 34	....	174	214	 114	....	....	214	274	 134	....	....	274
181	81	41	....	181	221	121	....	....	221	281	141	....	....	281
182	82	42	....	182	222	122	....	....	222	282	142	....	....	282
183	83	43	....	183	223	123	....	....	223	283	143	....	....	283
184	84	44	....	184	224	124	....	....	224	284	144	....	....	284
190	90	...... − 30	....	190	230	...... −110	....	....	230	290	...... −130	....	....	290
191	 91	 41	....	191	231	 121	....	....	231	291	 141	....	....	291
192	 92	 42	....	192	232	 122	....	....	232	292	 142	....	....	292
193	 93	 43	....	193	233	 123	....	....	233	293	 143	....	....	293
194	 94	 44	....	194	234	 124	....	....	234	294	 144	....	....	294
201	101	31	....	201	241	111	....	....	241	301	131	....	....	301
202	102	32	....	202	242	112	....	....	242	302	132	....	....	302
203	103	33	....	203	243	113	....	....	243	303	133	....	....	303
204	104	34	....	204	244	114	....	....	244	304	134	....	....	304
210	110	....	....	210	270	130	....	....	270					
211	 111	....	....	211	271	 131	....	....	271					

The following examples will show the use of the preceding Table, in forming the equations of condition which serve to determine the coefficients of the inequalities of the reciprocal of the radius vector and of the longitude.

$$-\frac{d^2 . r^3 \delta \frac{1}{r}}{d t^2} - \mu \delta . \frac{1}{r} + 2 \int d R + r \left(\frac{d R}{d r}\right) = 0$$

$$r^3 = a^3 \left\{ 1 + 3\,e^2 \left(1 + \frac{e^2}{8}\right) - 3\,e \left(1 + \frac{3}{8}\,e^2\right) \cos\left(n\,t + \varepsilon - \varpi\right) + \frac{e^3}{8} \cos\left(3\,n\,t + 3\,\varepsilon - 3\,\varpi\right)\right.$$

$$\frac{(n - n_{\prime})^2}{n^2} \left\{ (1 + 3\,e^2)\,r_1 - \frac{3\,e^2}{2}\,(r_{11} + r_{21}) \right\} - r_1 + \frac{m_{\prime}}{a}\,q_1 = 0$$

$$\frac{4\,(n - n_{\prime})^2}{n^2} \left\{ (1 + 3\,e^2)\,r_2 - \frac{3}{2}\,e^2\,(r_{12} + r_{22}) \right\} - r_2 + \frac{m_{\prime}}{a}\,q_2 = 0$$

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\,t} = \frac{h}{r^2} + \frac{2\,h}{r}\,\delta \,.\, \frac{1}{r} - \frac{1}{r^2} \int \frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\,\mathrm{d}\,t$$

$$\frac{a^2}{r^2} = 1 + \frac{e^2}{2} + 2\,e \left(1 + \frac{3\,e^2}{8}\right) \cos\left(n\,t + e - \varpi\right) + \frac{5\,e^2}{2} \cos\left(2\,n\,t + 2\,\varepsilon - 2\,\varpi\right)$$

$$+ \frac{13}{4}\,e^3 \cos\left(3\,n\,t + 3\,\varepsilon - 3\,\varpi\right)$$

$$\frac{a}{r} = 1 + e \left(1 - \frac{e^2}{8}\right) \cos\left(n\,t + \varepsilon - \varpi\right) + e^2 \cos\left(2\,n\,t + 2\,\varepsilon - 2\,\varpi\right) + \frac{9}{8}\,e^3 \cos\left(3\,n\,t + 3\,\varepsilon - 3\,\varpi\right)$$

$$\lambda = n \{ 1 + 2\,r_0 \}\,t + \varepsilon$$

$$+ \left\{ 2 \left\{ r_1 + \frac{e^2}{2}\,(r_{11} + r_{21}) \right\} \right.$$

$$\left. - \frac{m_{\prime}}{\mu} \left\{ \left(1 + \frac{e^2}{2}\right) \frac{a\,n\,R_1}{(n - n_{\prime})} + \frac{e^2}{n_{\prime}}\,a\,n\,R_{11} + \frac{e^2\,a\,n\,R_{21}}{(2\,n - n_{\prime})} \right\} \right\} \frac{n}{(n - n_{\prime})} \sin\left(n\,t - n_{\prime}t + \varepsilon - \varepsilon_{\prime}\right)$$

$$+ \left\{ 2 \left\{ r_2 + \frac{e^2}{2} (r_{12} + r_{22}) \right\} \right.$$

$$\left. - \frac{m_{\prime}}{\mu} \left\{ \left(1 + \frac{e^2}{2}\right) \frac{a\,n\,R_2}{(n - n_{\prime})} + \frac{2\,e^2 a\,n\,R_{12}}{(n - 2\,n_{\prime})} + \frac{2\,e^2\,a\,n\,R_{22}}{(3\,n - 2\,n_{\prime})} \right\} \right\} \frac{n}{2\,(n - n_{\prime})} \sin\left(2\,n\,t - 2\,n_{\prime}t + \varepsilon - \varepsilon_{\prime}\right)$$

In the same way, by means of the Table, all the other coefficients may be found.

The great inequality of Jupiter consists of the arguments 155, 174, 213, 273, and 312, the variable part of which is $2\,n - 5\,n_{\prime}$, and arises, as is well known, from the introduction of the square of this quantity, which is small, by successive integrations in the denominators of the coefficients of the sines in the expression for the longitude, of which the above named are the arguments.

The following are the equations which have reference to these arguments, and which may be found at once by Table II.

$$\frac{(2\,n - 5\,n_{\prime})^2}{n^2} \left\{ r_{155} - \frac{3}{2}\,r_{54} + \frac{1}{16}\,r_4 \right\} - r_{155} + \frac{m_{\prime}\,a}{\mu}\,q_{155} = 0$$

$$\frac{(2\,n - 5\,n_{\prime})^2}{n^2} \left\{ r_{174} - \frac{3}{2}\,r_{74} \right\} - r_{174} + \frac{m_{\prime}\,a}{\mu}\,q_{174} = 0$$

$$\frac{(2\,n - 5\,n_{\prime})^2}{n^2} \left\{ r_{213} - \frac{3}{2}\,r_{113} \right\} - r_{213} + \frac{m_{\prime}\,a}{\mu}\,q_{214} = 0$$

$$\frac{(2n-5n_{\prime})^2}{n^2}\left\{r_{273}-\frac{3}{2}r_{133}\right\}-r_{273}+\frac{m_{\prime}a}{\mu}q_{273}=0$$

$$\frac{(2n-5n_{\prime})^2}{n^2}\left\{r_{312}-r_{312}\right\}+\frac{m_{\prime}a}{\mu}q_{312}=0$$

$$\delta\lambda=\left\{2\left\{r_{155}+\frac{1}{2}\left(r_{55}+r_{15}+\frac{9}{8}r_4\right)\right\}\right.$$

$$-\frac{m_{\prime}}{\mu}\left\{\frac{5na}{(2n-5n_{\prime})}R_{155}+\frac{5naR_{55}}{(3n-5n_{\prime})}+\frac{5.5naR_{15}}{4(3n-4n_{\prime})}+\frac{13.5naR_5}{8.5(n-n_{\prime})}\right\}\Bigg\}\frac{ne^3}{(2n-5n_{\prime})}\sin(2nt-5n_{\prime}t+3\varpi$$

$$+\left\{2\left\{r_{174}+\frac{1}{2}(r_{74}+r_{34})\right\}\right.$$

$$-\frac{m_{\prime}}{\mu}\left\{\frac{4naR_{174}}{(2n-5n_{\prime})}+\frac{4naR_{74}}{(3n-5n_{\prime})}+\frac{5.4.naR_{34}}{4(4n-5n_{\prime})}\right\}\Bigg\}\frac{ne^2e_{\prime}}{(2n-5n_{\prime})}\sin(2nt-5n_{\prime}t+2\varpi+\varpi_{\prime})$$

$$+\left\{2\left\{r_{213}+\frac{1}{2}r_{113}\right\}-\frac{m_{\prime}}{\mu}\left\{\frac{3naR_{213}}{(2n-5n_{\prime})}+\frac{3naR_{113}}{(3n-5n_{\prime})}\right\}\right\}\frac{nee_{\prime}^2}{(2n-5n_{\prime})}\sin(2nt-5n_{\prime}t+\varpi+2\varpi_{\prime})$$

$$+\left\{2\left\{r_{273}+\frac{1}{2}r_{133}\right\}-\frac{m_{\prime}}{\mu}\left\{\frac{3naR_{273}}{(2n-5n_{\prime})}+\frac{3naR_{133}}{(3n-5n_{\prime})}\right\}\right\}\frac{ne\sin^2\frac{\iota_{\prime}}{2}}{(2n-5n_{\prime})}\sin(2nt-5n_{\prime}t+\varpi+2\nu_{\prime})$$

$$+\left\{2r_{312}-\frac{2m_{\prime}naR_{312}}{\mu(2n-5n_{\prime})}\right\}\frac{ne_{\prime}\sin^2\frac{\iota_{\prime}}{2}}{(2n-5n_{\prime})}\sin(2nt-5n_{\prime}t+\varpi_{\prime}+2\nu_{\prime})$$

The quantities r_{55}, r_{74}, r_{113} and r_{133} have the quantity $2n-5n_{\prime}$ in the denominator, rejecting those quantities in the value of $\delta\lambda$ which have not $(2n-5n_{\prime})^2$ in the denominator.

$$r_{155}=-\frac{4m_{\prime}n^3aR_{155}e^3}{\mu(n-5n_{\prime})(3n-5n_{\prime})(2n-5n_{\prime})}$$

$$r_{174}=-\frac{4m_{\prime}n^3aR_{174}e^2e_{\prime}}{\mu(n-5n_{\prime})(3n-5n_{\prime})(2n-5n_{\prime})}$$

$$r_{213}=-\frac{4m_{\prime}n^3aR_{213}ee_{\prime}^2}{\mu(n-5n_{\prime})(3n-5n_{\prime})(2n-5n_{\prime})}$$

$$r_{273}=-\frac{4m_{\prime}n^3aR_{273}e\sin^2\frac{\iota_{\prime}}{2}}{\mu(n-5n_{\prime})(3n-5n_{\prime})(2n-5n_{\prime})}$$

$$r_{312}=-\frac{4m_{\prime}n^3aR_{312}e_{\prime}\sin^2\frac{\iota_{\prime}}{2}}{\mu(n-5n_{\prime})(3n-5n_{\prime})(2n-5n_{\prime})}$$

$$\delta\lambda=\left\{2r_{155}+r_{55}-\frac{5m_{\prime}naR_{155}}{\mu(2n-5n_{\prime})}\right\}\frac{ne^3}{(2n-5n_{\prime})}\sin(2nt-5n_{\prime}t+3\varpi)$$

$$+\left\{2r_{174}+r_{74}-\frac{4m_{\prime}naR_{174}}{\mu(2n-5n_{\prime})}\right\}\frac{ne^2e_{\prime}}{(2n-5n_{\prime})}\sin(2nt-5n_{\prime}t+2\varpi+\varpi_{\prime})$$

$$+\left\{2r_{213}+r_{113}-\frac{3m_{,}anR_{213}}{\mu(2n-5n_{,})}\right\}\frac{nee_{,}^2}{(2n-5n_{,})}\sin(2nt-5n_{,}t+\varpi+2\varpi_{,})$$

$$+\left\{2r_{273}+r_{133}-\frac{3m_{,}anR_{273}}{\mu(2n-5n_{,})}\right\}\frac{ne\sin^2\frac{\iota_{,}}{2}}{(2n-5n_{,})}\sin(2nt-5n_{,}t+\varpi+2\nu_{,})$$

$$+\left\{2r_{312}-\frac{2m_{,}anR_{312}}{\mu(2n-5n_{,})}\right\}\frac{ne_{,}\sin^2\frac{\iota_{,}}{2}}{(2n-5n_{,})}\sin(2nt-5n_{,}t+\varpi_{,}+2\nu_{,})$$

The coefficients of the terms in the development of R multiplied by the cubes of the eccentricities, as regards the quantities b_5 and b_7, (they also contain the quantities b_3,) may be found by changing b_3 into b_5, in the terms in R multiplied by the eccentricities, and multiplying the result by

$$-\underset{[0]}{\frac{9}{8}\frac{(a^2e^2+a_{,}^2e_{,}^2)}{a_{,}^2}}+\underset{[50]}{\frac{3}{8}\frac{a^2}{a_{,}^2}e^2\cos 2x}-\underset{[1]}{\frac{3}{4}\frac{a}{a_{,}}\left(e^2+e_{,}^2+2\sin^2\frac{\iota_{,}}{2}\right)\cos t}+\underset{[61]}{\frac{9}{16}\frac{a}{a_{,}}e^2\cos(t+2x)}$$

$$-\underset{[111]}{\frac{9}{16}\frac{a}{a_{,}}e_{,}^2\cos(t-2z)}+\underset{[51]}{\frac{3}{16}\frac{a}{a_{,}}e^2\cos(t-2x)}+\underset{[121]}{\frac{3}{4}\frac{a}{a_{,}}e_{,}^2\cos(t+2z)}+\underset{[91]}{\frac{27}{8}\frac{a}{a_{,}}ee_{,}\cos(t-x+z)}$$

$$-\underset{[81]}{\frac{9}{8}\frac{a}{a_{,}}ee_{,}\cos(t+x+z)}-\underset{[71]}{\frac{9}{8}\frac{a}{a_{,}}ee_{,}\cos(t-x-z)}+\underset{[101]}{\frac{3}{8}\frac{a}{a_{,}}ee_{,}\cos(t+x-z)}$$

$$+\underset{[141]}{\frac{3}{2}\frac{a}{a_{,}}\sin^2\frac{\iota_{,}}{2}\cos(t+2y)}+\underset{[110]}{\frac{3}{8}e_{,}^2\cos 2z}$$

and changing b_5 into b_7, in the terms in R multiplied by the squares and products of the eccentricities, and multiplying the result by

$$-\frac{5}{6}\text{ and }-\underset{[10]}{\frac{2a^2}{a_{,}^2}e\cos x}+\underset{[11]}{\frac{3a}{a_{,}}e\cos(t-x)}+\underset{[41]}{\frac{3a}{a_{,}}e_{,}\cos(t+z)}-\underset{[21]}{\frac{a}{a_{,}}e\cos(t+x)}$$

$$-\underset{[31]}{\frac{a}{a_{,}}e_{,}\cos(t-z)}-\underset{[30]}{2e_{,}\cos z}$$

and changing b_3 into b_5 in the terms in R multiplied by the squares and products of the eccentricities, and multiplying the result by $-\frac{3}{4}$ and the same quantity.

Thus R_{155} results from the combination of the arguments

51 × 14, 50 × 15, 61 × 16, 10 × 55, and 11 × 54.

$$51\times 14\text{ gives }+\frac{3}{32}\frac{a}{a_{,}}\left\{\frac{3a}{4a_{,}^2}b_{5,3}-\frac{a^2}{2a_{,}^3}b_{5,4}-\frac{a}{4a_{,}^2}b_{5,5}\right\}$$

$$50 \times 15 \text{ gives } + \frac{3}{16}\frac{a^2}{a_{\prime}^2}\left\{\frac{3\,a}{4\,a_{\prime}^2}b_{5,4} - \frac{a^2}{2\,a_{\prime}^3}b_{5,5} - \frac{a}{4\,a_{\prime}^2}b_{5,6}\right\}$$

$$61 \times 16 \text{ gives } + \frac{9}{32}\frac{a}{a_{\prime}}\left\{\frac{3\,a}{4\,a_{\prime}^2}b_{5,5} - \frac{a^2}{2\,a_{\prime}^3}b_{5,6} - \frac{a}{4\,a_{\prime}^2}b_{5,7}\right\}$$

$$R_{55} = -\frac{a}{16\,a_{\prime}^2}b_{3,4} - \frac{a^2}{8\,a_{\prime}^3}b_{3,5} - \frac{3\,a}{16\,a_{\prime}^2}b_{3,6} - \frac{3\,.\,9}{2\,.\,4\,.\,4}\frac{a^2}{a_{\prime}^3}b_{5,3} + \frac{3\,.\,3}{2\,.\,4}\frac{a^3}{a_{\prime}^4}b_{5,4}$$

$$-\frac{3\,a^2}{2\,.\,4\,.\,2}\frac{(2\,a^2 - 3\,a_{\prime}^2)}{a_{\prime}^5}b_{5,5} - \frac{3}{2\,.\,4}\frac{a^3}{a_{\prime}^4}b_{5,6} - \frac{3\,a^2}{2\,.\,4\,.\,4\,a_{\prime}^3}b_{5,7}$$

changing b_3 into $-\frac{3}{4}b_5$, and b_5 into $-\frac{5}{6}b_7$, we have

$$\frac{3\,a}{64\,a_{\prime}^2}b_{5,4} + \frac{3\,a^2}{32\,a_{\prime}^3}b_{5,5} + \frac{9}{64}\frac{a}{a_{\prime}^2}b_{5,6} + \frac{3\,.\,9\,.\,5}{2\,.\,4\,.\,4\,.\,6}\frac{a^2}{a_{\prime}^3}b_{7,3} - \frac{3\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{a^3}{a_{\prime}^4}b_{7,4}$$

$$+\frac{3\,.\,5\,a^2}{2\,.\,4\,.\,2}\frac{(2\,a^2 - 3\,a_{\prime}^2)}{a^5}b_{7,5} + \frac{3\,.\,5}{2\,.\,4\,.\,6}\frac{a^3}{a_{\prime}^4}b_{7,6} + \frac{3\,.\,5}{2\,.\,4\,.\,4\,.\,6}\frac{a^2}{a_{\prime}^3}b_{7,7}$$

$$= \frac{3}{64}\frac{a}{a_{\prime}^2}b_{5,4} + \frac{3}{32}\frac{a^2}{a_{\prime}^3}b_{5,5} + \frac{9}{64}\frac{a}{a_{\prime}^2}b_{5,6} + \frac{3\,.\,5}{8\,.\,6}\frac{a^2}{a_{\prime}^3}\left\{\frac{a^2 + a_{\prime}^2}{a_{\prime}^2}b_{7,5} - \frac{a}{a_{\prime}}b_{7,4} - \frac{a}{a_{\prime}}b_{7,6}\right\}$$

$$+ \frac{3\,.\,9\,.\,5}{8\,.\,4\,.\,6}\frac{a^2}{a_{\prime}^3}\left\{b_{7,3} - b_{7,5}\right\} - \frac{3\,.\,5}{4\,.\,6}\frac{a^3}{a_{\prime}^4}\left\{b_{7,4} - b_{7,6}\right\} - \frac{3\,.\,5}{32\,.\,6}\frac{a^2}{a_{\prime}^3}\left\{b_{7,5} - b_{7,7}\right\}$$

and since $b_{5,5} = \frac{a^2 + a_{\prime}^2}{a_{\prime}^3}b_{7,5} - \frac{a}{a_{\prime}}b_{7,4} - \frac{a}{a_{\prime}}b_{7,6}$

$$4\,b_{5,4} = \frac{5}{2}\frac{a}{a_{\prime}}\left\{b_{7,3} - b_{7,5}\right\} \quad 5\,b_{5,5} = \frac{5}{2}\frac{a}{a_{\prime}}\left\{b_{7,4} - b_{7,6}\right\} \quad 6\,b_{5,5} = \frac{5}{2}\frac{a}{a_{\prime}}\left\{b_{7,5} - b_{7,7}\right\}$$

$$= \frac{3}{64}\frac{a}{a_{\prime}^2}b_{5,4} + \frac{3}{32}\frac{a^2}{a_{\prime}^3}b_{5,5} + \frac{9}{64}\frac{a}{a_{\prime}^2}b_{5,6} + \frac{15}{48}\frac{a^2}{a_{\prime}^3}b_{5,5} + \frac{27}{24}\frac{a}{a_{\prime}^2}b_{5,4} - \frac{15}{12}\frac{a^2}{a_{\prime}^3}b_{5,5} - \frac{3}{16}\frac{a}{a_{\prime}^2}b_{5,6}$$

$$= \frac{75}{64}\frac{a}{a_{\prime}^2}b_{5,4} - \frac{27}{32}\frac{a^2}{a_{\prime}^3}b_{5,5} - \frac{3}{64}\frac{a}{a_{\prime}^2}b_{5,6}$$

$$R_{54} = -\frac{a}{16\,a_{\prime}^2}b_{3,3} - \frac{a^2}{8\,a_{\prime}^3}b_{3,4} - \frac{3\,a}{16\,a_{\prime}^2}b_{3,5} - \frac{3\,.\,9\,a^2}{2\,.\,4\,.\,4\,a_{\prime}^3}b_{5,2} + \frac{3\,.\,3\,a^3}{2\,.\,4\,a_{\prime}^4}b_{5,3}$$

$$-\frac{3\,a^2\,(2\,a^2 - 3\,a_{\prime}^2)}{2\,.\,4\,.\,2\,a_{\prime}^5}b_{5,4} - \frac{3\,a^3}{2\,.\,4\,a_{\prime}^4}b_{5,6} - \frac{3\,a^2}{2\,.\,4\,.\,4\,a_{\prime}^3}b_{5,6}$$

Similar changes and reductions give

$$\frac{57\,a}{64\,a_{\prime}^2}b_{5,3} - \frac{19\,a^2}{32\,a_{\prime}^3}b_{5,4} - \frac{a}{64\,a_{\prime}^2}b_{5,5}$$

$$R_{155} = \frac{3}{32}\frac{a}{a_{\prime}}\left\{\frac{3}{4}\frac{a}{a_{\prime}^2} b_{5,3} - \frac{a^2}{2a_{\prime}^3} b_{5,4} - \frac{a}{4a_{\prime}^2} b_{5,5}\right\} + \frac{3}{16}\frac{a^2}{a_{\prime}^2}\left\{\frac{3}{4}\frac{a}{a_{\prime}^2} b_{5,4} - \frac{a^2}{2a_{\prime}^3} b_{5,5} - \frac{a}{4a_{\prime}^2} b_{5,6}\right\}$$

$$+ \frac{9a}{32a_{\prime}}\left\{\frac{3a}{4a_{\prime}^2} b_{5,5} - \frac{a^2}{2a_{\prime}^3} b_{5,6} - \frac{a}{4a_{\prime}^2} b_{5,7}\right\} - \frac{a^2}{a_{\prime}^2}\left\{\frac{75}{64}\frac{a}{a_{\prime}^2} b_{5,4} - \frac{27}{32}\frac{a^2}{a_{\prime}^3} b_{5,5} - \frac{3}{64}\frac{a}{a_{\prime}^2} b_{5,6}\right\}$$

$$+ \frac{3a}{2a_{\prime}}\left\{\frac{57}{64}\frac{a}{a_{\prime}^2} b_{5,3} - \frac{19}{32}\frac{a^2}{a_{\prime}^3} b_{5,4} - \frac{a}{64a_{\prime}^2} b_{5,5}\right\}$$

and adding the terms which depend upon b_3;

$$R_{155} = \frac{a}{96a_{\prime}^2} b_{3,4} - \frac{a^2}{16a_{\prime}^3} b_{3,5} + \frac{a}{12a_{\prime}^2} b_{3,6} + \frac{45}{32}\frac{a^2}{a_{\prime}^3} b_{5,3} - \frac{63}{32}\frac{a^3}{a_{\prime}^4} b_{5,4} + \frac{(21a_{\prime}^2 + 96a^2)}{128a_{\prime}^5} a^2 b_{5,5}$$

$$- \frac{9}{64}\frac{a^3}{a_{\prime}^4} b_{5,6} - \frac{9}{128}\frac{a^2}{a_{\prime}^3} b_{5,7}$$

which may be still further reduced. R_{174}, R_{213}, R_{273}, and R_{312} may be obtained in a similar manner.

The following Table shows the arguments which, by their combination with the arguments 1, 2, 3, 12, 13, 31, 32, 64, 65, 73, 74, 112, and 113, by addition and subtraction produce the arguments 155, 174, 213, 273, and 312.

	1	2	3	12	13	31	32	64	65	73	74	112	113	
155	154	153	152	53	52	......	......	11	......	192	191	......	......	155
	156	157	158	......	......	......	......	......	− 10	......	......	......	......	
174	173	172	171	72	71	53	52	......	......	11	......	192	191	174
	175	176	177	......	......	......	......	− 30	− 41	......	− 10	......	......	
213	212	211	−210.	111	......	72	71	......	......	......	......	11	......	213
	214	215	216	......	−110	......	......	−231	−232	− 30	− 41	......	− 10	
273	272	271	−270.	131	......	......	......	......	......	330	......	......	......	273
	274	275	276	......	−130	......	......	−291	−292	......	−331	......	......	
312	311	−310.	−321.	......	......	131	......	......	......	......	......	330	......	312
	313	314	315	......	......	......	−130	......	......	−291	−292	......	−331	

If

$$r\,\delta.\frac{1}{r} = r'_1\cos(nt - n_{\prime}t) + r'_2\cos(2nt - 2n_{\prime}t) + r'_3\cos(3nt - 3n_{\prime}t) + e r'_{12}\cos(nt - 2n_{\prime}t + \varpi)$$

$$+ e r'_{13}\cos(2nt - 3n_{\prime}t + \varpi) + \&c.$$

$$r_{\prime}\,\delta.\frac{1}{r_{\prime}} = r_{\prime}{}'_1\cos(nt - n_{\prime}t) + r_{\prime}{}'_2\cos(2nt - 2n_{\prime}t) + r_{\prime}{}'_3\cos(3nt - 3n_{\prime}t) + e r_{\prime}{}'_{12}\cos(nt - 2n_{\prime}t + \varpi)$$

$$+ e r_{\prime}{}'_{13}\cos(2nt - 3n_{\prime}t + \varpi) + \&c.$$

$$\delta\lambda = \lambda_1\sin(nt - n_{\prime}t) + \lambda_2\sin(2nt - 2n_{\prime}t) + \lambda_3\sin(3nt - 3n_{\prime}t) + e\lambda_{12}\sin(nt - 2n_{\prime}t + \varpi)$$

$$+ e\lambda_{13}\sin(2nt - 3n_{\prime}t + \varpi) + \&c.$$

$$\delta\lambda_{,} = \lambda_{,1}\sin(nt - n_{,}t) + \lambda_{,2}\sin(2nt - 2n_{,}t) + \lambda_{,3}\sin(3nt - 3n_{,}t) + e\lambda_{,12}\sin(nt - 2n_{,}t + \varpi)$$
$$+ e\lambda_{,13}\sin(2nt - 3n_{,}t + \varpi) + \&c.$$

Supposing that the arguments 1, 2, 3, 12, 13, 31, 32, 64, 65, 73, 74, 112, 113, 155, 174, 213, 273, and 312 are alone sensible in $\delta \, . \, \frac{1}{r}$, $\delta\lambda$, $\delta\frac{1}{r_{,}}$ and $\delta\lambda_{,}$ the coefficient of $e^3\cos(2nt - 5n_{,}t + 3\varpi)$ in the expression for δR or δR_{155}

$$= -\frac{1}{2}\left\{\frac{a\,d\,.\,R_{154}}{d\,a} + \frac{a\,d\,.\,R_{156}}{d\,a}\right\}r'_{1} + \left\{2R_{154} - 3R_{156}\right\}\left\{\lambda_{1} - \lambda_{,1}\right\} - \frac{1}{2}\left\{\frac{a\,d\,R_{153}}{d\,a} + \frac{a\,d\,R_{157}}{d\,a}\right\}r'_{2}$$
$$+ \frac{1}{2}\left\{3R_{153} - 7R_{157}\right\}\left\{\lambda_{2} - \lambda_{,2}\right\} - \frac{1}{2}\left\{\frac{a\,d\,.\,R_{152}}{d\,a} + \frac{a\,d\,.\,R_{158}}{d\,a}\right\}r'_{3}$$
$$+ \left\{R_{152} - 4R_{158}\right\}\left\{\lambda_{3} - \lambda_{,3}\right\} - \frac{a\,d\,.\,R_{53}}{2\,d\,a}r'_{12} + \frac{3}{2}R_{53}\left\{\lambda_{12} - \lambda_{,12}\right\}$$
$$- \frac{a\,d\,.\,R_{52}}{2\,d\,a}r'_{13} + R_{52}\left\{\lambda_{13} - \lambda_{,13}\right\} - \frac{a\,d\,.\,R_{64}}{2\,d\,a}r'_{11} + 2R_{64}\left\{\lambda_{11} - \lambda_{,11}\right\}$$
$$- \frac{a\,d\,.\,R_{65}}{2\,d\,a}r'_{10} - \frac{5}{2}R_{65}\left\{\lambda_{10} - \lambda_{,10}\right\} - \frac{a\,d\,R_{192}}{2\,d\,a}r'_{73} - R_{192}\left\{\lambda_{73} - \lambda_{,73}\right\} - \frac{a\,d\,R_{193}}{2\,d\,a}r'_{74}$$
$$- \frac{1}{2}R_{193}\left\{\lambda_{74} - \lambda_{,74}\right\} - \frac{a\,d\,.\,R_{0}}{d\,a}r'_{155} - \frac{1}{2}\left\{\frac{a_{,}d\,.\,R_{154}}{d\,a_{,}} + \frac{a_{,}d\,.\,R_{156}}{d\,a_{,}}\right\}r_{,}'_{1}$$
$$- \frac{1}{2}\left\{\frac{a_{,}d\,.\,R_{153}}{d\,a_{,}} + \frac{a_{,}d\,.\,R_{157}}{d\,a_{,}}\right\}r_{,}'_{2} - \frac{1}{2}\left\{\frac{a_{,}d\,R_{152}}{d\,a_{,}} + \frac{a_{,}d\,R_{158}}{d\,a_{,}}\right\}r_{,}'_{3} - \frac{a_{,}d\,.\,R_{53}}{2\,d\,a_{,}}r_{,}'_{12}$$
$$- \frac{a_{,}d\,.\,R_{52}}{2\,d\,a_{,}}r_{,}'_{13} - \frac{a_{,}d\,.\,R_{64}}{2\,d\,a_{,}}r_{,}'_{11} - \frac{a_{,}d\,.\,R_{65}}{2\,d\,a_{,}}r_{,}'_{10} - \frac{a_{,}d\,R_{192}}{2\,d\,a_{,}}r_{,73} - \frac{a_{,}d\,R_{193}}{2\,d\,a_{,}}r_{,74} - \frac{a_{,}d\,.\,R_{0}}{d\,a_{,}}r_{,}'_{155}$$

In the same way the expression for $\delta \, . \, R_{174}$, $\delta \, . \, R_{213}$, $\delta \, . \, R_{273}$, and $\delta \, . \, R_{312}$ may be found from the preceding Table.

If $a < a_{,}$ and

$$\left\{1 - \frac{a}{a_{,}}\cos\theta + \frac{a^2}{a_{,}^2}\right\}^{-\frac{1}{2}} = \tfrac{1}{2}b_{1,0}{}^{*} + b_{1,1}\cos\theta + b_{1,2}\cos 2\theta \; \&c.$$

$$\left\{1 - \frac{a}{a_{,}}\cos\theta + \frac{a^2}{a_{,}^2}\right\}^{-\frac{3}{2}} = \tfrac{1}{2}b_{3,0}{}^{*} + b_{3,1}\cos\theta + b_{3,2}\cos 2\theta \; \&c.$$

$$R = m_{,}\left\{\frac{a}{a_{,}^2}\left(\cos^2\frac{\iota_{,}}{2} - \frac{e^2 + e_{,}^2}{2}\right\}\cos(nt - n_{,}t) \dagger\right.$$

$$- \frac{3m_{,}}{2}\frac{a}{a_{,}^2}e\cos(n_{,}t - \varpi) + \frac{m_{,}a}{a_{,}^2}e\cos(2nt - n_{,}t - \varpi) + \frac{2m_{,}a}{2a_{,}^2}e_{,}\cos(nt - 2n_{,}t + \varpi_{,})$$

* The notation is slightly changed from that used before.

† ε and $\varepsilon_{,}$ which accompany $n\,t$ and $n_{,}\,t$ are omitted for convenience.

$$+ \frac{m_{\prime} a}{8 a_{\prime}^2} e^2 \cos (n t + n_{\prime} t - 2 \varpi) + \frac{3 m_{\prime} a}{8 a_{\prime}^2} e^2 \cos (3 n t - n_{\prime} t - 2 \varpi) - \frac{3 m_{\prime} a}{a_{\prime}^2} e e_{\prime} \cos (2 n_{\prime} t - \varpi - \varpi_{\prime})$$

General expression for the development of R.

$$+ \frac{m_{\prime} a}{a_{\prime}^2} e e_{\prime} \cos (2 n t - 2 n_{\prime} t - \varpi + \varpi_{\prime}) + \frac{27}{8} \frac{m_{\prime} a}{a_{\prime}^2} e_{\prime}^2 \cos (n t - 3 n_{\prime} t + 2 \varpi_{\prime})$$

$$+ \frac{m_{\prime} a}{8 a_{\prime}^2} e_{\prime}^2 \cos (n t + n_{\prime} t - 2 \varpi_{\prime}) + \frac{m_{\prime} a}{a_{\prime}^2} \sin^2 \frac{\iota_{\prime}}{2} \cos (n t + n_{\prime} t - 2 \nu_{\prime})$$

$$+ m_{\prime} \Sigma \left\{ - \frac{b_{1,i}}{2 a_{\prime}} + \frac{a}{4 a_{\prime}^2} \sin^2 \frac{\iota_{\prime}}{2} \left(b_{3,i-1} + b_{3,i+1} \right) \right.$$

$$\left. + \frac{a (e^2 + e_{\prime}^2)}{16 a_{\prime}^2} \Big((3 i - 1) b_{3,i-1} - (3 i + 1) b_{3,i+1} \Big) \right\} \cos i (n t - n_{\prime} t)$$

$$+ m_{\prime} \Sigma \left\{ - \frac{a}{4 a_{\prime}^2} b_{3,i-1} - \frac{a^2}{2 a_{\prime}^3} b_{3,i} + \frac{3 a}{4 a_{\prime}^2} b_{3,i+1} \right\} e \cos \Big(i (n t - n_{\prime} t) + n t - \varpi \Big)$$

$$+ m_{\prime} \Sigma \left\{ \frac{3}{4} \frac{a}{a_{\prime}^2} b_{3,i-1} - \frac{1}{2 a_{\prime}} b_{3,i} - \frac{a}{4 a_{\prime}^2} b_{3,i+1} \right\} e_{\prime} \cos \Big(i (n t - n_{\prime} t) + n_{\prime} t - \varpi_{\prime} \Big)$$

$$+ m_{\prime} \Sigma \left\{ - \frac{(2 + i)}{16} \frac{a}{a_{\prime}^2} b_{3,i-1} - \frac{(1 + i)}{2} \frac{a^2}{a_{\prime}^3} b_{3,i} \right.$$

$$\left. + \frac{(8 + 9 i)}{16} \frac{a}{a_{\prime}^2} b_{3,i+1} \right\} e^2 \cos \Big(i (n t - n_{\prime} t) + 2 n t - 2 \varpi \Big)$$

$$+ m_{\prime} \Sigma \left\{ \frac{(3 + 9 i)}{8} \frac{a}{a_{\prime}^2} b_{3,i-1} - \frac{i}{a_{\prime}} b_{3,i} \right.$$

$$\left. - \frac{(1 + i)}{8} \frac{a}{a_{\prime}^2} b_{3,i+1} \right\} e e_{\prime} \cos \Big(i (n t - n_{\prime} t) + n t + n_{\prime} t - \varpi - \varpi_{\prime} \Big)$$

$$+ m_{\prime} \Sigma \left\{ - \frac{(1 + 3 i)}{8} \frac{a}{a_{\prime}^2} b_{3,i-1} \right.$$

$$\left. + \frac{3 (1 + i)}{8} \frac{a}{a_{\prime}^2} b_{3,i+1} \right\} e e_{\prime} \cos \Big(i (n t - n_{\prime} t) + n t - n_{\prime} t - \varpi + \varpi_{\prime} \Big)$$

$$+ m_{\prime} \Sigma \left\{ \frac{(8 - 9 i)}{16} \frac{a}{a_{\prime}^2} b_{3,i-1} + \frac{(1 - i)}{2 a_{\prime}} b_{3,i} \right.$$

$$\left. - \frac{(2 - i)}{16} \frac{a}{a_{\prime}^2} b_{3,i+1} \right\} e_{\prime}^2 \cos \Big(i (n t - n_{\prime} t) + 2 n_{\prime} t - 2 \varpi_{\prime} \Big)$$

$$- m_{\prime} \Sigma \frac{a}{2 a_{\prime}^2} b_{3,i-1} \sin^2 \frac{\iota_{\prime}}{2} \cos \Big(i (n t - n_{\prime} t) + 2 n_{\prime} t - 2 \nu_{\prime} \Big)$$

i being every whole number, positive and negative and zero, and observing that $b_{m,n} = b_{m,-n}$. Considering only the terms multiplied by e and $e_{\prime}$,

$$r \left(\frac{\mathrm{d} R}{\mathrm{d} r} \right) = - \frac{3 m_{\prime}}{2} \frac{a}{a_{\prime}^2} e \cos (n_{\prime} t - \varpi) + \frac{m_{\prime} a}{2 a_{\prime}^2} e \cos (2 n t - n_{\prime} t - \varpi)$$

$$+ \frac{m_, a}{2 a_,^2} e_, \cos (n t - 2 n_, t + \varpi_,)$$

$$+ m_, \Sigma \left\{ - \frac{i}{4} \frac{a}{a_,^2} b_{3,i-1} + \frac{(1+2i)}{2} \frac{a^2}{a_,^3} b_{3,i} - \frac{3i}{4} \frac{a}{a_,^2} b_{3,i+1} \right\} e \cos \Big(i (n t - n_, t) + n t - \varpi \Big)$$

$$+ m_, \Sigma \left\{ - \frac{3(1+i)}{4} \frac{a}{a_,^2} b_{3,i-1} + \frac{i a}{a_,} b_{3,i} + \frac{(1-i)}{4} b_{3,i+1} \right\} e_, \cos \Big(i (n t - n_, t) + n_, t - \varpi_, \Big)$$

$$\frac{a}{r} = - \frac{m_,}{\mu} \frac{n^2}{(3n - n_,)(n - n_,)} \left\{ \frac{2n}{2n - n_,} + \frac{1}{2} \right\} \frac{a^2}{a_,^2} e \cos (2 n t - n_, t - \varpi)$$

$$- \frac{m_,}{\mu} \frac{3 n^2}{2 (n - n_,)(n + n_,)} \frac{a^2}{a_,^2} e \cos (n_, t - \varpi)$$

$$+ \frac{m_,}{\mu} \frac{n^2}{n_, (2n - 2n_,)} \left\{ \frac{2n}{(n - 2n_,)} + 1 \right\} \frac{a^2}{a_,^2} e_, \cos (n t - 2 n_, t + \varpi_,)$$

$$+ \Sigma \frac{n^2}{\Big(i (n - n_,) + 2 n \Big) i (n - n_,)} \left[\frac{3 \Big(i (n - n_,) + n \Big)}{2 n^2} 2 r_i {}^* \right.$$

$$- \frac{m_,}{\mu} \left\{ \frac{2(1+i) n}{i (n - n_,) + n} \left\{ - \frac{a^2}{4 a_,^2} b_{3,i-1} - \frac{a^3}{2 a_,^3} b_{3,i} + \frac{3 a^2}{4 a_,^2} b_{3,i+1} \right\} \right.$$

$$\left. \left. - \frac{i}{4} \frac{a^2}{a_,^2} b_{3,i-1} + \frac{(1+2i)}{2} \frac{a^3}{a_,^3} b_{3,i} - \frac{3i}{4} \frac{a^2}{a_,^2} b_{3,i+1} \right\} \right] e \cos \Big(i (n t - n_, t) + n t - \varpi \Big)$$

$$+ \frac{m_,}{\mu} \Sigma \frac{n^2}{(1-i)(n - n_,) \Big((i+1)(n - n_,) + 2 n_, \Big)} \left\{ \frac{2 i n}{i (n - n_,) + n_,} \left\{ \frac{3 a^2}{4 a_,^2} b_{3,i-1} \right. \right.$$

$$\left. - \frac{a}{2 a_,} b_{3,i} - \frac{a^2}{4 a_,^2} b_{3,i+1} \right\} - \frac{3(1+i)}{4} \frac{a^2}{a_,^2} b_{3,i-1}$$

$$\left. + \frac{i a}{a_,} b_{3,i} + \frac{(1-i)}{4} \frac{a^2}{a_,^2} b_{3,i+1} \right\} e_, \cos \Big(i (n t - n_, t) + n_, t - \varpi_, \Big)$$

$$\lambda = - \left\{ \frac{3 n^2}{2 n_,^2} + \frac{n^2}{n_, (n - n_,)} \frac{m_,}{\mu} \right\} \frac{a^2}{a_,^2} e \sin (n_, t - \varpi)$$

$$- \left\{ \frac{n^2}{(2n - n_,)^2} + \frac{n^2}{(2n - n_,)(n - n_,)} \right\} \frac{m_,}{\mu} \frac{a^2}{a_,^2} e \sin (2 n t - n_, t - \varpi)$$

$$- \frac{2 n^2}{(n - 2 n_,)^2} \frac{m_,}{\mu} \frac{a^2}{a_,^2} e_, \sin (n t - 2 n_, t + \varpi_,)$$

* r_i being the coefficient of $\cos \Big(i (n t - n_, t) \Big)$ in the expression for $\frac{a}{r}$.

$$+ \Sigma \dagger \frac{n}{i(n - n_{\prime}) + n} \left\{ 2 \left(r^* + \frac{r_i}{2} \right) - \frac{m_{\prime} n i}{\mu \left(i(n - n_{\prime}) + n \right)} \left(- \frac{a^2}{4 a_{\prime}^2} b_{3,i-1} - \frac{a^3}{2 a_{\prime}^3} b_{3,i} \right. \right.$$

$$\left. \left. + \frac{3 a^2}{4 a_{\prime}^2} b_{3,i+1} \right) + \frac{m_{\prime} n}{\mu (n - n_{\prime})} \frac{a}{a_{\prime}} b_{1,i} \right\} e \sin \left(i(n t - n_{\prime} t) + n t - \varpi \right)$$

$$+ \Sigma \frac{n}{i(n - n_{\prime}) + n_{\prime}} \left\{ 2 r^* - \frac{m_{\prime} n i}{\mu \left(i(n - n_{\prime}) + n_{\prime} \right)} \left(\frac{3}{4} \frac{a^2}{a_{\prime}^2} b_{3,i-1} - \frac{a}{2 a_{\prime}} b_{3,i} \right. \right.$$

$$\left. \left. - \frac{a^2}{4 a_{\prime}^2} b_{3,i+1} \right) \right. e_{\prime} \sin \left(i(n t - n_{\prime} t) + n_{\prime} t - \varpi_{\prime} \right)$$

If $a > a_{\prime}$, and

$$\left\{ 1 - \frac{a_{\prime}}{a} \cos \theta + \frac{a_{\prime}^2}{a^2} \right\}^{-\frac{1}{2}} = \tfrac{1}{2} b_{1,0} + b_{1,1} \cos \theta + b_{1,2} \cos 2\theta + \&c.$$

$$\left\{ 1 - \frac{a_{\prime}}{a} \cos \theta + \frac{a_{\prime}^2}{a^2} \right\}^{-\frac{3}{2}} = \tfrac{1}{2} b_{3,0} + b_{3,1} \cos \theta + b_{3,2} \cos 2\theta + \&c.$$

the value of R may be easily inferred from the value which it has in the former case. Considering only the terms multiplied by the eccentricities

$$r \left(\frac{dR}{dr} \right) = - \frac{3 m_{\prime}}{2} \frac{a}{a_{\prime}^2} e \cos (n t - \varpi) + \frac{m_{\prime}}{2} \frac{a}{a_{\prime}^2} e \cos (2 n t - n_{\prime} t - \varpi)$$

$$+ \frac{m_{\prime}}{2} \frac{a}{a_{\prime}^2} e_{\prime} \cos (n t - 2 n_{\prime} t + \varpi_{\prime})$$

$$+ m_{\prime} \Sigma \left\{ - \frac{i}{4} \frac{a_{\prime}}{a^2} b_{3,i-1} + \frac{(1 + 2i)}{2a} b_{3,i} \right.$$

$$\left. - \frac{3i}{4} \frac{a_{\prime}}{a^2} b_{3,i+1} \right\} e \cos \left(i(n t - n_{\prime} t) + n t - \varpi \right)$$

$$+ m_{\prime} \Sigma \left\{ - \frac{3(1 + i)}{4} \frac{a_{\prime}}{a^2} b_{3,i-1} + \frac{i a_{\prime}^2}{a^3} b_{3,i} \right.$$

$$\left. + \frac{(1 - i)}{4} \frac{a_{\prime}}{a^2} b_{3,i+1} \right\} e_{\prime} \cos \left(i(n t - n_{\prime} t) + n_{\prime} t - \varpi_{\prime} \right)$$

All these expressions are to a certain extent arbitrary, on account of the equation which connects $b_{3,i-1}$, $b_{3,i}$, and $b_{3,i+1}$

$$\frac{(2i + 1)}{2} \frac{a}{a_{\prime}} b_{3,i+1} = \frac{i(a^2 + a_{\prime}^2)}{a_{\prime}^2} b_{3,i} - \frac{(2i - 1)}{2} \frac{a}{a_{\prime}} b_{3,i-1}$$

† r^* being the coefficient of the cosine of the same argument in the expression for $\frac{a}{r}$ and excluding the case of $i = 0$.

The reader is requested to make the following corrections.

Page 50, line 4, read $q_6 = -\frac{3\,a}{2\,a_{\prime}^2} + \frac{3}{2}\,\frac{a}{a_{\prime}^2}\,b_{3,0} - \frac{a^2}{2\,a_{\prime}^3}\,b_{3,1} + \frac{a}{4\,a_{\prime}^2}\,b_{3,2}$

Page 53, line 3, read $= \frac{m_{\prime}}{\mu}\left\{\frac{2\,a^3}{a_{\prime}^3}\,b_{3,0} - \frac{5}{4}\,\frac{a^2}{a_{\prime}^2}\,b_{3,1}\right\}$

Page 247, line 1, read

$$\begin{aligned}\lambda = {} & n\,t \\ & + \lambda_1 \sin 2\,t \\ & + e\,\lambda_2 \sin x \\ & + e\,\lambda_3 \sin (2\,t - x) \\ & + e\,\lambda_4 \sin (2\,t + x) \\ & + e_{\prime}\,\lambda_5 \sin z \quad \&c.\ \&c.\end{aligned}$$

for

$$\begin{aligned}\lambda = {} & n\,t \\ & + \lambda_1 \cos 2\,t \\ & + e\,\lambda_2 \cos x \quad \&c.\ \&c.\end{aligned}$$

Page 254, line 1, read $-\frac{3}{2}\,e^2 e_{\prime} \cos (2\,t + 2\,x + z)$
[25] [30]

Page 260, line 6, read $+\left\{3 - \frac{15}{2}\right\}\,e\,e_{\prime} \cos (x - z - 2\,y)$
[89]

Page 262, line 6, read $-\frac{15}{32}\,e\,e_{\prime}^3 \cos (2\,t + x - 3\,z)$
[58]

Page 265, line 1, read $+\frac{25}{64}\,\frac{a^2}{a_{\prime}^3}\,e^3\,e_{\prime} \cos (2\,t + 3\,x + z) + \frac{3}{32}\,\frac{a^2}{a_{\prime}^3}\,e^3\,e_{\prime} \cos (3\,x - z)$
[43] [44]

Page 274, line 6, read $+\left\{2\,r_3 + r_1 - \left\{\frac{9}{2\,(2 - m - c)}\right.\right.$ &c.

Page 274, line 7, read $+\left\{2\,r_4 + r_1 - \left\{-\frac{3}{2\,(2 - m + c)}\right.\right.$ &c.

Page 291, line 9, read $+\frac{3}{16}\,\frac{a}{a_{\prime}^2}\,e_{\prime}^2 \cos (t + 2\,z)$

Page 294, line 20, read $+\frac{m_{\prime}\,a}{2\,a_{\prime}^2} \cos (2\,n\,t - n_{\prime}\,t - \varpi) + \frac{2\,m_{\prime}\,a}{a_{\prime}^2}\,e_{\prime} \cos (n\,t - 2\,n_{\prime}\,t + \varpi_{\prime})$

XVII. *On a peculiar class of Acoustical Figures; and on certain Forms assumed by groups of particles upon vibrating elastic Surfaces.* *By* M. Faraday, *F.R.S. M.R.I., Corr. Mem. Royal Acad. Sciences of Paris, &c. &c.*

Read May 12, 1831.

1. THE beautiful series of forms assumed by sand, filings, or other grains, when lying upon vibrating plates, discovered and developed by Chladni, are so striking as to be recalled to the minds of those who have seen them by the slightest reference. They indicate the quiescent parts of the plates, and visibly figure out what are called the nodal lines.

2. Afterwards M. Chladni observed that shavings from the hairs of the exciting violin bow did not proceed to the nodal lines, but were gathered together on those parts of the plate the most violently agitated, i. e. at the centres of oscillation. Thus when a square plate of glass held horizontally was nipped above and below at the centre, and made to vibrate by the application of a violin bow to the middle of one edge, so as to produce the lowest possible sound, sand sprinkled on the plate assumed the form of a diagonal cross; but the light shavings were gathered together at those parts towards the middle of the four portions where the vibrations were most powerful and the excursions of the plate greatest.

3. Many other substances exhibited the same appearance. Lycopodium, which was used as a light powder by Oersted, produced the effect very well. These motions of lycopodium are entirely distinct from those of the same substance upon plates or rods in which longitudinal vibrations are excited.

4. In August 1827, M. Savart read a paper to the Royal Academy of Sciences *, in which he deduced certain important conclusions respecting the subdivision of vibrating sonorous bodies from the forms thus assumed by light powders. The arrangement of the sand into lines in Chladni's experiments

* Annales de Chimie, xxxvi. p. 187.

shows a division of the sounding plate into parts, all of which vibrate isochronously, and produce the same tone. This is the principal mode of division. The fine powder which can rest at the places where the sand rests, and also accumulate at other places, traces a more complicated figure than the sand alone, but which is so connected with the first, that, as M. SAVART states, "the first being given, the other may be anticipated with certainty; from which it results that every time a body emits sounds, not only is it the seat of many modes of division which are superposed, but amongst all these modes there are always two which are more distinctly established than all the rest. My object in this memoir is to put this fact beyond a doubt, and to study the laws to which they appear subject."

5. M. SAVART then proceeds to establish a secondary mode of division in circular, rectangular, triangular and other plates; and in rods, rings, and membranes. This secondary mode is pointed out by the figures delineated by the lycopodium or other light powder; and as far as I can perceive, its existence is assumed, or rather proved, exclusively from these forms. Hence much of the importance which I attach to the present paper. A secondary mode of division, so subordinate to the principal as to be always superposed by it, might have great influence in reasonings upon other points in the philosophy of vibrating plates; to prove its existence therefore is an important matter. But its existence being assumed and supported by such high authority as the name of SAVART, to prove its non-existence, supposing it without foundation, is of equal consequence.

6. The essential appearances, as far as I have observed them, are as follows. Let the plate before mentioned (2), which may be three or four inches square, be nipped and held in a horizontal position by a pair of pincers of the proper form, and terminated, at the part touching the glass, by two pieces of cork; let lycopodium powder be sprinkled over the plate, and a violin bow be drawn downwards against the middle of one edge so as to produce a clear full tone. Immediately the powder on those four parts of the plate towards the four edges will be agitated, whilst that towards the two diagonal cross lines will remain nearly or quite at rest. On repeating the application of the bow several times, a little of the loose powder, especially that in small masses, will collect upon the diagonal lines, and thus, showing one of the figures which CHLADNI dis-

covered, will also show the principal mode of division of the plate. Most of the powder which remains upon the plate will, however, be collected in four parcels; one placed near to each edge of the plate, and evidently towards the place of greatest agitation. Whilst the plate is vibrating (and consequently sounding) strongly, these parcels will each form a rather diffuse cloud, moving rapidly within itself; but as the vibration diminishes, these clouds will first contract considerably in bulk, and then settle down into four groups, each consisting of one, two, or more hemispherical parcels (53), which are in an extraordinary condition; for the powder of each parcel continues to rise up at the centre and flow down on every side to the bottom, where it enters the mass to ascend at the centre again, until the plate has nearly ceased to vibrate. If the plate be made to vibrate strongly, these parcels are immediately broken up, being thrown into the air, and form clouds, which settle down as before; but if the plate be made to vibrate in a smaller degree, by a more moderate application of the bow, the little hemispherical parcels are thrown into commotion without being sensibly separated from the plate, and often slowly travel towards the quiescent lines. When one or more of them have thus receded from the place over which the clouds are always formed, and a powerful application of the bow is made, sufficient to raise the clouds, it will be seen that these heaps rapidly diminish, the particles of which they are composed being swept away from them, and passing back in a current over the glass to the cloud under formation, which ultimately settles as before into the same four groups of heaps. These effects may be repeated any number of times, and it is evident that the four parts into which the plate may be considered as divided by the diagonal lines are repetitions of one effect.

7. The form of the little heaps, and the involved motion they acquire, are no part of the phenomena under consideration at present. They depend upon the adhesion of the particles to each other and to the plate, combined with the action of the air or surrounding medium, and will be resumed hereafter (53). The point in question is the manner in which fine particles do not merely remain at the centres of oscillation, or places of greatest agitation, but are actually driven towards them, and that with so much the more force as the vibrations are more powerful.

8. That the agitated substance should be in very fine powder, or very light, appears to be the only condition necessary for success; fine scrapings from a

common quill, even when the eighth of an inch in length or more, will show the effect. Chemically pure and finely divided silica rivals lycopodium in the beauty of its arrangement at the vibrating parts of the plate, although the same substance in sand or heavy particles proceeds to the lines of rest. Peroxide of tin, red lead, vermilion, sulphate of baryta, and other heavy powders when highly attenuated, collect also at the vibrating parts. Hence it is evident that the nature of the powder has nothing to do with its collection at the centres of agitation, provided it be dry and fine.

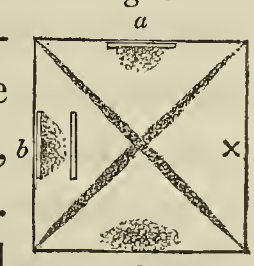
9. The cause of these effects appeared to me, from the first, to exist in the medium within which the vibrating plate and powder were placed, and every experiment which I have made, together with all those in M. Savart's paper, either strongly confirm, or agree with this view. When a plate is made to vibrate (2), currents (24) are established in the air lying upon the surface of the plate, which pass from the quiescent lines towards the centres or lines of vibration, that is, towards those parts of the plates where the excursions are greatest, and then proceeding outwards from the plate to a greater or smaller distance, return towards the quiescent lines. The rapidity of these currents, the distance to which they rise from the plate at the centre of oscillation, or any other part, the blending of the progressing and returning air, their power of carrying light or heavy particles, and with more or less rapidity or force, are dependent upon the intensity or force of the vibrations, the medium in which the vibrating plate is placed, the vicinity of the centre of vibration to the limit or edge of the plate, and other circumstances, which a simple experiment or two will immediately show must exert much influence on the phenomena.

10. So strong and powerful are these currents, that when the vibrations were energetic, the plate might be inclined 5°, 6°, or 8° to the horizon and yet the gathering clouds retain their places. As the vibrations diminished in force, the little heaps formed from the cloud descended the hill; but on strengthening the vibrations they melted away, the particles ascending the inclined plane on those sides proceeding upwards, and passing again to the cloud. This took place when neither sand nor filings could rest on the quiescent or nodal lines. Nothing could remain upon the plate except those particles which were so fine as to be governed by the currents, which (if they exist at all) it is evident would exist in whatever situation the plate was placed.

11. M. Savart seems to consider that the reason why the powder gathers together at the centres of oscillation is, "that the amplitude of the oscillations being very great, the middle of each of those centres (of vibration) is the only place where the plate remains nearly plane and horizontal, and where, consequently, the powder may reunite, whilst the surface being inclined to the right or left of this point, the parcels of powder cannot stop there." But the inclination thus purposely given to the plate, was very many times that which any part acquires by vibration in a horizontal position, and consequently proves that the horizontality of any part of the plate is not the cause of the powder collecting there, although it may be favourable to its remaining there when collected.

12. Guided by the idea of what ought to happen, supposing the cause now assigned were the true one, the following amongst many other experiments were made. A piece of card about an inch long and a quarter of an inch wide was fixed by a little soft cement on the face of the plate near one edge, the plate held as before at the middle, lycopodium or fine silica strewed upon it, and the bow applied at the middle of another edge; the powder immediately advanced close to the card, and the place of the cloud was much nearer to the edge than before. Fig. 1 represents the arrangement; the diagonal lines being those which sand would have formed, the line at the top *a* representing the place of the card, and the × to the right the place where the bow was applied. On applying a second piece of card as at *b*, the powder seemed indifferent to it or nearly so, and ultimately collected as in the first figure: *c* represents the place of the cloud when no card is present.

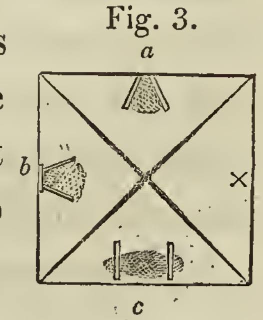
Fig. 1.



13. Pieces of card were then fixed on the glass in the three angular forms represented in fig. 2; upon vibrating the plate the fine powder always went into the angle, notwithstanding its difference of position in the three experiments, but perfectly in accordance with the idea of currents intercepted more or less by the card. When two pieces of card were fixed on the plate as in fig. 3. *a*, the powder proceeded into the angle but not to the edge of the glass, remaining about $\frac{1}{8}$th of an inch from it; but on closing up that opening, as at *b*, the powder went quite up into the corner.

Fig. 2.

Fig. 3.



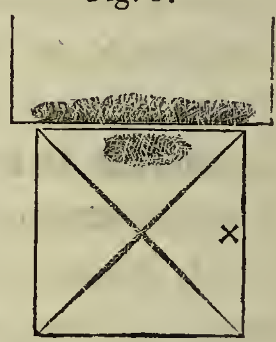
14. Upon fixing two pieces of card on the plate as at *c* fig. 3, the powder between them collected in the middle very nearly as if no card had been present; but that on the outside of the cards gathered close up against them, being able to proceed so far in its way to the middle, but no further.

15. In all these experiments the sound was very little lowered, the form of the cross was not changed, and the light powders collected on the other three portions of the plate, exactly as if no card walls had been applied on the fourth; so that no reason appears for supposing that the mode in which the plate vibrated was altered, but the powders seem to have been carried forward by currents which could be opposed or directed at pleasure by the card stops.

Fig. 4.

16. A piece of gold-leaf being laid upon the plate, so that it did not overlap the edge, fig. 4, the current of air towards the centre of vibration was beautifully shown; for, by its force, the air crept in under the gold-leaf on all sides, and raised it up into the form of a blister; that part of the gold-leaf corresponding to the centre of the locality of the cloud, when light powder was used, being frequently a sixteenth or twelfth of an inch from the glass. Lycopodium or other fine powder sprinkled round the edge of the gold-leaf, was carried in by the entering air, and accumulated underneath.

Fig. 5.



17. When silica was placed on the edge of another glass plate, or upon a book, or block of wood, and the edge of the vibrating plate brought as nearly as possible to the edge of the former, fig. 5, part of the silica was always driven on to the vibrating plate, and collected in the usual place; as if in the midst of all the agitation of the air in the neighbourhood of the two edges, there was still a current towards the centre of vibration, even from bodies not themselves vibrating.

18. When a long glass plate is supported by bridges or strings at the two nodal lines represented in fig. 6, and made to vibrate, the lycopodium collects in three divisions; that between the nodal lines does not proceed at once into a line equidistant from the nodal lines and parallel to them, but advances from the edges of the plate towards the middle by paths, which are a little curved and oblique to the edges where they occur near the nodal lines, but are almost

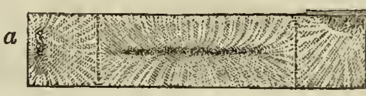
Fig. 6.

perpendicular to it elsewhere, and the powder gradually forms a line along the middle of the plate ; it is only by continuing the experiment for some time that it gathers up into a heap or cloud equidistant from the nodal lines. But upon fixing card walls upon this plate, as in fig. 7, the course of the powder within the cards was directly parallel to them and to the edge, instead of being perpendicular, and also directly towards the centre of oscillation. To prove that it was not as a weight that the card acted, but as an obstacle to the currents of air formed, it was not moved from its place, but bent flat down outwards, and then the fine powder resumed the courses it took upon the plate when without the cards. Upon raising the cards the first effect was reproduced.

Fig. 7.

19. The lycopodium sprinkled over the extremities of such a plate proceeds towards places equidistant from the sides and near the ends, as at *a* fig. 8 ; but on cementing a piece of paper to the edge, so as to form a wall about one quarter or one third of an inch high, *b*, the powder immediately moved up to it, and retained this new place. In a longer narrow plate, similarly arranged, the powder could be made to pass to either edge, or to the middle, according as paper interceptors to the currents of air were applied.

Fig. 8.

a  *b*

20. Plates of tin, four or five inches long, and from an inch to two inches wide, fixed firmly at one end in a horizontal position, and vibrated by applying the fingers, show the progress of the air and the light powders well. The vibrations are of comparatively enormous extent, and the appearances are consequently more instructive.

21. If a tuning-fork be vibrated, then held horizontally with the broad surface of one leg uppermost, and a little lycopodium be sprinkled upon it, the collection of the powder in a cloud along the middle, and the formation of the involving heaps also in a line along the middle of the vibrating steel bar, may be beautifully observed. But if a piece of paper be attached by wax to the side of the limb, so as to form a fence projecting above it, as in the former experiments (19), then the powder will take up its place close to the paper ; and if pieces of paper be attached on different parts of the same leg, the powder will go to the different sides, in the different parts, at the same time.

22. The effects under consideration are exceedingly well shown and illus-

trated by membranes. A piece of parchment was stretched and tightly tied, whilst moist, over the aperture of a funnel five or six inches in diameter; a small hole was made in the middle, and a horse-hair passed through it, but with a knot at the extremity that it might thereby be retained. Upon fixing the funnel in an upright position, and after applying a little powdered resin to the thumbs and fore-fingers, drawing them upward over the horse-hair, the membrane was thrown into vibration with more or less force at pleasure. By supporting the funnel on a ring, passing the horse-hair in the opposite direction through the hole in the membrane, and drawing the fingers over it downwards, the direction in which the force was applied could be varied according to circumstances.

23. When lycopodium or light powders were sprinkled upon this surface, the rapidity with which they ran to the centre, the cloud formed there, the involving heaps, and many other circumstances, could be observed very advantageously.

24. The currents which I have considered as existing upon the surface of the plate, membranes, &c. from the quiescent parts towards the centres or lines of vibration (9), arise necessarily from the mechanical action of that surface upon the air. As any particular part of the surface moves upwards in the course of its vibration, it propels the air and communicates a certain degree of force to it, perpendicular or nearly so to the vibrating surface; as it returns, in the course of its vibration, it recedes from the air so projected, and the latter consequently tends to return into the partial vacuum thus formed. But as of two neighbouring portions of air, that over the part of the plate nearest to the centre of oscillation has had more projectile force communicated to it than the other, because the part of the plate urging it was moving with greater velocity, and through a greater space, so it is in a more unfavourable condition for its immediate return, and the other, i. e. the portion next to it towards the quiescent line, presses into its place. This effect is still further favoured, because the portion of air thus displaced is urged from similar causes at the same moment into the place left vacant by the air still nearer the centre of oscillation; so that each time the plate recedes from the air, an advance of the air immediately above it is made from the quiescent towards the vibrating parts of the plates.

25. It will be evident that this current is highly favourable for the transference of light powders towards the centre of vibration. Whilst the air is forced forward, the advance of the plate against the particles holds them tight; but when the plate recedes, and the current exists, the particles are at that moment left unsupported except by the air, and are free to move with it.

26. The air which is thus thrown forward at and towards the centre of oscillation, must tend by the forces concerned to return towards the quiescent lines, forming a current in the opposite direction to the first, and blending more or less with it. I endeavoured, in various ways, to make the extent of this system of currents visible. In the experiment already referred to, where gold-leaf was placed over the centre of oscillation (16), the upward current at the most powerful part was able to raise the leaf about one tenth of an inch from the plate. The higher the sounds with the same plate or membrane, i. e. the greater the number of vibrations, the less extensive must be the series of currents; the slower the vibrations, or the more extensive the excursion of the parts from increased force applied, the greater the extent of disturbance. With glass plates (2. 12) the cloud is higher and larger as the vibrations are stronger, but still not so extensive as they are upon the stretched membrane (22), where the cloud may frequently be seen rising up in the middle and flowing over towards the sides.

27. When the membrane stretched upon the funnel (22) was made to vibrate by the horse-hair proceeding downwards, and a large glass tube, as a cylindrical lamp-glass, was brought near to the centre of vibration, no evidence of a current entirely through the lamp-glass could be perceived; but still the most striking proofs were obtained of the existence of carrying currents by the effects upon the light powder, for it flew more rapidly under the edge, and tended to collect towards the axis of the tube; it could even be diverted somewhat from its course towards the centre of oscillation. A piece of upright paper, held with its edge equally near, did not produce the same effect; but immediately that it was rolled into a tube, it did. When the glass chimney was suspended very carefully, and at but a small distance from the membrane, the powder often collected at the edge, and revolved there; a complicated action between the currents and the space under the thickness of the glass taking place, but still tending to show the influence of the air in arranging and disposing the powders.

28. A sheet of drawing-paper was stretched tightly over a frame so as to form a tense elastic surface nearly three feet by two feet in extent. Upon placing this in a horizontal position, throwing a spoonful of lycopodium upon it, and striking it smartly below with the fingers, the phenomena of collection at the centre of vibration, and of moving heaps, could be obtained upon a magnificent scale. When the lycopodium was uniformly spread over the surface, and any part of the paper slightly tapped by the hand, the lycopodium at any place chosen could be drawn together merely by holding the lamp-glass over it. It will be unnecessary to enter into the detail of the various actions combining to produce these effects; it is sufficiently evident, from the mode in which they may be varied, that they depend upon currents of air.

29. A very interesting set of effects occurred when the stretched parchment upon the funnel (22) was vibrated under plates; the horse-hair was directed downwards, and the membrane, after being sprinkled over with light powder, was covered by a plate of glass resting upon the edge of the funnel; upon throwing the membrane into a vibratory state, the powder collected with much greater rapidity than without the plate; and instead of forming the semi-globular moving heaps, it formed linear arrangements, all concentric to the centre of vibration. When the vibrations were strong, these assumed a revolving motion, rolling towards the centre at the part in contact with the membrane, and from it at the part nearest the glass; thus illustrating in the clearest manner the double currents caged up between the glass and the membrane. The effect was well shown by carbonate of magnesia.

30. Sometimes when the plate was held down very close and tight, and the vibrations were few and large, the powder was all blown out at the edge; for then the whole arrangement acted as a bellows; and as the entering air travelled with much less velocity than the expelled air, and as the forces of the currents are as the squares of the velocity, the issuing air carried the powder more forcibly than the air which passed in, and finally threw it out.

31. A thin plate of mica laid loosely upon the vibrating membrane showed the rotating concentric lines exceedingly well.

32. From these experiments on plates and surfaces vibrating in air, it appears that the forms assumed by the determination of light powders towards the places of most intense vibration, depend, not upon any secondary mode of

division, or upon any immediate and peculiar action of the plate, but upon the currents of air necessarily formed over its surface, in consequence of the extra-mechanical action of one part beyond another. In this point of view the nature of the medium in which those currents were formed ought to have great influence over the phenomena; for the only reason why silica as sand should pass towards the quiescent lines, whilst the same silica as fine powder went from them, is, that in its first form the particles are thrown up so high by the vibrations as to be above the currents, and that if they were not thus thrown out of their reach they would be too heavy to be governed by them; whilst in the second form they are not thrown out of the lower current, except near the principal place of oscillation, and are so light as to be carried by it in whatever direction it may proceed.

33. In the exhausted receiver of the air-pump therefore the phenomena ought not to occur as in air; for as the force of the currents would be there excessively weakened, the light powders ought to assume the part of heavier grains in the air. Again, in denser media than air, as in water for instance, there was every reason to expect that the heavier powder, as sand and filings, would perform the part of light powders in air, and be carried from the quiescent to the vibrating parts.

34. The experiments in the air-pump receiver were made in two ways. A round plate of glass was supported on four narrow cork legs upon a table, and then a thin glass rod with a rounded end held perpendicularly upon the middle of the glass. By passing the moistened fingers longitudinally along this rod the plate was thrown into a vibratory state; the cork legs were then adjusted in the circular nodal line occurring with this mode of vibration; and when their places were thus found they were permanently fixed. The plate was then transferred into the receiver of an air-pump, and the glass rod by which it was to be thrown into vibration passed through collars in the upper part of the receiver, the entrance of air there being prevented by abundance of pomatum. When fine silica was sprinkled upon the plate, and the plate vibrated by the wet fingers applied to the rod, the receiver not being exhausted, the fine powder travelled from the nodal line, part collecting at the centre, and other part in a circle, between the nodal line and the edge. Both these situations were places of vibration, and exhibited themselves as such by the agitation of the powder. Upon again sprinkling fine silica uniformly over the plate, ex-

hausting the receiver to twenty-eight inches, and vibrating the plate, the silica went from the middle towards the nodal line or place of rest, performing exactly the part of sand in air. It did not move at the edges of the plate, and as the apparatus was inconvenient and broke during the experiment, the following arrangement was adopted in its place.

35. The mouth of a funnel was covered (22) with a well-stretched piece of fine parchment, and then fixed on a stand with the membrane horizontal; the horse-hair was passed loosely through a hole in a cork, fixed in a metallic tube on the top of the air-pump receiver; the tube above the cork was filled to the depth of half an inch with pomatum, and another perforated cork put over that; a cup was formed on the top of the second cork, which was filled with water. In this way the horse-hair passed first through pomatum and then water, and by giving a little pressure and rotatory motion to the upper cork during the time that the horse-hair was used to throw the membrane into vibration, it was easy to keep the pomatum below perfectly in contact with the hair, and even to make it exude upwards into the water above. Thus no possibility of the entrance of air by and along the horse-hair could exist, and the tightness of all the other and fixed parts of the apparatus was ascertained by the ordinary mode of examination. A little paper shelf was placed in the receiver under the cork to catch any portion of pomatum that might be forced through by the pressure, and prevent its falling on to the membrane.

36. This arrangement succeeded: when the receiver was full of air, the lycopodium gathered at the centre of the membrane with great facility and readiness, exhibiting the cloud, the currents, and the involving heaps. Upon exhausting the receiver until the barometrical gauge was at twenty-eight inches, the lycopodium, instead of collecting at the centre, passed across the membrane towards one side which was a little lower than the other. It passed by the middle just as it did over any other part; and when the force of the vibrations was much increased, although the powder was more agitated at the middle than elsewhere, it did not collect there, but went towards the edges or quiescent parts. Upon allowing air to enter until the barometer stood at twenty-six inches, and repeating the experiments, the effect was nearly the same. When the vibrations were very strong, there were faint appearances of a cloud, consisting of the very finest particles, collecting at the centre of vibration;

but no sensible accumulation of the powder took place. At twenty-four inches of the barometer the accumulation at the centre began to appear, and there was a sensible, though very slight effect visible of the return of the powder from the edges. At twenty-two inches these effects were stronger; and when the barometer was at twenty inches, the currents of air within the receiver had force enough to cause the collection of the principal part of the lycopodium at the centre of vibration. Upon again, however, restoring the exhaustion to twenty-eight inches, all the effects were reproduced as at first, and the lycopodium again proceeded to the lower or the quiescent parts of the membrane. These alternate effects were obtained several times in succession before the apparatus was dismounted.

37. In this form of experiment there were striking proofs of the existence of a current upwards from the middle of the membrane when vibrating in air, (24), and the extent of the system of currents (26) was partly indicated. The powder purposely collected at the middle by vibrations, when the receiver was full of air, was observed as to the height to which it was forced upwards by the vibrations; and then the receiver being exhausted, the height to which the powder was thrown by similar vibrations was again observed. In the latter cases it was nothing like so great as in the former, the height not being two-thirds, and barely one-half, the first height. Had the powder been thrown up by mere propulsion, it should have risen far higher in vacuo than in air: but the reverse took place; and the cause appears to be, that in air the current had force enough to carry the fine particles up to a height far beyond what the mere blow which they received from the vibrating membrane could effect.

38. For the experiments in a denser medium than air, water was chosen. A circular plate of glass was supported upon four feet in a horizontal position, surrounded by two or three inches of water, and thrown into vibration by applying a glass rod perpendicular to the middle, as in the first experiment in vacuo (34); the feet were shifted until the arrangement gave a clear sound, and the moistened brass filings sprinkled upon the plate formed regular lines or figures. These lines were not however lines of rest, as they would have been in the air, but were the places of greatest vibration; as was abundantly evident from their being distant from that nodal line determined and indicated by the contact of the feet, and also from the violent agitation of the filings.

In fact, the filings proceeded from the quiescent to the moving parts, and there were gathered together; not only forming the cloud of particles over the places of intense vibration, but also settling down, when the vibrations were weaker, into the same involving groups, and in every respect imitating the action of light powders in air. Sand was affected exactly in the same manner; and even grains of platina could be in this way collected by the currents formed in so dense a medium as water.

39. The experiments were then made under water with the membranes stretched over funnels (22) and thrown into vibration by horse-hairs drawn between the fingers. The space beneath the membrane could be retained, filled with air, whilst the upper surface was covered two or three inches deep with water; or the space below could also be filled with water, or the force applied to the membrane by the horse-hair could be upwards or downwards at pleasure. In all these experiments the sand or filings could be made to pass with the utmost facility to the most powerfully vibrating part, that being either at the centre only, or in addition, in circular lines, according to the mode in which the membrane vibrated. The edge of the funnel was always a line of rest; but circular nodal lines were also formed, which were indicated, not by the accumulation of filings upon them, but by the tranquil state of those filings which happened to be there, and also by being between those parts where the filings, by their accumulation and violent agitation, indicated the parts in the most powerful vibratory state.

40. Even when by the relaxation of the parchment from moisture, and the force upwards applied by the horse-hair, the central part of the membrane was raised the eighth of an inch or more above the edges, the circle not being four inches in diameter, still the filings would collect there.

41. When in place of parchment common linen was used, as becoming tighter rather than looser when wetted, the same effects were obtained.

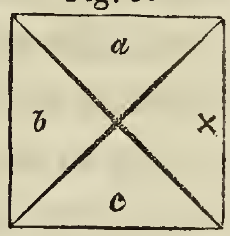
42. Both the reasoning adopted and the effects described were such as to lead to the expectation that if the plate vibrating in air was covered with a layer of liquid instead of sand or lycopodium, that liquid ought to be determined from the quiescent to the vibrating parts and be accumulated there. A square plate was therefore covered with water, and vibrated as in the former experiments (2. 6.); but all endeavours to ascertain whether accumulation

occurred at the centres of oscillation, either by direct observation, or the reflection from its surface of right-lined figures, or by looking through the parts, as through a lens, at small print and other objects, failed.

43. As however when the plate was strongly vibrated, the well-known and peculiar crispations which form on water at the centres of vibration, occurred and prevented any possible decision as to accumulation, it was only when these were absent and the vibration weak, and the accumulation therefore small, that any satisfactory result could be expected; but as even then no appearance was perceived, it was concluded that the force of gravity combined with the mobility of the fluid was sufficient to restore the uniform condition of the layer of water after the bow was withdrawn, and before the eye had time to observe the convexity expected.

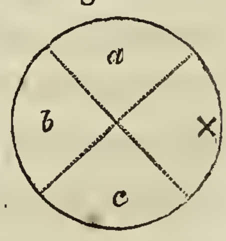
44. To remove in part the effect of gravity, or rather to make it coincide with, instead of oppose the convexity, the under surface of the plate was moistened instead of the upper, and by inclining the plate a little, the water made to hang in drops at *a* or *b* or *c*, fig. 9, at pleasure. On applying the bow at ×, and causing the plate to vibrate, the drops instantly disappeared, the water being gathered up and expanded laterally over the parts of the plate from which it had flowed. On stopping the vibration, it again accumulated in hanging drops, which instantly disappeared as before on causing the plate to vibrate, the force of gravity being entirely overpowered by the superior forces excited by the vibrating plate. Still, no visible evidence of convexity at the centres of vibration were obtained, and the water appeared rather to be urged from the vibrating parts than to them.

Fig. 9.



45. The tenacity of oil led to the expectation that better results would be obtained with it than with water. A round plate, held horizontally by the middle (6. 42), was covered with oil over the upper surface, so as to be flooded, except at ×, fig. 10, and the bow applied at × as before, to produce strong vibration. No crispation occurred in the oil, but it immediately accumulated at *a*, *b*, and *c*, forming fluid lenses there, rendered evident by their magnifying power when print was looked at through them. The accumulations were also visible on putting a sheet of white paper beneath, in consequence of the colour of the oil

Fig. 10.



being deeper at the accumulations than elsewhere; and they were also rendered beautifully evident by making the experiment in sunshine, or by putting a candle beneath the plate, and placing a screen on the opposite side to receive the images formed at the focal distance.

46. When the vibration of the plate ceased, the oil gradually flowed back until of uniform depth. On renewing the vibration, the accumulations were re-formed, the phenomena of accumulation occurring with as much certainty and beauty as if lycopodium powder had been used.

47. To remove every doubt of the fluid passing from the quiescent to the agitated parts, centres of vibration were used, nearly surrounded by nodal lines. A square plate, fig. 11, being held at *c*, and the bow applied at ×, gave with sand, nodal lines, resembling those in the figure. Then clearing off the sand, putting oil in its place, and producing the same mode of vibration as before, the oil accumulated at *a* and *b*, forming two heaps or lenses as in the former experiment (45).

Fig. 11.

48. The experiment made with water on the under surface (44) was now repeated with oil, the round plate being used (45). The hanging drop of oil rose up as the water did before, but the lateral diffusion was soon limited; for lenses were formed at the centres of vibration just as when the oil was upon the upper surface, and, as far as could be ascertained by general examination, of the same form and power. On stopping the vibration, the oil gathered again into hanging drops; and on renewing it, it was again disposed in the lens-like accumulations.

49. With white of egg the same observable accumulation at the centres of vibration could be produced.

50. Hence it is evident that when a surface vibrating normally, is covered with a layer of liquid, that liquid is determined from the quiescent to the vibrating parts, producing accumulation at the latter places; and that this accumulation is limited, so that if purposely rendered too great by gravity or other means, it will quickly be diminished by the vibrations until the depth of fluid at any one part has a certain and constant relation to the velocity there and to the depth elsewhere.

51. From the accumulated evidence which these experiments afford, I think there can remain no doubt of the cause of the collection of fine powders at the

centres or lines of vibration of plates, membranes, &c. under common circumstances; and that no secondary mode of division need be assumed to account for them. I have been the more desirous of accumulating experimental evidence, because I have thought on the one hand that the authority of SAVART should not be doubted on slight grounds, and on the other, that if by accident it be placed in the wrong scale, the weight of evidence against it should be such as fully to establish the truth and prevent a repetition of the error by others.

52. It must be evident that the phenomena of collection at the centres or lines of greatest vibration are exhibited in their purest form at those places which are surrounded by nodal lines; and that where the centre or place of vibration is at or near to an edge, the effects must be very much modified by the manner in which the air is there agitated. It is this influence, which, in the square plates (6. 12) and other arrangements, prevents the clouds being at the very edge of the glass. They may be well illustrated by vibrating tin plates under water over a white bottom, and sprinkling dark-coloured sand or filings upon various parts of the plates.

On the peculiar Arrangement and Motions of the heaps formed by particles lying on vibrating surfaces.

53. The peculiar manner in which the fine powder upon a vibrating surface is accumulated into little heaps, either hemispherical or merely rounded, and larger or smaller in size, has already been described (6. 28), as well also as the singular motion which they possess, as long as the plate continues in vibration. These heaps form on any part of the surface which is in a vibratory state, and not merely under the clouds produced at the centres of vibration, although the particles of the clouds always settle into similar heaps. They have a tendency, as heaps, to proceed to the nodal or quiescent lines, but are often swept away in powder by the currents already described (6). When on a place of rest, they do not acquire the involving motion. When two or more are near together or touch, they will frequently coalesce and form but one heap, which quickly acquires a rounded outline. When in their most perfect and final form, they are always round.

54. The moving heaps formed by lycopodium on large stretched drawing-

paper (28), are on so large a scale as to be very proper for critical examination. The phenomena can be exhibited also even by dry sand on such a membrane, the sand being in large quantity and the vibrations slow. When the surface is thickly covered by sand from a sieve, and the paper tapped with the finger, the manner in which the sand draws up into moving heaps is very beautiful.

55. When a single heap is examined, which is conveniently done by holding a vibrating tuning-fork in a horizontal position, and dropping some lycopodium upon it, it will be seen that the particles of the heap rise up at the centre, overflow, fall down upon all sides, and disappear at the bottom, apparently proceeding inwards; and this evolving and involving motion continues until the vibrations have become very weak.

56. That the medium in which the experiment is made has an important influence, is shown by the circumstance of heavy particles, such as filings, exhibiting all these peculiarities when they are placed upon surfaces vibrating in water (39); the heaps being even higher at the centre than a heap of equal diameter formed of light powder in the air. In water, too, they are formed indifferently upon any part of the plate or membrane which is in a vibratory state. They do not tend to the quiescent lines; but that is merely from the great force of the currents formed in water as already described (38), and the power with which they urge obstacles to the place of greatest vibration.

57. If a glass plate be supported and vibrated (6), its surface having been covered with sand enough to hide the plate, and water enough to moisten and flow over the sand, the sand will draw together in heaps, and these will exhibit the peculiar and characteristic motion of the particles in a very striking manner.

58. The aggregation and motion of these heaps, either in air or other fluids, is a very simple consequence of the mechanical impulse communicated to them by the joint action of the vibrating surface and the surrounding medium. Thus in air, when, in the course of a vibration, the part of a plate under a heap rises, it communicates a propelling force upwards to that heap, mingled as it is with air, greater than that communicated to the surrounding atmosphere, because of the superior specific gravity of the former; upon receding from the heap, therefore, in performing the other half of its vibration, it forms a partial

vacuum, into which the air, round the heap, enters with more readiness than the heap itself; and as it enters, carries in the powder at the bottom edge of the heap with it. This action is repeated at every vibration, and as they occur in such rapid succession that the eye cannot distinguish them, the centre part of the heap is continually progressing upwards; and as the powder thus accumulates above, whilst the base is continually lessened by what is swept in underneath, the particles necessarily fall over and roll down on every side.

59. Although this statement is made upon the relation of the heap, as a mass, to the air surrounding it, yet it will be seen at once that the same relation exists between any two parts of the heap at different distances from the centre; for the one nearest the centre will be propelled upward with the greatest force, and the other will be in the most favourable state for occupying the partial vacuum left by the receding plate.

60. This view of the effect will immediately account for all the appearances; the circular form, the fusion together of two or more heaps, their involving motion, and their existence upon any vibrating part of the plate. The manner in which the neighbouring particles would be absorbed by the heaps is also evident; and as to their first formation, the slightest irregularities in the powder or surface would determine a commencement, which would then instantly favour the increase.

61. It is quite true, that if the powder were coherent, that force alone would tend to produce the same effect, but only in a very feeble degree. This is sufficiently shown by the experiments made in the exhausted receiver (36). When the barometer of the air-pump was at twenty-eight inches, that in the air being about 29.2 inches, the heaps, or rather parcels, formed very beautifully over the whole surface of the membrane; but they were very flat and extensive compared with the heaps in air, and the involving motion was very weak. As the air was admitted, the vibration being continued, the heaps rose in height, contracted in diameter, and moved more rapidly. Again, in the experiments with filings and sand in water, no cohesive action could assist in producing the effect; it must have been entirely due to the manner in which the particles were mechanically urged in a medium of less density than themselves.

62. The conversion of these round heaps into linear concentric involving parcels, in the experiment already described (29. 31), when the membrane was

covered by a plate of glass, is a necessary consequence of the arrangements there made, and tends to show how influential the action of the air or other including medium is in all the phenomena considered in this paper. No incompatible principles are assumed in the explication given of the arrangement of the forces producing the two classes of effects in question, and though by variation of the force of vibration and other circumstances, the one effect can be made, within certain limits, to pass into the other, no anomaly or contradiction is thus involved, nor any result produced, which, as it appears to me, cannot be immediately accounted for by reference to the principles laid down.

Royal Institution,
March 21, 1831.

APPENDIX.

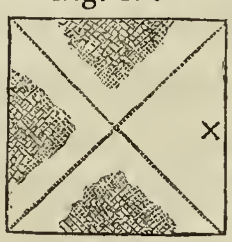
On the Forms and States assumed by Fluids in contact with vibrating elastic surfaces.

63. When the upper surface of a plate vibrating so as to produce sound (2. 6) is covered with a layer of water, the water usually presents a beautifully crispated appearance in the neighbourhood of the centres of vibration. This appearance has been observed by OERSTED*, WHEATSTONE †, WEBER ‡, and probably others. It, like the former phenomena which I have endeavoured to explain, has led to false theory, and being either not understood or misunderstood, has proved an obstacle to the progress of acoustical philosophy.

64. On completing the preceding investigation, I was led to believe that the principles assumed would, in conjunction with the cohesion of fluids, account for these phenomena. Experimental investigation fully confirmed this expectation, but the results were obtained at too late a period to be presented to the Royal Society before the close of the Session; and it is only because the philosophy and the subject itself is a part of that received into the Philosophical Transactions in the preceding paper, that I am allowed, by the President and Council, the privilege of attaching the present paper in the form of an Appendix.

65. The general phenomenon now to be considered is easily produced upon a square plate nipped in the middle, either by the fingers or the pincers (2. 6), held horizontally, covered with sufficient water on the upper surface to flow freely from side to side when inclined, and made to vibrate strongly by a bow applied to one edge, ×, fig. 12, in the usual way. Crispations appear on the surface of the water, first at the centres of vibration, and extend more or less towards the nodal lines, as the vibrations are stronger or weaker. The crispation presents the appearance of small conoidal elevations of equal lateral extent, usually arranged

Fig. 12.



* LIEBER's Hist. of Natural Phenomena for 1813. † Annals of Philosophy, N. S. vi. p. 82.

‡ Wellenlehre, p. 414.

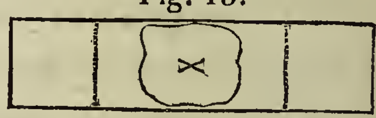
rectangularly with extreme regularity; permanent* (in appearance), so long as a certain degree of vibration is sustained; increasing and diminishing in height, with increased or diminished vibration; but not affected in their lateral extent by such variations, though the whole crispated surface is enlarged or diminished at those times. If the plate be vibrated, so as to produce a different note, the crispations still appear at the centre of vibration, but are smaller for a high note, larger for a low one. The same note produced on different sized plates, by different modes of vibration, appears to produce crispations of the same dimension, other circumstances being the same.

66. These appearances are beautifully seen when ink diluted with its bulk of water is used on the plate.

67. It was necessary, for examination, both to prolong and enlarge the effect, and the following were found advantageous modes of producing it. Plates of crown-glass, from eighteen to twenty-two inches long, and three or four inches wide, were supported each by two triangular pieces of wood acting as bridges (18), and made to vibrate by a small glass rod or tube resting perpendicularly at the middle, over which the moist fingers were passed. By sprinkling dry sand on the plates, and shifting the bridges, the nodal lines were found (usually about one fifth of the whole length from each end), and their places marked by a file or diamond. Then clearing away the sand, putting water or ink upon the plate, and applying the rod or fingers, it was easy to produce the crispations and sustain them undisturbed, and with equal intensity for any length of time.

68. By making a broad mark, or raising a little ledge of bee's wax, or a mixture of bee's wax and turpentine, it was easy to confine the pool of water to the middle part of the plate, fig. 13, where, of course, the crispations were most powerfully produced. Such a barrier is often useful to separate the wet and dry parts of the glass, especially when a violin bow is used as the exciter.

Fig. 13.



69. In other experiments, deal laths, two, three, or four feet long, one inch and a half wide, and three eighths or more of an inch in thickness, were used instead of the glass plates. These could be made to vibrate by the fingers and wet rod (67), and by either shifting the bridges or changing the lath an almost

* WEBER's Wellenlehre, p. 414.

unlimited change of isochronous vibrations, from that producing a high note to those in which not more than five or six occurred in a second, could be obtained. The crispations were formed upon a glass plate attached to the middle of the lath, by two or three little pellets of soft cement*.

70. Obtained in this way the appearances were very beautiful, and the facilities very great. A glass plate, from four to eight inches square, could be covered uniformly with crispations of the utmost regularity; for, by attaching the plate with a little method, and at points equidistant from the centre of the bar, it was easy to make every part travel with the same velocity, and in that respect differ from and surpass the bar which sustained it. The conoidal heaps constituting the crispation could be so enlarged by slowness of vibration, that three or four occupied a linear inch. The glass plate could be removed, and another of different form or substance, and with other fluids, as mercury, &c., substituted in an instant.

71. In using laths, it is necessary to confine the parts bearing upon the bridges, either by slight pressure of the fingers, or by loops of string, or by weights. The exciting glass rod need not necessarily rest upon the middle of the bar or plate, but may be applied with equal effect at some distance from it. Long laths may be made to subdivide in their mode of vibration, according as the rod is applied to different places, and the pressure given by the exciting moist fingers is varied; with each change of this kind an immediate change of the crispation is observed.

72. This form of apparatus was enlarged until a board eighteen feet long was used, the layer of water being now three fourths of an inch in depth and twenty-eight inches by twenty inches in extent. The sides of the cistern were very much inclined, so that the water should gradually diminish in depth, and thus reflected waves be prevented. The vibrations were so slow as to be produced by the direct application of the hand, and the heaps were each from an inch to two inches in extent. Though of this magnitude, they were identical in their nature with those forming crispations on so small a scale as to appear merely like a dullness on the surface of the water.

73. In these experiments the proportion of water requires a general adjustment, the crispations being produced more readily and beautifully when there

* Equal parts of yellow wax and turpentine.

is a certain quantity than when there is less. For small crispations, the water should flow upon the surface freely. Large crispations require more water than small ones. Too much water sometimes interferes with the beauty of the appearance, but the crispation is not incompatible with much fluid, for the depth may amount to eight, ten, or twelve inches (111), and is probably unlimited.

74. These crispations are equally produced upon the under with the upper surface of vibrating plates. When the lower surface is moistened, and the bow applied (65), the drops which hang down by the force of gravity are rippled; but being immediately gathered up as described in the former paper (44), a certain definite layer is produced, which is beautifully rippled or crispated at the centres of vibration.

75. Most fluids, if not all, may be used to produce these crispations, but some with particular advantages; alcohol, oil of turpentine, white of egg*, ink, and milk produce them. White of egg, notwithstanding its viscosity, shows them readily and beautifully. Ink has great advantages, because, from its colour and opacity, the surface form is seen undisturbed by any reflection from the glass beneath; its appearance in sunshine is exceedingly beautiful. When diluted ink is used for large crispations, upon tin plate or over white paper, or mercury, the different degrees of colour or translucency corresponding to different depths of the fluid, give important information relative to the true nature of the phenomena (78. 85. 97). Milk is, for its opacity, of similar advantage, especially when a light is placed beneath, and being more viscid than water is better for large arrangements (72. 98), because it produces less splashing.

76. Oil does not show small crispations readily (120), and was supposed to be incapable of forming them, but when warmed (by which its liquidity is increased) it produces them freely. Cold oil will also produce large crispations, and for very large ones would probably be better than water, because of its cohesion. The difference between oil and white of egg is remarkable; for the latter, from common observation, would appear to be a thicker fluid than oil: but the qualities of cohesion differ in the two, the apparent thickness of white of egg depending upon an elastic power (probably due to an approach to

* WHEATSTONE.

structure), which tends to restore its particles to their first position, and co-existing with great freedom to move through small spaces, whilst that of oil is due to a real difficulty in removing the particles one by another. It is possible that the power of assuming, more or less readily, the crispated state may be a useful and even important indication of the internal constitution of different fluids.

77. With mercury the crispations are formed with great facility, and of extreme beauty, when a piece of amalgamated tin or copper plate being fixed on a lath (69), is flooded with the fluid metal, and then vibrated. A film quickly covers the metal, and then the appearances are not so regular as at first; but on removing the film by a piece of paper, their regularity and beauty are restored. It is more convenient to cover the mercury with a little very dilute acetic or nitric acid; for then the crispations may be produced and maintained for any length of time with a surface of perfect brilliancy.

78. When a layer of ink was put over the mercury, the acid of the ink removed all film, and the summits of the metallic heaps, by diminishing the thickness of the ink over them, became more or less visible, producing the appearance of pearls of equal size beautifully arranged in a black medium. When mercury covered with a film of dilute acid was vibrated in the sunshine, and the light reflected from its surface received on a screen, it formed a very beautiful and regular image; but the screen required to be placed very near to the metal, because of the short focal lengths of the depressions on the mercurial surface.

79. It is sometimes difficult to arrive by inspection at a satisfactory conclusion of the forms and arrangements thus presented, because of multiplied reflection and the particular condition of the whole, which will be described hereafter (95). When observed, well formed with vibrations so slow as to produce three or four elevations in a linear inch (70), they are seen to be conoidal heaps rounded above, and apparently passing into each other below by a curvature in the opposite direction. When arranged regularly, each is surrounded by eight others, so that, a single light being used, nine images may be sent from each elevation to the eye. These are still further complicated, when transparent fluids are used, by reflections from the glass beneath. The use of ink

(75) removes a good deal of the difficulty experienced, and the production of slow, regular, sustained vibrations, more (67. 69).

80. These elevations I will endeavour to distinguish henceforth by the term *heaps*.

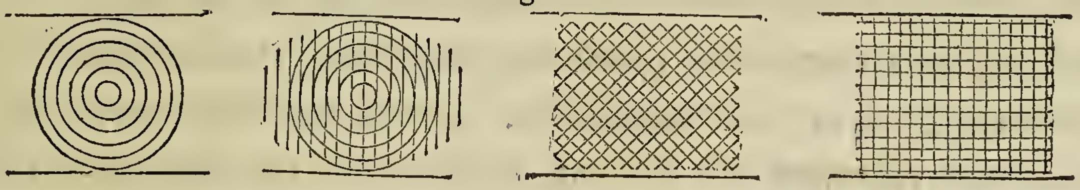
81. The crispation on the long plate of glass described (67) always ultimately assumed a rectangular arrangement, i. e. the heaps were equidistant, and in rows parallel or at right angles to each other. The rows usually form angles of 45° to the sides of the plate at the commencement; but if the vibration be continued, the whole system usually wheels round through 45° until the rows coincide with the edges of the plate.

82. The lateral dimension of the heaps remained constant notwithstanding considerable variations in the force of vibration. But it was soon found that variation in the depth of water affected their number; that with less water the heaps were smaller, and with more water larger, though the sound and therefore the number of vibrations in a given period remained the same. The number of heaps could be reduced to eight or increased to eleven and a half in the three inches by a change in no other condition than the depth of fluid.

83. With the above plate (67. 81) the appearances were usually in the following order, the pool of water being quadrangular or nearly so, and the exciting rod resting in the middle of it. Ring-like linear heaps concentric to the exciting rod first form to the number of six or seven; these may be retained by a moderated state of vibration, and produce intervals which measured across the diameter of the rings are to the number of ten in three inches, with a certain constant depth of water. By increasing the force of vibration the altitude of these elevations increases, but not their lateral dimension, and then linear heaps form across these circles and the plate, and parallel to the bridges, having an evident relation to the manner in which the whole plate vibrates. These, which like all other of these phenomena are strongest at the part most strongly vibrating, soon break up the circles, and are themselves broken up, producing independent heaps, which at first are irregular and changeable, but soon become uniform and produce the quadrangular order; first at angles of 45° to the edges of the plate, but gradually moving round until parallel to them. So the arrangement continues, unless the force be so violent as to break

it up altogether: if the vibratory force be gradually diminished, then the heaps as gradually fall, but without returning through the order in which they were produced. The following lines may serve to indicate the course of the phenomena.

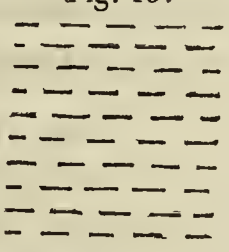
Fig. 14.



When perfectly formed, the heaps are also to the number of ten in three inches with the same depth of water as that which produced the rings. The intervals between the rings and the heaps are the same, other influential circumstances remaining unaltered.

84. Then another form of heaps occasionally occurred, but always passing ultimately into those described. These heaps were grouped in an arrangement still very nearly rectangular, and at angles of 45° to the sides of the plate, but were contracted in one direction, and elongated in the other; these directions being parallel to the sides and ends of the plate. If the marks in fig. 15 be supposed to represent the tops of the heaps, an idea of the whole will be obtained. Three inches along these heaps included eight, but across them it included fifteen nearly. These numbers are therefore the relation of length to breadth. But along the lines of the quadrilateral arrangement three inches included eleven heaps, which, notwithstanding the difference in form, is the same number that was produced by the same plate, with the same depths of water, when the heaps were round; therefore an equal number of heaps existed in the same area in both cases; and the departure from perfect rectangular arrangement, and also the ratio of 1 : 2, is probably due to some slight influence of the sides of the plate.

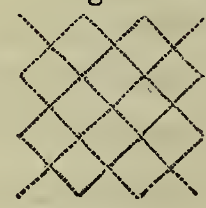
Fig. 15.



85. When mercury covered with a film of very dilute nitric acid is vibrated (77), the rectangular arrangement is constantly obtained. When vibrated under dilute ink (78), it is still more beautifully seen and distinguished. The tin plate sustaining the mercury was square, and when the whole surface was covered with crispations, the lines of the rectangular arrangement were always at angles of 45° to its edges.

86. When sand is sprinkled uniformly over a plate on which large water crispations are produced, i. e. four, five or six in the inch, it gives some very important indications. It immediately becomes arranged under the water, and with a little method may be made to yield very regular forms. It is always removed from under the heaps, passing to the parts between them, and frequently producing therefore the accompanying form, fig. 16, of great regularity. As the sand figure remains when the vibration has ceased, it allows of the determination of position, the measurement of intervals, &c. very conveniently.

Fig. 16.



87. Very often the lines of sand are not continuous, but separated with extreme regularity into portions as represented fig. 17. The portions of these lines were sometimes, with little sand on the plate, very small, fig. 18; and when more sand was present they were thickened occasionally, fig. 19; then assuming the appearance of heaps arranged in straight lines at angles of 45° to the lines regulating the position of the water-heaps which formed them, and just double in number to the latter. At other times the sand instead of being deficient at the intersecting angle would accumulate there only, fig. 20; and at other times would accumulate there principally, but still show the original form by a few connecting particles, fig. 21.

Fig. 17.

Fig. 18.

Fig. 19.

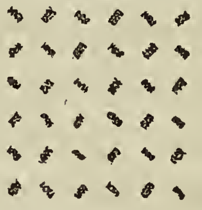


Fig. 20.

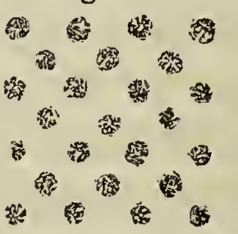
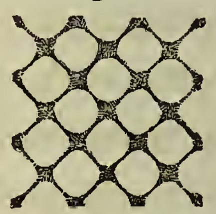
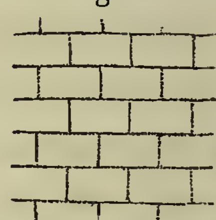


Fig. 21.



88. When the heaps were of the form described (84), the sand was still washed from under them; it did not however assume lines parallel to the rectangular arrangement of the heaps, but was arranged as in fig. 22.

Fig. 22.



89. When only the circular linear heaps (83) were produced, the sand assumed similar circular forms, concentric and alternating with the water elevations.

90. On strewing a little lycopodium over the water for the purpose of gaining information relative to what occurred at the surface during the cris-

pation, it moved about over the fluid in every possible direction, whilst the crispations existed of the utmost steadiness beneath. The same thing occurred with pieces of cork on very large crispations (98). But when much lycopodium was put on, so that the particles retained each other in a steady position, then it formed lines * parallel to the arrangement of the heaps, the powder being displaced from the parts over the heaps, and taking up an arrangement perpendicularly over the sand beneath. As the lycopodium forms float on the water they are easily disturbed, and in no respect approach as to beauty and utility to the forms produced by the sand; but lycopodium may be used with smaller crispations than sand.

91. The crispations are much influenced by various circumstances. They tend to commence at the place of greatest vibration; but if the quantity of fluid is too little there, and more abundant elsewhere, they will often commence at the latter place first. Their final arrangement is also much affected by the form of the plate, or of the pool of water on which they occur. When the plates or pools are rectangular, and all parts vibrate with equal velocity, the lines of heaps are at angles of 45° to the edges. But when semicircular and other plates were used, the arrangement, though quadrangular, was unsteady, often breaking up and starting by pieces into different and changing positions.

92. When mercury was used (77), the film formed on it after a few moments had great power, according to the manner in which it was puckered, of modifying the general arrangement of new crispations.

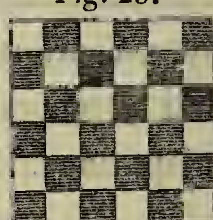
93. When a circular plate, supported by cork feet attached where a single nodal line would occur, was covered with water and vibrated by a rod resting upon the middle, the crispations extended from the middle towards the nodal line; these were sometimes arranged rectangularly, but had no steadiness of position, and changed continually. At other times the heaps appeared as if hexagonal, and were arranged hexagonally, but these also shifted continually. This and many other experiments (83) showed that the direction and nature of the vibration of the plate (i. e. of the lines of equal or varying vibrating force), had a powerful influence over the regularity and final arrangement of the crispations.

* Wheatstone.

94. The beautiful appearance exhibited when the crispations are produced in sunshine, or examined by a strong concentrated artificial light, has been already referred to (78. 79). When the reflected image from any one heap is examined, (for which purpose ink (75) or mercury (77) is very convenient,) it will be found not to be stationary, as would happen if the heap was permanent and at rest, nor yet to form a vertical line, as would occur if the heap were permanent but travelled to and fro with the vibrating plate; but it moves so as to re-enter upon its course, forming an endless figure, like those produced by Dr. Young's piano-forte wires, or Wheatstone's kaleidophone, varying with the position of the light and the observer, but constant for any particular position and velocity of vibration. Upon placing the light and the eye in positions nearly perpendicular to the general surface of the fluid, so as to avoid the direct influence of the motion of vibration, still the luminous, linear, endless figure was produced, extending more or less in different directions, according to the relation of the light and eye to the crispated surface, and occasionally corresponding in its extent one way to the width of the heap, i. e. to the distance between the summit of one heap and its neighbours, but never exceeding it. The figure produced by one heap was accurately repeated by all the heaps when the vibrating force of the plate was equal (70) and the arrangement regular.

95. The view which I had been led to anticipate of the nature of the heaps, from the effects described in the former paper, were, that each heap was a permanent elevation, like the cones of lycopodium powder (53. 58), the fluid rising at the centre, but descending down the inclined sides, the whole system being influenced, regulated, and connected by the cohesive force of the fluid. But these characters of the reflected image, with others of the effects already described, led to the conclusion, that notwithstanding the apparent permanency of the crispated surface, especially when produced on a small scale, as by the usual method, the heaps were not constant, but were raised and destroyed with each vibration of the plate; and also that the heaps did not all exist at once, but (referring to locality) formed two sets of equal number and arrangement, fig. 23, never existing together, but alternating with, and being resolved into each other, and by their rapidity of recurrence giving the appearance of simultaneous and

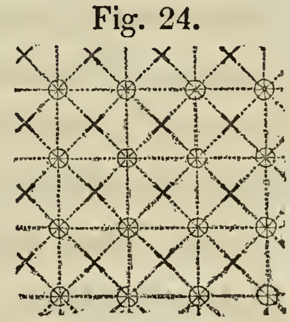
Fig. 23.



even permanent existence. Provided this view were confirmed, it seemed as if it would be easy to explain the production of the heaps, their regular arrangement, &c., and to deduce their recurrence, dimensions, and many other points relative to their condition.

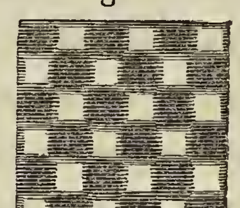
96. On producing a water crispation, having four or five heaps in a linear inch, placing a candle beneath, and a screen of French tracing paper above it, the phenomena were very beautiful, and such as supported the view taken. By placing the screen at different distances, it could be adapted to the focal length due to the curvature at different parts of the surface of fluid, so that by observing the luminous figure produced and its transitions as the screen was moved nearer or further, the general form of the surface could be deduced. Each heap with a certain distance of screen gave a star of light ⊕, fig. 24, which twinkled, i. e. appeared and disappeared alternately, as the heap rose and fell. At the corners × equidistant from these, fainter starred lights appeared; and by putting the screen nearer to or further from the surface, lines of light, in two or even four directions, appeared intersecting the luminous centres and apparently permanent, whilst circumstances remained unchanged. These effects could be magnified to almost any scale (72).

Fig. 24.



97. When heaps of similar magnitude were produced, with diluted ink on glass (75), and white paper or an illuminated screen looked at through them, a chequered appearance was observed. In one position, lines of a certain intensity separated the heaps from each other, but the square places representing the heaps looked generally lighter. In another position, when but little reflected light came from the surface of the heaps, their places could be perceived as dark, from the greater depth of ink there. By care, another position could be found in which the whole surface looked like an alternate arrangement of light and dark chequers, fig. 25, not steady, but with a quivering motion, which further attention could trace as due to a rapid alternation in which the light spaces became dark and the dark light, simultaneously. When, instead of glass, a bright tin plate was used under the diluted ink, the chequered spaces and their alternations could be seen still more beautifully.

Fig. 25.



98. It was in consequence of these effects that very large arrangements were

made (72), giving heaps that were two inches and a half wide each*; and now it was evident, by ordinary inspection, that the heaps were not stationary, but rose and fell; and also that there were two sets regularly and alternately arranged, the one set rising as the other descended.

99. Sand gave no indications of arrangement with these large heaps (86); but when some coarse saw-dust was soaked, so as to sink in water, and then distributed in the fluid, its motions were beautifully illustrative of the whole philosophy of the phenomena. It was immediately washed away from under the rising and falling heaps, and collected in the places equidistant between these spots, as the sand did in the former experiments (86), and by its vibratory motion to and fro, it showed distinctly how the water oscillated from one heap towards another, as the heaps sunk and rose.

100. When milk (75) was used instead of water for these large arrangements in a dark room, and a candle was placed beneath, the appearances also were very beautiful, resembling in character those described (97).

101. Each heap (identified by its locality) recurs or is re-formed in two complete vibrations of the sustaining surface †; but as there are two sets of heaps, a set occurs for each vibration. The maximum and minimum of height for the heaps appears to be alternately, almost immediately after the supporting plate has begun to descend in one complete vibration.

102. Many of these results are beautifully confirmed by the appearances produced, when regular crispations have been sustained for a short time with mercury, on which a certain degree of film has been allowed to form (77). On examining the film afterwards in one light, lines could be seen on it, coinciding with the intervals of the heaps in one direction; in another light, lines coinciding with the other direction came into sight, whilst the first disappeared; and in a third light, both sets of lines could be seen cutting out the square places where the heaps had existed: in these spaces the film was minutely wrinkled and bagged, as if it had there been distended; at the lines it was only a little wrinkled, giving the appearance of texture; and at the crossing

* This estimate is given in accordance with the mode of estimating the former and smaller heaps, as if the heaps were formed simultaneously; but it is evident that if only half the number exist at once, each heap will have twice the width or four times the area of those which can be formed if all exist together.

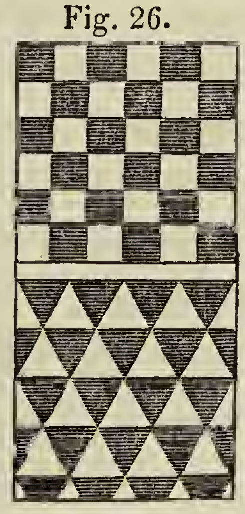
† A vibration is here considered as the motion of the plate, from the time that it leaves its extreme position until it returns to it, and not the time of its return to the intermediate position.

of the lines themselves, it was quite free from mark, and fully distended. All these are natural consequences, if the film be considered as a flexible but inelastic envelope formed over the whole surface whilst the heaps were rising and falling.

103. The mode of action by which these heaps are formed is now very evident, and is analogous in some points to that by which the currents and the involving heaps already described are produced. The plate in rising tends to lift the overlying fluid, and in falling to recede from it; and the force which it is competent to communicate to the fluid can, in consequence of the physical qualities of the latter, be transferred from particle to particle in any direction. The heaps are at their maximum elevation just after the plate begins to recede from them; before it has completed its motion downwards, the pressure of the atmosphere and that part of the force of the plate which through cohesion is communicated to them, has acted, and by the time the plate has begun to return, it meets them endowed with momentum in the opposite direction, in consequence of which they do not rise as a heap, but expand laterally, all the forces in action combining to raise a similar set of heaps, at exactly intermediate distances, which attain their maximum height just after the plate again begins to recede; these therefore undergo a similar process of demolition, being resolved into exact duplicates of the first heaps. Thus the two sets oscillate with each vibration of the plate, and the action is sustained so long as the plate moves with a certain degree of force; much of that force being occupied in sustaining this oscillation of the fluid against the resistance offered by the cohesion of the fluid, the air, the friction on the plate, and other causes.

104. A natural reason now appears for the quadrangular and right-angled arrangement which is assumed, when the crispation is most perfect. The hexagon, the square, and the equilateral triangle are the only regular figures that can fill an area perfectly. The square and triangle are the only figures that can allow of one half alternating symmetrically with the other, in conformity with what takes place between the two reciprocating sets of heaps, fig. 26; and of these two the boundary lines between squares are of shorter extent than those between equilateral triangles of equal area. It is evident therefore that one of these two will be finally assumed, and that that will be the square arrangement; because then the fluid

Fig. 26.



will offer the least resistance in its undulations to the motions of the plate, or will pass most readily to those positions into which the forces it receives from the plate conspire to impel it.

105. All the phenomena observed and described may, as it appears to me, be now comprehended. The fluid may be considered as a pendulum vibrating to and fro under a given impulse; the various circumstances of specific gravity, cohesion, friction, intensity of vibrating force, &c. determining the extent of oscillation, or, what is the same thing, the number of heaps in a given interval. When the number of vibrations in a given time is increased, these heaps are more numerous, because the oscillation, to be more rapid, must occur in a shorter space. The necessity of a certain depth of fluid (73) is evident, and also the reason why, by varying the depth (82), the lateral extent of the heaps is changed. The arrangement of the sand and lycopodium, by the crispations, and the occurrence of the latter at centres of vibration, and only upon surfaces vibrating normally, are all evident consequences. The permanency of the lateral extension of the heaps, when the velocity of the vibrating plate varies, is a very marked effect, and it is probable that the investigation of these phenomena may hereafter importantly facilitate inquiries into the undulations of fluids, their physical qualities, and the transmission of forces through them.

106. As to the origin or determination of crispations, no difficulty can arise; the smallest possible difference in almost any circumstance, at any one part, would, whilst the plate is vibrating, cause an elevation or depression in the fluid there; the smallest atom of dust falling on the surface, or the smallest elevation in the plate, or the smallest particle in the fluid of different specific gravity to the liquid itself, might produce this first effect; this would, by each vibration of the plate, be increased in amount, and also by each vibration extended the breadth of a heap, in at least four directions: so that in less than a second a large surface would be affected, even under the improbable supposition that only one point should at first be disturbed.

107. I have thought it unnecessary to dwell upon the explanation of the circular linear heaps (83. 93. 110) produced on long or circular plates by feeble vibration. They are explicable upon the same principles, account being at the same time taken of the arrangement and proportion of vibrating force in the various parts of the plates.

108. The heaps which constitute crispation (as the word has been used in

this paper) are in form, quality, and motion of their parts, the same with what are called stationary undulations; and if the mercury in a small circular basin be tapped at the middle, stationary undulations, resembling the ring-like heaps (83. 110), will be obtained; or if a rectangular frame be made to beat at equal intervals of time on mercury or water, heaps like those of the crispations, arranged quadrangularly at angles of 45° to the frame, will be produced. These effects are in fact the same with those described, but are produced by a cause differing altogether. The first are the result of two progressing and opposed undulations, the second of four: but the heaps of crispations are produced by the power impressed on the fluid by the vibrating plate; are due to vibrations of that fluid occurring in twice the time of the vibrations of the plate; and have no dependence on progressive undulations, originating laterally, as many of the phenomena described prove. Thus, when the edges were bevelled (72. 110), or covered with cloth, or wet saw-dust, so that waves reaching the side should be destroyed, or when the limits of the water or plates were round (91) or irregular, still the heaps were produced, and their arrangement square. When the round plate (93) was used, regular crispations were still produced, though, as the water extended over the nodal line, and was there perfectly undisturbed, no progressing and opposed undulations could originate to produce them. Vellum stretched over a ring, and rendered concave by the pressure of the exciting rod, produced the same effect.

109. When a plate of tin, rendered very slightly concave, was attached to a lath (69), so as to have equality of vibratory motion in all its parts, and a little dilute alkali (which would wet the surface) put into it, the crispations formed in the middle, but ceased towards the sides, where, though well wetted, there was not depth enough of water, and from whence also no waves could be reflected to produce stationary undulations in the ordinary manner.

110. When a similar arrangement was made with mercury on a concave tin plate, the effects were still more beautiful and convincing. The centre portion was covered with one regular group of quadrangular crispations; at some distance from the centre, and where the mercury was less in depth, these passed into concentric, ring-like heaps, of which there were a great many; and outside of these there was a part wet with mercury, but with too little fluid to give either lines or heaps. Here there could be no reflected waves; or, if that were thought possible, those waves could not have formed both the circular rings and

the square crispation. When this plate was vibrated, the mercury spread in all directions up the side, a natural consequence of the production of powerful oscillations at the middle, which would extend their force laterally, but quite against their being due to the opposition and crossing of waves originating at the sides.

111. A limited depth of fluid is by no means necessary to produce crispations on the surface (73). A circular glass basin about five inches in diameter and four inches deep was attached to a lath (69), filled with water and vibrated, the exciting rod being applied at the side (71). The surface of the water was immediately covered with the most regular crispations, i. e. heaps arranged quadrangularly. On taking out part of the water and filling it up with oil, the oil assumed the same superficies. On putting an inch in depth of mercury under the water, the mercury became crispated. The experiment was finally made with water fourteen inches in depth. Particles at a very moderate depth in the water seemed to have no motion except the general motion of the fluid, and the whole of the lower part of the water may be considered as performing the part of a solid mass upon which the superficial undulating portion reposed. In fact it matters not to the fluid, what is beneath, provided it has sufficient cohesion, is uniform in relation to the surface fluid, and can transmit the vibrations to it in an undisturbed manner *.

112. The beautiful action thus produced at the limits of two immiscible fluids, differing in density or some other circumstances, by which the denser was enabled most readily to accommodate itself to rapid, regular and alternating displacements of its support when that support was horizontal, suggested an inquiry into the probable arrangement of the fluid when the displacements were lateral or even superficial.

Fig. 27.

113. On arranging the long plate (67. 81) vertically, so that the lower extremity dipped about one third of an inch into water, fig. 27, and causing it to vibrate by applying the rod at ×, or by tapping the plate with the finger, undulations of a peculiar character were observed: those passing from the plate towards the sides of the basin were scarcely visible though the plate vibrated strongly, but in place of such appeared others, in the production of which the mechanical force

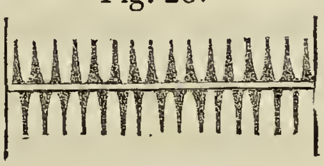
* I have seen the water in a pail placed in a barrow, and that on the head of an upright cask in a brewer's van passing over stones, exhibit these elevations.

of the vibrating plate exerted upon the fluid was principally employed. These were apparently permanent elevations, at regular intervals, strongest at the plate, projecting directly out from it over the surface of the water, like the teeth of a coarse comb gradually diminishing in height, and extending half or three quarters of an inch in length. These varied in commencing at the glass, or having intervening ridges, or in height, or in length, or in number, or in breaking up into violently agitated pimples and drops, &c. according as the plate dipped more or less into the water, or vibrated more or less violently, or subdivided whilst vibrating into parts, or changed in other circumstances. But when the plate (sixteen or seventeen inches long) dipped about one sixth of an inch, then four of these linear heaps occupied as nearly as possible the same space as four heaps formed with the same plate in the former way (83) and accompanied with the same sound.

114. By fixing a wooden lath (69) perpendicularly downwards in a vice, plates of any size or form could be attached to its lower end and immersed more or less in water; and by varying the immersion of the plate, or the length of the lath, or the place against which the exciting rod (71) was applied, the vibrations could be varied in rapidity to any extent.

115. On using a piece of board at the extremity of the lath, eight inches long and three inches deep, with pieces of tin plate four inches by five, fixed on at the ends in a perpendicular position to prevent lateral disturbance at those parts, very regular and beautiful ridges were obtained of any desired width, fig. 28. These ridges, as before, formed only on the wood, and were parallel to the direction of its vibration. They occurred on each side of the vibrating plane with equal regularity, force and magnitude, but seemed to have no connection, for sometimes they corresponded in position, and at other times not; the one set shifting a little, without the others being displaced.

Fig. 28.



116. It could now be observed that the ridges on either side the vibrating plane consisted of two alternating sets; the one set rising as the other fell. For each fro and to motion of the plane, or one complete vibration, one of the sets appeared, so that in two complete vibrations the cycle of changes was complete. Pieces of cork and lycopodium powder showed that there was no important current setting in the direction of the ridges; towards the heads

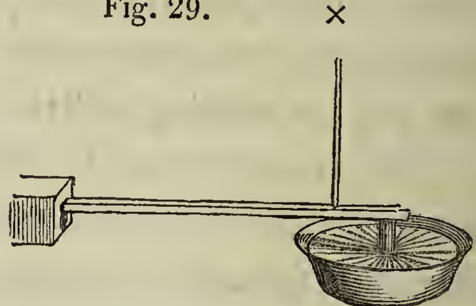
of the ridges pieces of cork oscillated from one ridge towards its neighbour, and back again. The lycopodium sometimes seemed to move on the ridges from the wood, and between them to it; but the motion was irregular, and there was no general current outwards or inwards. There was not so much disturbance as amongst the heaps (90).

117. A very simple arrangement exhibits these ripples beautifully. If an oval or circular pan, fifteen or eighteen inches in diameter, be filled with water, and a piece of lath (69) twelve or fifteen inches long be held in it, edge upwards, so as to bear against the sides of the pan as supporting points, and cut the surface of the water, then on being vibrated horizontally by the glass rod and wet finger, the phenomenon immediately appears with ripples an inch or more in length. When the upper edge of the lath was an inch below the surface, the ripples could be produced. When the vessel had a glass bottom, the luminous figures produced by a light beneath and a screen above, were very beautiful (96). Glass, metal and other plates could thus be easily experimented with.

118. These ripple-like stationary undulations are perfectly analogous as to cause, arrangement and action with the heaps and crispations already explained, i. e. they are the results of that vibrating motion in directions perpendicular to the force applied (105), by which the water can most readily accommodate itself to rapid, regular, and alternating changes in bulk in the immediate neighbourhood of the oscillating parts.

119. From this view of the effect it was evident that similar phenomena would be produced if a substance were made to vibrate in contact with and normally to the surface of a fluid, or indeed in any other direction. A lath was therefore fixed horizontally in a vice by one end, so that the other could vibrate vertically; a cork was cemented to the under surface of the free end, and a basin of water placed beneath with its surface just touching the cork; on vibrating the lath by means of the glass rod and fingers (67), a beautiful and regular star of ridges two, three, or even four inches in length, was formed round the cork, fig. 29. These ridges were more or less numerous according to the number of vibrations, &c. As the water was raised, and more

Fig. 29.



of the cylinder immersed, the ridges diminished in strength, and at last disappeared: when the cylinder of cork just touched the surface, they were most powerfully developed. This is a necessary consequence of the dependence of the ridges upon the portion of water which is vertically displaced and restored at each vibration. When that, being partial in relation to the whole surface, is at or near the surface, the ridges are freely formed in the immediate vicinity; when at a greater depth (being always at the bottom of the cork), the displacement is diffused over a larger mass and surface, each particle moves through less space and with less velocity, and consequently the vibrations must be stronger or the ridges be weaker or disappear altogether. The refraction of a light through this star produces a very beautiful figure on a screen.

120. A heavy tuning-fork vibrating, but not too strongly, if placed with the end of one limb either vertical, inclined, or in any other position, just touching the surface of water, ink, milk, &c. (75), shows the effect very well for a moment. It also shows the ridges on mercury, but the motion and resistance of so dense a body quickly bring the fork to rest. It formed ridges in hot oil, but not in cold oil (76). With cold oil a very inclined fork produced a curious pump-like action, throwing up four streams, easily explained when witnessed, but not so closely connected with the present phenomena as to require more notice here.

121. There is a well known effect of crispation produced when a large glass full of water is made to sound by passing the wet finger round the edges. The glass divides into four vibrating parts opposite to which the crispations are strongest, and there are four nodal points considered in relation to a horizontal section, at equal distances from each other, the finger always touching at one of them. If the vessel is a large glass jar, and soft sounds are produced, the surface of the water exhibits the ridges at the centres of vibration; as the sound is rendered louder, these extend all round the glass, and at last break up at the centres of vibration into irregular crispations, but both the ridges and crispations are effects of the kind already described, and require no further explanation.

122. There are some other effects, one of which I wish here briefly to notice, as connected more or less with the vibratory phenomena that have been described. If, during a strong steady wind, a smooth flat sandy shore, with

enough water on it, either from the receding tide or from the shingles above, to cover it thoroughly, but not to form waves, be observed in a place where the wind is not broken by pits or stones, stationary undulations will be seen over the whole of the wet surface, forming ridges like those already described, and each several inches long. These are not waves of the ordinary kind; they are accurately parallel to the course of the wind; they are of uniform width whatever the extent of surface, varying in width only as the force of the wind and the depth of the stratum of water varies. They may be seen at the windward side of the pools on the sand, but break up so soon as waves appear. If the waves be quelled by putting some oil on the water to windward, these ripples then appear on those parts. They are often seen, but so confused that their nature could not be gathered from such observations, on the pavements, roads, and roofs when sudden gusts of wind occur with rain. The character of these ripples, and their identity with stationary undulations, may be ascertained by exerting the eye and the mind to resolve them into two series of ordinary advancing waves moving directly across the course of the wind in opposite directions. But as such series could not be caused by the wind exerted in a manner similar to that by which ordinary waves are produced, (the direction being entirely opposed to such an idea,) I think the effect is due to the water acquiring an oscillatory condition similar to those described, probably influenced in some way by the elastic nature of the air itself (124) and analogous to the vibration of the strings of the Æolian harp, or even to the vibration of the columns of air in the organ-pipe and other instruments with embouchures.

These ridges were strong enough to arrange the sand beneath where ordinary waves had not been powerful enough to give form to the surface.

123. All the phenomena as yet described are such as take place at the *surfaces* of those fluids in common language considered as inelastic, and in which the elasticity they possess performs no necessary part; nor is it possible that they could be produced within their mass. But on extending the reasoning, it does not seem at all improbable that analogous effects should take place in gases and vapour, their elasticity supplying that condition necessary for vibration which in liquids is found in an abrupt termination of the mass by an unconfined surface.

124. If this be so, then a plate vibrating in the atmosphere may have the air immediately in contact with it separated into numerous portions, forming two alternating sets like the heaps described (95); the one denser, and the other rarer than the ordinary atmosphere; these sets alternating with each other by their alternate expansion and condensation with each vibration of the plate.

125. With the hope of discovering some effect of this kind, a flat circular tin plate had a raised edge of tin three quarters of an inch high fixed on all round, and the plate was then attached to a lath (69), a little lycopodium put on to it, and vibrated powerfully, so that the powder should form a mere cloud in the air, which, in consequence of the raised edge and the equal velocity (70) of all parts of the plate, had no tendency to collect. Immediately it was seen that in place of a uniform cloud it had a misty honeycomb appearance, the whole being in a quivering condition; and on exerting the attention to perceive waves as it were travelling across the cloud in opposite directions, they could be most distinctly traced. This is exactly the appearance that would be produced by a dusty atmosphere lying upon the surface of a plate and divided into a number of alternate portions rapidly expanding and contracting simultaneously.

126. But the spaces were very many times too small to represent the interval through which the air by its elasticity would vibrate laterally once for two vibrations of the plate, in analogy with the phenomena of liquids; and this forms a strong objection to its being an effect of that kind. But it does not seem impossible that the air may have vibrated in subdivisions like a string or a long column of air; and the air itself also being laden with particles of lycopodium would have its motions rendered more sluggish thereby. I have not had time to extend these experiments, but it is probable that a few, well chosen, would decide at once whether these appearances of the particles in the air are due to real lateral vibrations of the atmosphere, or merely to the direct action of the vibrating plate upon the particles.

127. If the atmosphere vibrates laterally in the manner supposed, the effect is probably not limited to the immediate vicinity of the plate, but extends to some distance. The vertical plates intersecting the surface of water and vibrating in a horizontal plane (117) produced ripples proceeding directly out from them five or six inches long; whilst the waves parallel to the vibrating plate

were hardly sensible; and something analogous to this may take place in the atmosphere. If so, it would seem likely that these vibrations occurring conjointly with those producing sound, would have an important influence upon its production and qualities, upon its apparent direction, and many other of its phenomena.

128. Then by analogy these views extend to the undulatory theory of light, and especially to that theory as modified by M. Fresnel. That philosopher, in his profound investigations of the phenomena of light, especially when polarized, has conceived it necessary to admit that the vibrations of the ether take place transversely to the ray of light, or to the direction of the wave causing its phenomena. "In fact we may conceive direct light to be an assemblage, or rather a rapid succession, of an infinity of systems of waves polarized (i. e. vibrating transversely) in all azimuths, and so that there is as much polarized light in any one plane as in a plane perpendicular to it." Herschel says that Fresnel supposes the eye to be affected *only* by such vibrating motions of the ethereal molecules as are performed in planes perpendicular to the direction of the rays. Now the effects in question seem to indicate how the direct vibration of the luminous body may communicate transversal vibration in every azimuth to the molecules of the ether, and so account for that condition of it which is required to explain the phenomena.

129. When the star of ridges formed by a vibrating cylinder (119) upon the surface of water is witnessed instead of the series of circular waves that might be expected, it seems like the instant production of the phenomena of radiation by means of vibratory action. Whether the contiguous rarified and condensed portions which I have supposed in air, gases, vapour and the ether, are arranged radially like the ridges in the experiment just quoted, or whether rare and dense alternate in the direction of the radii as well as laterally, is a question which may perhaps deserve investigation by experiment or calculation.

Royal Institution,
July 30*th*, 1831.

XVIII. *A Table for facilitating the Computations relative to Suspension Bridges.* *By* Davies Gilbert, *Esq. V.P.R.S.*

Read, May 19, 1831.

THE following Table is supplementary to those accompanying the paper "On the Mathematical Theory of Suspension Bridges," printed in the Philosophical Transactions for 1826. It is deduced from the first Table there given, by the plain operations of common arithmetic; but this admits of a far more ready application than the former, to all cases of practical investigation.

The first column contains the deflections or versed sines of the curve, expressed in fractional parts of the double ordinate or Span. It is therefore $2y$ divided by x, and their reciprocals are added under each.

The second column gives the lengths of the chain without alteration from the former Table, except that the double ordinate or span is taken as the unit.

The third column has the tensions of the chain at the middle points or apices of the curve, when the tensions are least; taking the weight of the chain, or that weight augmented by the adjunct weight, or with the adventitious weight also, as unity. The numbers are obtained by dividing a by $2z$.

The fourth column gives the tensions in a similar manner for the extremities of the chain, where they are greatest; and it is made by dividing T by $2z$.

The fifth column gives the angles made by the chains at their extremities with the plane of the horizon, being the complements of those in the former Table.

As all these numbers are immediately derived from an existing Table, there would have been much additional trouble, and without any adequate advantage, in making the denominators of the fractions in the first column or their reciprocals (the decimal fractions), to succeed each other by equal differences. And I have thought it unnecessary to extend the Table further in either direction; since no deflection is likely to be so great as a seventh of the span;

and in the event of a short and light bridge being constructed with a deflection less than one part in forty, the columns may be continued with an accuracy quite sufficient, by taking the direct proportion of the denominators for columns three and four, and the inverse proportion for column five. The numbers in column two may be neglected, as not sensibly differing, in such a case, from unity, or they may be found in the Table from which this has been derived.

Deflections or Versed Sines.	Length of the Chains.	Tensions at the Middle Points.	Tensions at the Extremities.	Angles with the Horizon at the Extremities.
One in 39.97 .02502	1.00166	4.992	5.017	5° 43′
39.17 .02553	1.00173	4.892	4.917	5 50
38.37 .02607	1.00181	4.791	4.817	5 58
37.57 .02662	1.00189	4.691	4.718	6 5
36.76 .02720	1.00196	4.591	4.618	6 13
35.96 .02781	1.00206	4.491	4.519	6 21
35.16 .02844	1.00215	4.391	4.419	6 30
34.36 .02910	1.00225	4.290	4.319	6 39
33.56 .02980	1.00236	4.190	4.220	6 48
32.76 .03053	1.00247	4.090	4.121	6 58
31.96 .03129	1.00260	3.989	4.021	7 9
31.16 .03210	1.00273	3.889	3.921	7 22
30.36 .03294	1.00288	3.789	3.822	7 31
29.56 .03384	1.00304	3.689	3.723	7 43
28.75 .03478	1.00322	3.588	3.623	7 56
27.95 .03577	1.00340	3.488	3.524	8 9
27.15 .03683	1.00360	3.388	3.425	8 24
26.35 .03795	1.00383	3.287	3.325	8 39
25.55 .03914	1.00407	3.187	3.226	8 55
24.75 .04041	1.00434	3.087	3.126	9 12

TABLE (Continued).

Deflections or Versed Sines.	Length of the Chains.	Tensions at the Middle Points.	Tensions at the Extremities.	Angles with the Horizon at the Extremities.
One in 23.95 .04176	1.00463	2.986	3.027	9° 30′
23.14 .04321	1.00496	2.886	2.928	9 50
22.34 .04476	1.00532	2.785	2.830	10 11
21.54 .04642	1.00568	2.685	2.731	10 33
20.74 .04823	1.00617	2.584	2.632	10 57
19.93 .05017	1.00668	2.483	2.533	11 23
19.13 .05227	1.00725	2.383	2.435	11 51
18.33 .05456	1.00789	2.282	2.337	12 22
17.52 .05706	1.00863	2.181	2.238	12 55
16.72 .05980	1.00947	2.080	2.140	13 31
15.92 .06282	1.01045	1.979	2.041	14 11
15.11 .06617	1.01158	1.878	1.943	14 54
14.31 .06989	1.01291	1.777	1.846	15 43
13.50 .07406	1.01448	1.676	1.749	16 28
12.70 .07876	1.01635	1.574	1.652	17 37
11.89 .08411	1.01862	1.473	1.555	18 45
11.08 .09024	1.02139	1.371	1.469	20 2
10.27 .09734	1.02484	1.269	1.363	21 31
9.47 .10563	1.02893	1.166	1.269	23 12
8.65 .11559	1.03474	1.063	1.174	25 11
7.84 .12762	1.04219	0.960	1.083	27 31
7.00 .14280	1.05343	0.854	0.990	30 20

XIX. *An Account of the Construction and Verification of a Copy of the Imperial Standard Yard made for the Royal Society. By Captain* Henry Kater, *F.R.S.*

Read May 19, 1831.

THE Royal Society having done me the honour to request that I would undertake the construction and verification of a copy of the Imperial Standard Yard for their use, it becomes necessary to place upon record the manner in which this was executed, in order that some judgement may be formed of the degree of confidence which may be placed in the result.

The scale in question is constructed in the manner which I have described in the Philosophical Transactions for 1830 * for diminishing the errors arising from the thickness of the bar upon which it has hitherto been customary to trace the divisions. The support of the scale is of brass, forty inches long, $1\frac{3}{4}$ inches wide, and $\frac{6}{10}$ths of an inch in thickness. A brass plate of seven hundredths of an inch thick was made to slide freely upon the support in a dovetail groove formed by two side plates, and was then fixed to the support by a screw passing through its middle.

This plate carries the divisions, which are fine dots upon gold disks let into the brass; the scale is divided into inches, and there is one inch to the left of zero, which is subdivided into tenths. The scale is the work of Mr. Dollond.

As the points designating the Imperial Standard Yard are upon a brass bar one inch in thickness, it was necessary to be extremely careful that the bar during the comparisons should be placed upon a surface as nearly as possible plane; since it has been shown in the paper before alluded to, that a curvature of which the versed sine is only one-hundredth of an inch in a yard would occasion a variation in the length of this standard amounting to nearly five-thousandths of an inch.

* Page 359.

The marble slab formerly used was employed on the present occasion. Its surface was examined by means of a wire, the diameter of which was one-hundredth of an inch, stretched by a bow with a force of about four pounds; but as this wire would suffer a deflexion by its own weight amounting to about four-thousandths of an inch, a wire of two-hundredths of an inch diameter was placed at each extremity of the marble slab, and the wire of the bow resting upon these; the distance of its middle point from the surface of the marble was found to be a little less than two-hundredths of an inch, estimated by passing beneath it a wire of one-hundredth of an inch in diameter. The marble slab being sixty-four inches long, its surface may therefore, perhaps for the extent of a yard, be considered as sufficiently approximating to a plane; and I may here remark that no new adjustment of the slab was found to be necessary, as its position appeared to have undergone no change since my last measurements.

The scale was placed upon the marble slab near the Imperial Standard Yard, and the comparisons were always made about nine o'clock in the morning, in order to ensure as far as possible an equality of temperature in the scale and the Standard Yard. It will be seen that seldom more than three comparisons were taken on the same morning, lest the proximity of the person of the observer might destroy the equality of temperature.

The microscopic apparatus used on the present occasion is that which was employed in the comparison of various British standards of linear measure, an account of which is given in the Philosophical Transactions for 1821, and the mode pursued in making the comparisons was the same as that which I have there detailed. The value of one division of the micrometer is $=.0000428742$ of an inch. As the microscopes invert, an increase in the readings indicates a corresponding deficiency in the length of the scale.

Date. 1831.	Imperial Standard Yard.	From 0 to 36 on the Royal Society's Scale.	Difference.	Difference in Inches.
	microm. readings.	microm. readings.	div.	inches.
April 5	46	62½	16½	.0007074
	48½	61	12½	.0005358
	46	59½	13½	.0005787
	43	59	16	.0006860
P.M.	32	45½	13½	.0005787
6	36	53	17	.0007288
	38½	53	14½	.0006216
	37	54	17	.0007288
	39	52	13	.0005573
7	26½	43	16½	.0007074
	29	44	15	.0006431
	31	44	13	.0005573
8	31	47	16	.0006860
	33½	47	13½	.0005787
	34	46	12	.0005144
9	51	67	16	.0006860
	51	68	17	.0007288
	52	67	15	.0006431
10	43	57	14	.0006002
11	35½	53½	18	.0007717
	36	54½	18½	.0007931
	41	52	11	.0004716
	37	51	14	.0006002
12	30	45	15	.0006431
	30	46	16	.0006860
	35	44½	9½	.0004072
	30	44	14	.0006002
13	39	55	16	.0006860
	37½	51½	14	.0006002
	36	50	14	.0006002
14	17	29	12	.0005144
	16	27½	11½	.0004930
	7	18	11	.0004716
15	86	103½	17½	.0007502
	91	106	15	.0006431
	92½	105	12½	.0005358
		Mean		.0006204

It will be seen in the above Table that the greatest difference between any one of the thirty-six comparisons and the mean is less than two ten-thousandths of an inch; the distance from 0 to 36 on the Royal Society's scale may therefore be considered as equal to 35.99938 inches of the Imperial Standard Yard.

XX. *On the Theory of the Elliptic Transcendents.* By James Ivory, *A.M. F.R.S. Instit. Reg. Sc. Paris, et Soc. Reg. Sc. Gottin., Corresp.*

Read June 9, 1831.

THE branch of the integral calculus which treats of elliptic transcendents originated in the researches of Fagnani, an Italian geometer of eminence. He discovered that two arcs of the periphery of a given ellipse may be determined in many ways, so that their difference shall be equal to an assignable straight line; and he proved that any arc of the lemniscata, like that of a circle, may be multiplied any number of times, or may be subdivided into any number of equal parts, by finite algebraic equations. These are particular results; and it was the discoveries of Euler that enabled geometers to advance to the investigation of the general properties of the elliptic functions. An integral in finite terms deduced by that geometer from an equation between the differentials of two similar transcendent quantities not separately integrable, led immediately to an algebraic equation between the amplitudes of three elliptic functions, of which one is the sum, or the difference, of the other two. This sort of integrals, therefore, could now be added or subtracted in a manner analogous to circular arcs, or logarithms; the amplitude of the sum, or of the difference, being expressed algebraically by means of the amplitudes of the quantities added or subtracted. What Fagnani had accomplished with respect to the arcs of the lemniscata, which are expressed by a particular elliptic integral, Euler extended to all transcendents of the same class. To multiply a function of this kind, or to subdivide it into equal parts, was reduced to solving an algebraic equation. In general, all the properties of the elliptic transcendents, in which the modulus remains unchanged, are deducible from the discoveries of Euler. Landen enlarged our knowledge of this kind of functions, and made a useful addition to analysis, by showing that the arcs of the hyperbola may be reduced, by a proper transformation, to those of the

ellipse. Every part of analysis is indebted to LAGRANGE, who enriched this particular branch with a general method for changing an elliptic function into another having a different modulus, a process which greatly facilitates the numerical calculation of this class of integrals. An elliptic function lies between an arc of the circle on one hand, and a logarithm on the other, approaching indefinitely to the first when the modulus is diminished to zero, and to the second when the modulus is augmented to unit, its other limit. By repeatedly applying the transformation of LAGRANGE, we may compute either a scale of decreasing moduli reducing the integral to a circular arc, or a scale of increasing moduli bringing it continually nearer to a logarithm. The approximation is very elegant and simple, and attains the end proposed with great rapidity.

The discoveries that have been mentioned occurred in the general cultivation of analysis; but LEGENDRE has bestowed much of his attention and study upon this particular branch of the integral calculus. He distributed the elliptic functions in distinct classes, and reduced them to a regular theory. In a Mémoire sur les Transcendantes Elliptiques, published in 1793, and in his Exercices de Calcul Intégral, which appeared in 1817, he has developed many of their properties entirely new; investigated the easiest methods of approximating to their values; computed numerical tables to facilitate their application; and exemplified their use in some interesting problems of geometry and mechanics. In a publication so late as 1825, the author, returning to the same subject, has rendered his theory still more perfect, and made many additions to it which further researches had suggested. In particular we find a new method of making an elliptic function approach as near as we please to a circular arc, or to a logarithm, by a scale of reduction very different from that of which LAGRANGE is the author, the only one before known. This step in advance would unavoidably have conducted to a more extensive theory of this kind of integrals, which, nearly about the same time, was being discovered by the researches of other geometers.

M. ABEL of Christiana, and M. JACOBI of Konigsberg, entirely changed the aspect of this branch of analysis by the extent and importance of their discoveries. The first of these geometers, whom, to the great loss of science, a premature death cut off in the beginning of a career of the highest expectations,

happily conceived the idea of expressing the amplitude of an elliptic function in terms of the function itself. By this procedure the sines and cosines of the amplitudes become periodical quantities like the sines and cosines of circular arcs; and analogy immediately points out many new and useful properties which it would be difficult to deduce by any other mode of investigation. This new way of considering the subject struck out by M. Abel, not only disclosed to him some interesting and original views, but it conducted him to the general and recondite theorems which, without his knowledge, had been previously discovered by the geometer of Konigsberg. M. Jacobi, following in his researches a different method from M. Abel, proved that an elliptic function may be transformed innumerable ways into another similar function to which it bears constantly the same proportion. In the solution of this problem the modulus and the amplitude sought are deduced from the like given quantities, by equations which depend upon the division into an odd number of equal parts of the definite integral, having its amplitude equal to 90°; and, as any odd number may be chosen at pleasure, the number of transformations is unlimited. In consequence of this discovery, an elliptic function can have its modulus augmented or diminished according to an infinite number of different scales. The new process for effecting the same reduction discovered by Legendre in 1825, is only the most simple case of the extensive theorem of M. Jacobi; and, although the older transformation of Lagrange is no part of the same theorem, it bears to it a close resemblance in every respect. Such is the principal addition made to this branch of analysis by M. Jacobi; but the new methods of investigation introduced by him and M. Abel, open a wide field of collateral research, which probably will long continue to furnish matter for exercising the ingenuity of mathematicians.

But it seldom happens that an inventor arrives by the shortest road at the results which he has created, or explains them in the simplest manner. The demonstrations of M. Jacobi require long and complicated calculations; and it can hardly be said that the train of deduction leads naturally to the truths which are proved, or presents all the conclusions which the theory embraces in a connected point of view. The theorem does not comprehend the transformation of Lagrange, which must be separately demonstrated. This is an imperfection of no great moment; but it is always satisfactory to contemplate

a theory in its full extent, and to deduce all the connected truths from the same principles. On a careful examination it will be found that the sines or cosines of the amplitudes used in the transformations are analogous to the sines or cosines of two circular arcs, one of which is a multiple of the other; insomuch that the former quantities are changed into the latter when the modulus is supposed to vanish in the algebraic expressions. We may therefore transfer to the elliptic transcendent the same methods of investigation that succeed in the circle. When this procedure is followed, there is no need to distinguish between an odd and an even number; the demonstrations are shortened; and the difficulties are mostly removed by the close analogy between the two cases. It is in this point of view that the subject is treated in this paper, in which it is proposed to demonstrate the principal theorems without going into the detail of the applications.

1. Elliptic functions of the first kind are of this form *, viz.

$$\int_0^{\phi} \frac{d\,\phi}{\sqrt{1-k^2\sin^2\phi}},$$

$$\int_0^{\psi} \frac{d\,\psi}{\sqrt{1-h^2\sin^2\psi}}:$$

the arcs ϕ and ψ being the amplitudes, and the quantities k and h, which are always less than unit, the moduli of the functions. For the sake of abridging, I shall denote the foregoing integrals by K (ϕ) and H (ψ), the prefixes K and H having reference to the moduli k and h; and, for the definite integral between the amplitudes 0 and $\frac{\pi}{2}$, I shall use indiscriminately either K $\left(\frac{\pi}{2}\right)$ and H $\left(\frac{\pi}{2}\right)$, or, more simply, K and H.

The general equation to be investigated is the following,

$$\int \frac{d\,\psi}{\sqrt{1-h^2\sin^2\psi}} = \beta \int \frac{d\,\phi}{\sqrt{1-k^2\sin^2\phi}}, \tag{1}$$

β being a constant quantity equal to the first ratio of the nascent arcs ψ and ϕ.

* In what follows, the terms 'elliptic functions' and 'elliptic transcendents' are to be understood as applying to those of the first kind only, which alone are treated of.

If we admit that this is a possible equation, and suppose that when ψ is successively equal to the arcs of the series,

$$0, \quad \frac{\pi}{2}, \quad 2\frac{\pi}{2}, \quad 3\frac{\pi}{2}, \text{ \&c.},$$

ϕ attains the respective values,

$$0, \lambda_1, \lambda_2, \lambda_3, \text{ \&c.};$$

we shall have,

$$\mathrm{H} = \beta\, \mathrm{K}(\lambda_1),\ 2\,\mathrm{H} = \beta\, \mathrm{K}(\lambda_2),\ 3\,\mathrm{H} = \beta\, \mathrm{K}(\lambda_3), \text{ \&c.};$$

and consequently,

$$\mathrm{K}(\lambda_2) = 2\,\mathrm{K}(\lambda_1), \quad \mathrm{K}(\lambda_3) = 3\,\mathrm{K}(\lambda_1), \text{ \&c.}$$

Thus the arcs λ_2, λ_3, &c. are the amplitudes of the multiples of the function $\mathrm{K}(\lambda_1)$, which itself remains indeterminate. We may therefore suppose $p \times \mathrm{K}(\lambda_1) = \mathrm{K}\left(\frac{\pi}{2}\right)$, p representing any integer number; and, in consequence, we shall have

$$\mathrm{K}(\lambda_1) = \frac{1}{p}\,\mathrm{K}, \quad \mathrm{K}(\lambda_2) = \frac{2}{p}\,\mathrm{K}, \ \ldots\ \mathrm{K}(\lambda_m) = \frac{m}{p}\,\mathrm{K}.$$

Any proposed number being assumed for p, we may determine the amplitudes λ_1, λ_2, λ_3, &c. by the theory for the multiplication and subdivision of elliptic functions: but as the equations to be solved are complicated and impracticable, the arcs λ_1, λ_2, &c. may be treated as known quantities without any attempt to compute them.

An elliptic function becomes equal to the arc of its amplitude, when the modulus vanishes: and in this case the arcs λ_1, λ_2, λ_3, &c. are obtained by the subdivision of the quadrant of the circle, and are respectively equal to $\frac{1}{p} \cdot \frac{\pi}{2}$, $\frac{2}{p} \cdot \frac{\pi}{2}$, $\frac{3}{p} \cdot \frac{\pi}{2}$, &c.

Having made these observations, we shall for the present dismiss all consideration of the equation to be demonstrated, and turn our attention to investigate two variable arcs ψ and ϕ, such that the first shall have the successive values,

$$0, \quad \frac{\pi}{2}, \quad 2 \times \frac{\pi}{2}, \quad 3 \times \frac{\pi}{2}, \text{ \&c.}$$

when the second becomes respectively equal to the several known amplitudes,

$$0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \&c.$$

2. As we shall have occasion to refer to the formulas for the addition and subtraction of elliptic functions, it will be convenient to premise them.

Let a and b represent any two amplitudes, and put

$$\mathrm{K}(a) + \mathrm{K}(b) = \mathrm{K}(s)$$
$$\mathrm{K}(a) - \mathrm{K}(b) = \mathrm{K}(\sigma):$$

then, according to the formulas of EULER *,

$$\sin s = \frac{\sin a \cos b \sqrt{1 - k^2 \sin^2 b} + \cos a \sin b \sqrt{1 - k^2 \sin^2 a}}{1 - k^2 \sin^2 a \sin^2 b}$$

$$\sin \sigma = \frac{\sin a \cos b \sqrt{1 - k^2 \sin^2 b} - \cos a \sin b \sqrt{1 - k^2 \sin^2 a}}{1 - k^2 \sin^2 a \sin^2 b}.$$

From these we immediately deduce,

$$\sin s \sin \sigma = \frac{\sin^2 a - \sin^2 b}{1 - k^2 \sin^2 a \sin^2 b} \dagger. \qquad \text{(A)}$$

It may be observed that if $a = \lambda_m$, $b = \lambda_n$; then $s = \lambda_{m+n}$, $\sigma = \lambda_{m-n}$: for it is obvious that

$$\mathrm{K}(\lambda_m) + \mathrm{K}(\lambda_n) = (m + n)\,\mathrm{K}(\lambda_1) = \mathrm{K}(\lambda_{m+n})$$
$$\mathrm{K}(\lambda_m) - \mathrm{K}(\lambda_n) = (m - n)\,\mathrm{K}(\lambda_1) = \mathrm{K}(\lambda_{m-n})$$

3. In order to avail ourselves of the analogy between the elliptic functions and the arcs of a circle, we must take that view of the matter first suggested by M. ABEL. Let

$$u = \int_0^\varphi \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \mathrm{K}(\varphi);$$

then, as u is a variable quantity depending upon the amplitude φ, reciprocally this latter quantity will depend upon the first; which dependance we shall express in this manner,

$$\varphi = \text{amplitude of } u = \mathrm{A}\,u,$$
$$\sin \varphi = \sin \mathrm{A}\,u.$$

* Traité des Fonctions Elliptiques, tom. i. p. 22.

† This equation is called by M. ABEL "la propriété fondamentale."

For the sake of abridging, let $\omega = \frac{1}{p} \mathrm{K}$; so that $\lambda_1, \lambda_2, \lambda_3$, &c. will be the respective amplitudes of ω, $2\,\omega$, $3\,\omega$, &c.; and λ_p = amplitude of $p\,\omega$ = amp. of $\mathrm{K} = 90°$; and λ_{2p} = amp. of $2\,p\,\omega$ = amp. of $2\,\mathrm{K} = 180°$. From the nature of the integral, it follows that when u receives an addition equal to $2\,p\,\omega$ or $2\,\mathrm{K}$, the amplitude of u will be increased by 180°.

To the indeterminate quantity u let there be added the several even multiples of ω less than $2\,p\,\omega$; and let us consider the sines of the amplitudes of the functions so formed, viz.

$$\sin \mathrm{A}\,u,\ \sin \mathrm{A}\,(u + 2\,\omega),\ \sin \mathrm{A}\,(u + 4\,\omega),\ \ldots\ \sin \mathrm{A}\,(u + 2\,p\,\omega - 2\,\omega):$$

in this series, if we substitute in place of u, the successive quantities $u + 2\,\omega$, $u + 4\,\omega$, $u + 6\,\omega$, &c., the same sines will constantly recur in periodical order, abstracting from the change of sign when an amplitude becomes greater than 180°, or than a multiple of 180°. Thus, if we put $u + 2\,\omega$ in place of u, the second term of the foregoing series will stand first, and the last term will be $\sin \mathrm{A}\,(u + 2\,p\,\omega) = - \sin \mathrm{A}\,(u)$. In like manner, if $u + 4\,\omega$ be substituted for u, the third term of the series will stand first, and the two last terms will be $- \sin \mathrm{A}\,u$, $- \sin \mathrm{A}\,(u + 2\,\omega)$; and so on.

Let us now put

$$\gamma = \sin \mathrm{A}\,\omega \times \sin \mathrm{A}\,(3\,\omega) \times \sin \mathrm{A}\,(5\,\omega) \ldots \times \sin \mathrm{A}\,(2\,p\,\omega - \omega)$$

or, which is the same thing,

$$\gamma = \sin \lambda_1 \times \sin \lambda_3 \times \sin \lambda_5 \ldots \times \sin \lambda_{2p-1};$$

and further, let us assume,

$$y = \frac{\sin \mathrm{A}\,u \times \sin \mathrm{A}\,(u + 2\,\omega) \times \sin \mathrm{A}\,(u + 4\,\omega) \ldots \times \sin \mathrm{A}\,(u + 2\,p\,\omega - 2\,\omega)}{\gamma}. \quad \text{(B)}$$

In this expression, if we substitute for u, the several odd multiples of ω in succession, viz.

ω, $3\,\omega$, $5\,\omega$, $7\,\omega$, &c.

it follows, from what has been said, that the products in the numerator will always be the same, and equal to the denominator, but that their signs will change alternately as the successive quantities are substituted. Thus, when

any odd multiple $(2n+1)\omega$ is substituted for u in the expression (B), the value of y is always $+1$ or -1, according as $(2n+1)\omega$ holds an odd or an even rank in the series of the odd multiples of ω.

On the other hand, when u is zero, or equal to $2n\omega$ any even multiple of ω, we shall have $y=0$, one of the factors of the numerator necessarily vanishing; for in a sequence of the even multiples of ω, of which the number is p, there must be one equal to $2p\omega$, or to a multiple of $2p\omega$; and therefore when $u=2n\omega$, one of the factors must be the sine of an amplitude equal to 180° or to a multiple of 180°.

Further, let $\omega - z$ be substituted for u in the expression (B), z being less than ω; then,

$$y = \frac{\sin A(\omega - z)\sin A(3\omega - z)\ldots\ldots\sin A(2p\omega - \omega - z)}{\gamma}.$$

Now, in the numerator, the partial products, of the first and last factors, of the second and last but one, and so on, are as follows:

$$\sin A(\omega - z)\sin A(2p\omega - \omega - z) = \sin A(\omega - z)\sin A(\omega + z),$$

$$\sin A(3\omega - z)\sin A(2p\omega - 3\omega - z) = \sin A(3\omega - z)\sin A(3\omega + z), \text{ \&c.}$$

to which we must add the single factor $\sin A(p\omega - z)$, when p is an odd number. All the partial products, it will be observed, have the same value whether z be positive or negative; and they are all greatest, when $z=0$, as will readily appear from what is proved in § 2. Wherefore y has the same value and the same sign, when u is at equal distances from the limits 0 and 2ω; and it attains its greatest magnitude, equal to 1, when $u=\omega$. And, if we substitute $(2n+1)\omega - z$ for u, this substitution will not change the foregoing factors, but only their order, and the sign of their product, which sign, while u is contained between the limits $2n\omega$ and $2n\omega + 2\omega$, will be $+$ or $-$, according as $(2n+1)\omega$ holds an odd or an even rank in the series of the odd multiples of ω.

We may now conclude, from what has been proved, that y, in the expression (B), represents the sine of an arc ψ, which increases from zero with the elliptic function u, and coincides with the successive terms of the series,

$$0,\ \frac{\pi}{2},\ 2\frac{\pi}{2},\ 3\frac{\pi}{2},\ \text{\&c. ad infinitum},$$

at the same time that u attains the values,

$$0, \ \omega, \ 2\,\omega, \ 3\,\omega, \ \text{\&c. ad infinitum,}$$

or, when the amplitude of u becomes equal to the several known arcs,

$$0, \ \lambda_1, \ \lambda_2, \ \lambda_3, \ \text{\&c. ad infinitum:}$$

and further, that there is but one value of y, or of $\sin \psi$, between the two consecutive terms $m \times \frac{\pi}{2}$ and $(m+1) \times \frac{\pi}{2}$, for any given value of u between the limits $m\,\omega$ and $(m+1)\,\omega$, or for any given amplitude between the arcs λ_m and λ_{m+1}.

4. In what has been proved, p may be either an odd or an even number; but we must now distinguish between the two cases, in like manner as it is necessary to do when we investigate the sine of a multiple of a circular arc. Representing the amplitude of u by ϕ, we shall have, $u = \mathrm{K}(\phi)$, and $\sin \phi = \sin \mathrm{A}\, u$. When p is odd, there will be an even number of factors after the first in the numerator of the expression of y or $\sin \psi$; and any one of these, as $\sin \mathrm{A}\,(u + 2\,n\,\omega)$, will have another, namely, $\sin \mathrm{A}\,(u + 2\,p\,\omega - 2\,n\,\omega) = \sin \mathrm{A}\,(2\,n\,\omega - u)$, answering to it; and the product of this pair of factors, viz. $\sin \mathrm{A}\,(u + 2\,n\,\omega) \times \sin \mathrm{A}\,(2\,n\,\omega - u)$, will be found by the formula (A) of § 2, observing that $\sin a = \sin \mathrm{A}\,(2\,n\,\omega) = \sin \lambda_{2n}$, $\sin b = \sin \mathrm{A}\, u = \sin \phi$, $\sin s = \sin \mathrm{A}\,(2\,n\,\omega + u)$, $\sin \sigma = \sin (2\,n\,\omega - u)$:

thus we have,

$$\sin \mathrm{A}\,(u + 2\,n\,\omega)\, \sin \mathrm{A}\,(u + 2\,p\,\omega - 2\,n\,\omega) = \frac{\sin^2 \lambda_{2n} - \sin^2 \phi}{1 - k^2 \sin^2 \lambda_{2n} \sin^2 \phi}.$$

Wherefore, by taking in all the factors and writing z for $\sin \phi$, we shall obtain,

$$\sin \psi = \frac{z}{\gamma} \cdot \frac{\sin^2 \lambda_2 - z^2}{1 - k^2 z^2 \sin^2 \lambda_2} \cdot \frac{\sin^2 \lambda_4 - z^2}{1 - k^2 z^2 \sin^2 \lambda_4} \cdots \frac{\sin^2 \lambda_{p-1} - z^2}{1 - k^2 z^2 \sin^2 \lambda_{p-1}}.$$

The expression of γ, viz.

$$\gamma = \sin \lambda_1 \,.\, \sin \lambda_3 \,.\, \sin \lambda_5 \ldots \sin \lambda_{2p-1},$$

may be written in this form,

$$\gamma = \sin^2 \lambda_1 \,.\, \sin^2 \lambda_3 \,.\, \sin^2 \lambda_5 \ldots \sin^2 \lambda_{p-1},$$

omitting the factor $\sin \lambda_p = 1$: wherefore, if we assume,

$$\beta = \frac{\sin^2\lambda_2 \,.\, \sin^2\lambda_4 \,.\, \sin^2\lambda_6 \ldots \sin^2\lambda_{p-1}}{\sin^2\lambda_1 \,.\, \sin^2\lambda_3 \,.\, \sin^2\lambda_5 \ldots \sin\lambda_{p-2}},$$

we shall have,

p being an odd number,

$$\sin\psi = \beta z \,.\, \frac{1 - \frac{z^2}{\sin^2\lambda_2}}{1 - k^2 z^2 \sin^2\lambda_2} \,.\, \frac{1 - \frac{z^2}{\sin^2\lambda_4}}{1 - k^2 z^2 \sin^2\lambda_4} \ldots \frac{1 - \frac{z^2}{\sin^2\lambda_{p-1}}}{1 - k^2 z^2 \sin^2\lambda_{p-1}}. \tag{2}$$

When p is an even number, if we leave out the first factor in the numerator of the expression of y or $\sin\psi$, there will remain an odd number of factors, that which occupies the middle place, being $\sin A\,(u + p\,\omega)$: and any factor, as $\sin A\,(u + 2\,n\,\omega)$, between the first and the middle one, will have another, viz. $\sin A\,(u + 2\,p\,\omega - 2\,n\,\omega)$, corresponding to it after the middle one; and the product of this pair of factors will be obtained as before, viz.

$$\sin A\,(u + 2\,n\,\omega)\sin A\,(u + 2\,p\,\omega - 2\,n\,\omega) = \frac{\sin^2\lambda_{2n} - \sin^2\phi}{1 - k^2\sin^2\lambda_{2n}\sin^2\phi}.$$

With regard to the middle factor, we shall have, in the formulas of § 2, $\sin a = \sin A\,(p\,\omega) = \sin 90°$, $\sin b = \sin A\,u = \sin\phi$, $\sin s = \sin A\,(u + p\,\omega)$; and

$$\sin A\,(u + p\,\omega) = \frac{\cos\phi}{\sqrt{1 - k^2\sin^2\phi}}$$

Wherefore, by proceeding as before, we shall have,

p being an even number,

$$\sin\psi = \frac{\beta z\sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \,.\, \frac{1 - \frac{z^2}{\sin^2\lambda_2}}{1 - k^2 z^2 \sin^2\lambda_2} \,.\, \frac{1 - \frac{z^2}{\sin^2\lambda_4}}{1 - k^2 z^2 \sin^2\lambda_4} \ldots \frac{1 - \frac{z^2}{\sin^2\lambda_{p-2}}}{1 - k^2 z^2 \sin^2\lambda_{p-2}} \tag{3}$$

$$\beta = \frac{\sin^2\lambda_2 \,.\, \sin^2\lambda_4 \,.\, \sin^2\lambda_6 \ldots \sin^2\lambda_{p-2}}{\sin^2\lambda_1 \,.\, \sin^2\lambda_3 \,.\, \sin^2\lambda_5 \ldots \sin^2\lambda_{p-1}}.$$

In both the formulas (2) and (3), it is obvious that β is the quotient of the product of the sines of all the even amplitudes, λ_2, λ_4, &c. between the limits 0 and 180°, divided by the product of the sines of all the odd amplitudes, λ_1, λ_3, &c. contained between the same limits. The general expression of β, common to the two cases, is therefore as follows,

$$\beta = \frac{\sin\lambda_2 \,.\, \sin\lambda_4 \,.\, \sin\lambda_6 \ldots \sin\lambda_{2p-2}}{\sin\lambda_1 \,.\, \sin\lambda_3 \,.\, \sin\lambda_5 \ldots \sin\lambda_{2p-1}}. \tag{4}$$

In the formula (2) let P and R stand for the products of the binomials in the numerator and denominator; then,

$$\sin\psi = \frac{\beta z \mathrm{P}}{\mathrm{R}};$$

and,

$$\cos^2\psi = \frac{\mathrm{R}^2 - \beta^2 z^2 \mathrm{P}^2}{\mathrm{R}^2}.$$

The numerator of this expression is a rational function of z^2, and it will vanish whenever $\cos^2\psi = 0$, or $\sin^2\psi = 1$, that is, when z^2 is equal to $\sin^2\lambda_{2n+1}$, $2n+1$ being any odd number less than $2p$. Suppose that $2n+1$ is any odd number less than p, the numerator of the value of $\cos^2\psi$ will be divisible by $\left(1 - \frac{z^2}{\sin^2\lambda_{2n+1}}\right)$, and also by $\left(1 - \frac{z^2}{\sin^2\lambda_{2p-2n-1}}\right)$; and as these binomials are equal, it will be divisible by their product $\left(1 - \frac{z^2}{\sin^2\lambda_{2n+1}}\right)^2$; and, p being itself an odd number, to the double divisors there must be added the single one $\left(1 - \frac{z^2}{\sin^2\lambda_p}\right) = 1 - z^2$. The numerator is therefore divisible by the product,

$$(1 - z^2) \cdot \left(1 - \frac{z^2}{\sin^2\lambda_1}\right)^2 \cdot \left(1 - \frac{z^2}{\sin^2\lambda_3}\right)^2 \cdots \left(1 - \frac{z^2}{\sin^2\lambda_{p-2}}\right)^2:$$

and, as the two expressions have the same absolute term and the same dimensions, they must be identical. Wherefore we have,

p being an odd number,

$$\cos\psi = \sqrt{1 - z^2} \cdot \frac{1 - \frac{z^2}{\sin^2\lambda_1}}{1 - k^2 z^2 \sin^2\lambda_2} \cdot \frac{1 - \frac{z^2}{\sin^2\lambda_3}}{1 - k^2 z^2 \sin^2\lambda_4} \cdots \frac{1 - \frac{z^2}{\sin^2\lambda_{p-2}}}{1 - k^2 z^2 \sin^2\lambda_{p-1}}. \quad (5)$$

In like manner, if P and R represent the rational binomial products in the numerator and denominator of the formula (3), we shall have

$$\sin\psi = \frac{\beta z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \times \frac{\mathrm{P}}{\mathrm{R}};$$

and

$$\cos^2\psi = \frac{(1 - k^2 z^2)\,\mathrm{R}^2 - \beta^2 z^2 (1 - z^2)\,\mathrm{P}^2}{(1 - k^2 z^2)\,\mathrm{R}^2}.$$

Proceeding as before, it will appear that the numerator of this expression is

divisible by the double divisor $\left(1 - \frac{z^2}{\sin^2 \lambda_{2n+1}}\right)^2$, $2n + 1$ being any odd number less than p; and in this case when p is an even number, all the divisors are double. Wherefore the product

$$\left(1 - \frac{z^2}{\sin^2 \lambda_1}\right)^2 \cdot \left(1 - \frac{z^2}{\sin^2 \lambda_3}\right)^2 \cdot \left(1 - \frac{z^2}{\sin^2 \lambda_5}\right)^2 \cdots \left(1 - \frac{z^2}{\sin^2 \lambda_{p-1}}\right)^2$$

will divide the numerator of the value of $\cos^2 \psi$; and it will be identical to it, because both the expressions have the same dimensions. Thus we obtain,

p being an even number,

$$\cos \psi = \frac{1 - \frac{z^2}{\sin^2 \lambda_1}}{\sqrt{1 - k^2 z^2}} \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_3}}{1 - k^2 z^2 \sin^2 \lambda_2} \cdots\cdots \frac{1 - \frac{z^2}{\sin^2 \lambda_{p-1}}}{1 - k^2 z^2 \sin^2 \lambda_{p-2}}. \tag{6}$$

From the equations (2) and (5) we deduce,

$$\tan \psi = \frac{\beta z}{\sqrt{1 - z^2}} \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_2}}{1 - \frac{z^2}{\sin^2 \lambda_1}} \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_4}}{1 - \frac{z^2}{\sin^2 \lambda_3}} \cdots \frac{1 - \frac{z^2}{\sin^2 \lambda_{p-1}}}{1 - \frac{z^2}{\sin^2 \lambda_{p-2}}}$$

but it will readily appear that

$$\frac{1 - \frac{\sin^2 \phi}{\sin^2 \lambda_{2n}}}{1 - \frac{\sin^2 \phi}{\sin^2 \lambda_{2n+1}}} = \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{2n}}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{2n+1}}}:$$

wherefore we obtain,

p being an odd number,

$$\tan \psi = \beta \tan \phi \times \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_2}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_1}} \cdot \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_4}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_3}} \cdots \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{p-1}}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{p-2}}}. \tag{7}$$

And in a similar manner we deduce from the equations (3) and (6),

p being an even number,

$$\tan \psi = \frac{\beta \tan \phi}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_1}} \cdot \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_2}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_3}} \cdot \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_4}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_5}} \cdots \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{p-2}}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{p-1}}}. \tag{8}$$

The formulas (2), (5), (7), in which p is an odd number, are those used in

the theorems of M. JACOBI; the other three, (3), (6), and (8), have been added here. All the formulas will be true in the circle, if we make $k = 0$, and derive the arcs λ_1, λ_2, &c. from the subdivision of the quadrant, in like manner as they have been obtained from the subdivision of the definite integral $k\left(\frac{\pi}{2}\right)$. The coefficient β is the expression of the first ratio of the nascent arcs ψ and ϕ; and it is equal to p in the circle.

All the formulas are, however, imperfect in one respect: they all suppose that the amplitudes λ_1, λ_2, &c., derived from the subdivision of the definite integral $k\left(\frac{\pi}{2}\right)$, are known. By means of these amplitudes, the general solution of the problem has been deduced from a particular case: but the formulas cannot be considered as complete till all the coefficients have been expressed in functions of the modulus k; and, with respect to this point, the researches of analysts have not yet been entirely successful.

5. Having now investigated the relation between the arcs ψ and ϕ, we have next to demonstrate that the equation (1) is true when these amplitudes are substituted in it, and a proper value is assigned to the indeterminate modulus h; but this requires some preparation, in order to avoid complicated operations.

First, p being an odd number, we have,

$$\sin\psi = \frac{\beta \,.\, z\mathrm{P}}{\mathrm{R}}, \quad \cos\psi = \frac{\sqrt{1-z^2}\,.\,\mathrm{Q}}{\mathrm{R}};$$

R, P, Q, representing the rational binomial products in the denominators and numerators of the equations (2) and (5): we therefore obtain,

$$\mathrm{R}^2 = \beta^2 z^2 \mathrm{P}^2 + (1 - z^2)\,\mathrm{Q}^2.$$

This equation has been found on the supposition that z is less than 1; but, as it contains no radical quantities, it will be true for all values of z. We may therefore substitute $\frac{1}{kz}$ for z; and, in the resulting equation, the symbol z will still represent a quantity unrestricted in its value. Now, the substitution of $\frac{1}{kz}$ for z being made, we shall obtain,

$$\mathrm{R}^2 - \beta^2 h^2 z^2 \mathrm{P}^2 = (1 - k^2 z^2)\,\mathrm{R}'^2,$$

in which expression R and P denote the same functions of z as before, and the values of the new symbols h and R' are as follows,

$$h = k^p \sin^4 \lambda_1 \, . \sin^4 \lambda_3 \, . \sin^4 \lambda_5 \, . \, . \, . \sin^4 \lambda_{p-2},$$

$$R' = (1 - k^2 z^2 \sin^2 \lambda_1)(1 - k^2 z^2 \sin^2 \lambda_3) \, . \, . \, . \, (1 - k^2 z^2 \sin^2 \lambda_{p-2}).$$

We thus have

$$\left.\begin{aligned} \sqrt{R^2 - \beta^2 z^2 P^2} &= \sqrt{1 - z^2} \, . \, Q \\ \sqrt{R^2 - \beta^2 h^2 z^2 P^2} &= \sqrt{1 - k^2 z^2} \, . \, R'. \end{aligned}\right\} \qquad (C)$$

Secondly, when p is an even number, R, P, Q will stand for the rational binomial products in the denominators and numerators of the equations (3) and (6): thus

$$\sin \psi = \frac{\beta z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \cdot \frac{P}{R}, \; \cos \psi = \frac{1}{\sqrt{1 - k^2 z^2}} \cdot \frac{Q}{R}:$$

consequently

$$(1 - k^2 z^2) R^2 = \beta^2 z^2 (1 - z^2) P^2 + Q^2.$$

And if in this equation we substitute $\frac{1}{kz}$ in place of z, we shall obtain this result,

$$(1 - k^2 z^2) R^2 = \beta^2 z^2 (1 - z^2) P^2 + R'^2$$

R and P representing the same functions of z as before, and the new symbols h and R' standing for these values,

$$h = k^p \, . \sin^4 \lambda_1 \, . \sin^4 \lambda_3 \, . \sin^4 \lambda_5 \, . \, . \, . \sin^4 \lambda_{p-1}$$

$$R' = (1 - k^2 z^2 \sin^2 \lambda_1)(1 - k^2 z^2 \sin^2 \lambda_3) \, . \, . \, . \, (1 - k^2 z^2 \sin^2 \lambda_{p-1}.$$

From what has been proved we now have

$$\left.\begin{aligned} \sqrt{(1 - k^2 z^2) R^2 - \beta^2 z^2 (1 - z^2) P^2} &= Q, \\ \sqrt{(1 - k^2 z^2) R^2 - \beta^2 h^2 z^2 (1 - z^2) P^2} &= R' \end{aligned}\right\} \qquad (D)$$

To these formulas we must add the following principle of analysis, on which the demonstration we have in view mainly turns. Let V and U denote rational functions of z: we shall have this identical equation,

$$\frac{d.\left(\frac{V}{U}\right)}{dz} U^2 = \frac{1}{a}\left\{ (V + aU)\frac{d.(V - aU)}{dz} - (V - aU)\frac{d.(V + aU)}{dz} \right\}:$$

from which it follows, that every double binomial factor either of $V + aU$, or of $V - aU$, is a simple binomial factor of $\frac{d.\left(\frac{V}{U}\right)}{dz} U^2$; and further, if $V + aU$ and $V - aU$ have no common divisor, that every double binomial factor of $(V + aU) \times (V - aU) = V^2 - a^2 U^2$, is a simple binomial factor of $\frac{d.\left(\frac{V}{U}\right)}{dz} . U^2$.

6. The differential of the equation (1) may now be readily demonstrated, supposing that β has the value investigated in § 4, and h, the value assigned to it in § 5. And first when p is an odd number, we obtain from the equation (2),

$$\sin \psi = \frac{\beta . z P}{R}, \quad z = \sin \phi :$$

and with these values the equation (1) will become

$$\frac{\frac{R^2}{dz} d.\left(\frac{zP}{R}\right)}{\sqrt{(R^2 - \beta^2 z^2 P^2)(R^2 - \beta^2 h^2 z^2 P^2)}} = \frac{1}{\sqrt{1 - z^2 . 1 - k^2 z^2}}:$$

and, on account of the formulas (C),

$$\frac{\frac{R^2}{dz} d.\left(\frac{zP}{R}\right)}{Q . R'} = 1.$$

Now it is evident that $R + \beta . zP$ and $R - \beta . zP$, have no common divisor: for, as R contains only the even powers of z, and zP only the odd powers, if $1 + cz$ be a factor of $R + \beta . zP$, $1 - cz$ will necessarily be a factor of $R - \beta . zP$. Wherefore, according to what has been proved above, every double binomial factor of $R^2 - \beta^2 z^2 P^2$, that is, every factor of Q, will be a factor of the function in the numerator of the left side of the last equation. In the very same manner it is proved that every double binomial factor of $R^2 - \beta^2 h^2 z^2 P^2$, that is, every factor of R', will be a factor of the same function.

Wherefore the numerator of the left side of the last equation is divisible by the product $Q \times R'$ in the denominator; and, as both the expressions have the same dimensions and the same absolute term, they are identical; which verifies the equation. Wherefore the equation (1) is demonstrated when p is an odd number.

Secondly, when p is an even number, we have by equation (3),

$$\sin \psi = \frac{\beta z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \cdot \frac{P}{R}, \sin \varphi = z:$$

and the differential of equation (1) will become by substitution,

$$\frac{\frac{(1 - k^2 z^2) R^2}{dz} \cdot d \cdot \left(\frac{z \sqrt{1 - z^2} \cdot P}{\sqrt{1 - k^2 z^2} \cdot R}\right)}{\sqrt{\left((1 - k^2 z^2) R^2 - \beta^2 z^2 (1 - z^2) P^2\right)\left((1 - k^2 z^2) R^2 - \beta^2 k^2 z^2 (1 - z^2) P^2\right)}}$$

$$= \frac{1}{\sqrt{1 - z^2 \cdot 1 - k^2 z^2}}: \text{ and, on account of the formulas (D),}$$

$$\frac{\frac{(1 - k^2 z^2) R^2}{dz} d \cdot \left(\frac{z \sqrt{1 - z^2} \cdot P}{\sqrt{1 - k^2 z^2} \cdot R}\right)}{Q \cdot R'} = \frac{1}{\sqrt{1 - z^2 \cdot 1 - k^2 z^2}}.$$

It will be proved, by the like reasoning as before, that the numerator of the left side of this equation is divisible by the product in the denominator. Now if we perform the differentiation indicated, we shall find,

$$S = (1 - 2 z^2 + k^2 z^4) P R + z (1 - z^2)(1 - k^2 z^2) R^2 \frac{d\left(\frac{P}{R}\right)}{dz},$$

$$\frac{(1 - k^2 z^2) R^2}{dz} d \cdot \left(\frac{z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \cdot \frac{P}{R}\right) = \frac{S}{\sqrt{1 - z^2 \cdot 1 - k^2 z^2}}:$$

and it is evident that all the rational factors of the left side of this last formula, and consequently all the factors of $Q \times R'$, will be factors of S. By substitution the differential equation (1) will now become

$$\frac{S}{Q \times R'} = 1,$$

which is manifestly verified: for, as $Q \times R'$ divides S, and the two expressions

have the same dimensions and the same absolute term, they are identical. The equation (1) is therefore demonstrated when p is an even number.

7. The transformation expressed by the equation,

$$\int_0^{\psi} \frac{d\psi}{\sqrt{1 - h^2 \sin^2 \psi}} = \beta \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

has now been demonstrated for any number whether odd or even, the constant β being determined by the formula (4), and the modulus h by the special formulas in § 5, or, generally without distinguishing whether p is odd or even, by this formula,

$$h = k^p . (\sin \lambda_1 \sin \lambda_3 \sin \lambda_5 \ldots \sin \lambda_{2p-1})^2, \qquad (9)$$

the sines multiplied together being those of all the odd amplitudes less than 180°. The relation between the variable amplitudes ψ and φ is expressed by the several equations in § 4.

In order to render the solution of the problem more complete, it may be proper to add a useful method of computing the amplitude ψ.

In § 5 we have obtained this equation,

$$R^2 = \beta^2 z^2 P^2 + (1 - z^2) Q^2.$$

And, if we represent by N and M the products of the binomials in the numerator and denominator of the equation (7), we shall have

$$\tan \psi = \frac{\beta \tan \varphi \, N}{M}.$$

Let $x = \tan \varphi$, then $z^2 = \sin^2 \varphi = \frac{x^2}{1 + x^2}$; and, observing that

$$1 - \frac{z^2}{\sin^2 \lambda_n} = \frac{1 - \frac{x^2}{\tan^2 \lambda_n}}{(1 + x^2)},$$

it will readily appear that

$$\beta z P = \frac{\beta x N}{(1 + x^2)^{\frac{p}{2}}}, \quad \sqrt{1 - z^2} . Q = \frac{M}{(1 + x^2)^{\frac{p}{2}}}.$$

These values being substituted in the foregoing equation, we get

$$M^2 + \beta^2 x^2 N^2 = (1 + x^2)^p . R^2 :$$

And if R be transformed into a function of x^2, we shall obtain

$$M^2 + \beta^2 x^2 N^2 = (1 + x^2)(1 + c_2^2 x^2)^2 (1 + c_4^2 x^2)^2 \ldots (1 + c_{p-2}^2 x^2)^2,$$

the new symbol c_{2n} being determined by this formula,

$$\overline{c_{2n}}|^2 = 1 - k^2 \sin^2 \lambda_{2n}.$$

The last equation may be resolved into these two,

$$M + \beta x N \sqrt{-1} = (1 + x\sqrt{-1})(1 + c_2 x \sqrt{-1})^2 \ldots (1 + c_{p-2} x \sqrt{-1})^2,$$

$$M - \beta x N \sqrt{-1} = (1 - x\sqrt{-1})(1 - c_2 x \sqrt{-1})^2 \ldots (1 - c_{p-2} x \sqrt{-1})^2:$$

the second of which being divided by the first, there will result,

$$\frac{1 - \tan\psi\sqrt{-1}}{1 + \tan\psi\sqrt{-1}} = \frac{1 - x\sqrt{-1}}{1 + x\sqrt{-1}} \cdot \left(\frac{1 - c_2 x\sqrt{-1}}{1 + c_2 x \sqrt{-1}}\right)^2 \ldots \left(\frac{1 - c_{p-2} x\sqrt{-1}}{1 + c_{p-2} x\sqrt{-1}}\right)^2.$$

Now u being an arc of a circle, we have this well known formula,

$$u = \frac{1}{2\sqrt{-1}} \times \log. \left(\frac{1 - \tan u \sqrt{-1}}{1 + \tan u \sqrt{-1}}\right):$$

wherefore, if we take the logarithms of the factors of the foregoing expression, and substitute the equivalent circular arcs, we shall obtain,

p being an odd number,

$$\psi = \varphi + 2\varphi_2 + 2\varphi_4 + 2\varphi_6 \ldots 2\varphi_{p-1}, \qquad (10)$$

the arc φ_{2n} being determined by the equation,

$$\tan \varphi_{2n} = c_{2n} \times \tan \varphi.$$

When p is an even number, we have this equation in § 5,

$$(1 - k^2 z^2) R^2 = \beta^2 z^2 (1 - z^2) P^2 + Q^2.$$

And, using N and M to denote the products of the binomials in the numerator and denominator of the equation (8), we have

$$\tan \psi = \frac{\beta \tan \phi \, N}{M}.$$

By the substitution of $\frac{x}{\sqrt{1 + x^2}}$ for z as before, it will be found that,

$$\beta z \sqrt{1 - z^2} \, . \, P = \frac{\beta x N}{(1 + x^2)^{\frac{p}{2}}}, \quad Q = \frac{M}{(1 + x^2)^{\frac{p}{2}}}.$$

Wherefore we have,

$$M^2 + \beta^2 x^2 N^2 = (1 + x^2)^p (1 - k^2 z^2) R^2;$$

and by converting $(1 - k^2 z^2) . R^2$ into a function of x^2, we get

$$M^2 + \beta^2 x^2 N^2 = (1 + x^2)(1 + k'^2 x^2)(1 + c_2^2 x^2)^2 \dots (1 + c^2_{p-2} x^2)^2,$$

$$k'^2 = 1 - k^2, \quad \overline{c_{2n}}|^2 = 1 - k^2 \sin^2 \lambda_{2n}.$$

By treating this equation as before, we get

$$\frac{1 - \tan\psi \sqrt{-1}}{1 + \tan\psi \sqrt{-1}} = \frac{1 - x\sqrt{-1}}{1 + x\sqrt{-1}} \cdot \frac{1 - k' x \sqrt{-1}}{1 + k' x \sqrt{-1}} \left(\frac{1 - c_2 x \sqrt{-1}}{1 + c_2 x \sqrt{-1}}\right)^2 \dots :$$

and from this we deduce,

p being an even number,

$$\psi = \varphi + \varphi' + \varphi_2 + \varphi_4 \dots + \varphi_{p-2}, \tag{11}$$

$$\tan\varphi' = k' \tan\varphi, \quad \tan\varphi_{2n} = c_{2n} \tan\varphi.$$

8. In what goes before, our attention has been confined to two related functions, which, for the sake of abridging, we have denoted by the prefixes H and K; but as we shall have occasion, in what follows, to compare several functions differing from one another in their moduli and amplitudes, it will be proper to adopt the usual and more general notation, by means of the characteristic F prefixed to the modulus and amplitude. According to this notation, the equation (1) will be thus written,

$$F(h, \psi) = \beta F(k, \varphi); \quad F(k, \varphi) = \frac{1}{\beta} F(h, \psi).$$

The modulus k being given, we can compute the amplitudes, λ_1, λ_2, &c., at least by approximation; and the amplitude φ being supposed known, the foregoing formulas will determine the modulus h, the multiplier β, and the amplitude ψ; so that the function $F(k, \varphi)$ will be reduced to the similar function $F(h, \psi)$, of which the modulus h is less than the given modulus k. And in like manner as the three quantities h, β, ψ were determined from the two k, φ, we can deduce, from the two h, ψ, three new quantities, $h_{,}$, $\beta_{,}$, $\psi_{,}$, which will satisfy the equations,

$$F(h_{,}, \psi_{,}) = \beta_{,} F(h, \psi); \quad F(k, \varphi) = \frac{1}{\beta\beta_{,}} \times F(h_{,}, \psi_{,});$$

the modulus $h_{\prime}$ being less than the modulus h. Continuing the like operations, we can pass along a scale of decreasing moduli, till we arrive at one which, being as small as we please, will make the function F (k, ϕ) approach to a circular arc as near as may be required.

If we wish to apply the same theorem to reduce the given function F (h, ϕ) to a logarithm, through a scale of increasing moduli, the process is not so direct. For, in the first place, the greater modulus k is not immediately deducible from the less h, by means of the formulas that have been investigated; and, in the second place, the amplitude ϕ cannot be found when ψ is given without solving an equation of p dimensions. The theorem is, no doubt, mathematically sufficient for effecting the reduction; but the operations required are practically impossible, except in a few cases when p is a small number. But the ingenuity of M. Jacobi has provided a remedy for this inconvenience by a new transformation, which we shall now briefly explain, as it discloses a new set of remarkable properties of the elliptic functions.

If we put $y = \tan \psi$, $x = \tan \phi$, $h'^2 = 1 - h^2$, $k'^2 = 1 - k^2$, the differential of the equation (1) will assume this form,

$$\frac{dy}{\sqrt{1 + y^2 \cdot 1 + h'^2 y^2}} = \frac{\beta\, dx}{\sqrt{1 + x^2 \cdot 1 + k'^2 x^2}}:$$

and, for solving this equation, we shall have by the formula (7),

p being an odd number,

$$y = \beta\, x \times \frac{1 - \dfrac{x^2}{\tan^2 \lambda_2}}{1 - \dfrac{x^2}{\tan^2 \lambda_1}} \cdot \frac{1 - \dfrac{x^2}{\tan^2 \lambda_4}}{1 - \dfrac{x^2}{\tan^2 \lambda_3}} \cdots \frac{1 - \dfrac{x^2}{\tan^2 \lambda_{p-1}}}{1 - \dfrac{x^2}{\tan^2 \lambda_{p-2}}}.$$

But if this value of y solve the differential equation, it will still solve it, if we change $+x^2$ and $+y^2$ into $-x^2$ and $-y^2$; for it is obvious that, if the expression of y make the two sides of the equation identical in one case, it will necessarily make them identical in the other case. Wherefore the equation

$$\frac{dy}{\sqrt{1 - y^2 \cdot 1 - h'^2 y^2}} = \frac{\beta\, dx}{\sqrt{1 - x^2 \cdot 1 - k'^2 x^2}},$$

will have for its solution.

$$y = \beta x \times \frac{1 + \frac{x^2}{\tan^2 \lambda_2}}{1 + \frac{x^2}{\tan^2 \lambda_1}} \cdot \frac{1 + \frac{x^2}{\tan^2 \lambda_4}}{1 + \frac{x^2}{\tan^2 \lambda_3}} \cdots \frac{1 + \frac{x^2}{\tan^2 \lambda_{p-1}}}{1 + \frac{x^2}{\tan^2 \lambda_{p-1}}}.$$

In this equation the values of y and x are between 0 and ± 1, which limits they both attain at the same time. If we make $x = \pm 1$, and attend to the value of β, we shall find $y = \pm 1$. Let $y = \sin \tau$, $x = \sin \sigma$: then the integral of the differential equation will be

$$\left.\begin{aligned} &F(h', \tau) = \beta F(k', \sigma) \\ &\sin \tau = \beta \sin \sigma \times \frac{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_2}}{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_1}} \cdot \frac{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_4}}{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_3}} \cdots \frac{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_{p-1}}}{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_{p-2}}} : \end{aligned}\right\} \quad (12)$$

the amplitudes τ and σ increasing together from zero, and becoming equal to one another at 90°, and at every multiple of 90°.

A property of considerable importance in this theory, results from the comparison of the equations (1) and (12). Recalling the notations before used, viz. $K = F\left(k, \frac{\pi}{2}\right)$ and $H = F\left(h, \frac{\pi}{2}\right)$, we obtain from what has already been said in § 1,

$$p \times H = \beta \times K:$$

and if we put similarly $K' = F\left(k', \frac{\pi}{2}\right)$ and $H' = F\left(h', \frac{\pi}{2}\right)$, and observe that in the equations (12), τ and σ are equal to 90° at the same time, we shall have,

$$H' = \beta K'.$$

By combining the two equations, we readily obtain, first,

$$\frac{H}{H'} = \frac{1}{p} \cdot \frac{K}{K'}; \quad \beta = p \cdot \frac{H}{K}; \quad (13)$$

and secondly,

$$\beta \beta' = p; \quad \frac{K'}{K} = \frac{1}{p} \cdot \frac{H'}{H}; \quad \beta' = p \cdot \frac{K'}{H'}. \quad (14)$$

For any number p, the first of the formulas (13) determines h, and the second determines β, when k is given. Both the formulas involve transcendent quantities; they are nevertheless of great practical utility in this theory; and they

express succinctly the conditions necessary, in order that the transformations in the equations (1) and (12) take place. A little attention will show that the formulas (14) and (13) are entirely similar, the quantities β', h', k' occupying the same places in the first, that β, k, h do in the other. From this we learn that the equations (1) and (12) will still be true if we change β, k, h for β', h', k', respectively. Thus we have,

$$F(k', \psi) = \beta' F(h', \varphi), \tag{15}$$

the letters ψ and φ, it need hardly be noticed, although used on a former occasion, here express simply the variable amplitudes of the related functions. If therefore we divide $H' = F\left(h', \frac{\pi}{2}\right)$ into p equal parts, and put μ_1, μ_2, μ_3, &c., for the respective amplitudes of $\frac{1}{p}$ H', $\frac{2}{p}$ H', $\frac{3}{p}$ H', &c.; we shall have by the formulas (4) and (9),

$$\left.\begin{aligned} \beta' &= \frac{\sin\mu_2 \sin\mu_4 \ldots\ldots \sin\mu_{2p-2}}{\sin\mu_1 \sin\mu_3 \ldots\ldots \sin\mu_{2p-1}}, \\ k' &= h'^{p}.\left(\sin\mu_1 \sin\mu_3 \ldots\ldots \sin\mu_{2p-1}\right)^2. \end{aligned}\right\} \tag{16}$$

The multipliers β and β' being similar functions, the first of the amplitudes λ_1, λ_2, λ_3, &c., and the other of the amplitudes μ_1, μ_2, μ_3, &c., the equation $\beta\beta' = p$, expresses a curious property of those functions.

And, in like manner, if we change β, k, h, respectively for β', h', k' in the equation (12); or, which is the same thing, if we derive an equation from (15) in the same manner that (12) was obtained from (1), we shall get

$$\left.\begin{aligned} F(k, \tau) &= \beta' F(h, \sigma), \\ \sin\tau &= \beta' \sin\sigma \times \frac{1 + \dfrac{\sin^2\sigma}{\tan^2\mu_2}}{1 + \dfrac{\sin^2\sigma}{\tan^2\mu_1}} \cdot \frac{1 + \dfrac{\sin^2\sigma}{\tan^2\mu_4}}{1 + \dfrac{\sin^2\sigma}{\tan^2\mu_3}} \ldots \frac{1 + \dfrac{\sin^2\sigma}{\tan^2\mu_{p-1}}}{1 + \dfrac{\sin^2\sigma}{\tan^2\mu_{p-2}}}. \end{aligned}\right\} \tag{17}$$

Although, in the investigations of this §, we have supposed that p is an odd number, yet it is obvious that they will succeed equally when p is an even number, the formula (8) being used instead of (7).

The analysis by which the equation (12), of which those that follow are consequences, has been deduced from the equation (1), is precisely that by which

the expression of a circular arc is made to pass into a logarithm; so that the whole of this analytical theory rests on one principle, namely, the analogy which an elliptic function bears to a circular arc and to a logarithm, which are its extreme limits.

10. Of the transformations in the last §, the principal one is contained in the formulas (17), which constitute what is called the second theorem of M. Jacobi. One of its chief uses is to supply the defect of the first theorem by furnishing a direct process for reducing an elliptic function to a logarithm, through a scale of increasing moduli. In the function $\mathrm{F}(h, \sigma)$, the modulus h being given, we know h' $(= \sqrt{1 - h^2})$ named the complement of h for the sake of abridging; we shall therefore obtain the amplitudes μ_1, μ_2, &c., by the subdivision of the function $\mathrm{H}' = \mathrm{F}\left(h', \frac{\pi}{2}\right)$; we next compute the quantities β' and k' by the formulas (16); and, the amplitude σ being given, we deduce from the formulas (17), the amplitude τ, which will satisfy the equation,

$$\mathrm{F}(h, \sigma) = \frac{1}{\beta'} \cdot \mathrm{F}(k, \tau),$$

the modulus k of the new function being greater than h, because the complement k' is less than the complement h'. Taking now k' the complement of k, we deduce from it, by means of the formulas (16) and (17), the three quantities $\beta_{\prime}'$, $k_{\prime}'$, $\tau_{\prime}$, in like manner as β', k', τ were deduced from h'; and we shall have these equations,

$$\mathrm{F}(k, \tau) = \beta_{\prime}' \, \mathrm{F}(k_{\prime}, \tau_{\prime})\,; \quad \mathrm{F}(h, \sigma) = \frac{1}{\beta' \beta_{\prime}'} \times \mathrm{F}(k_{\prime}, \tau_{\prime});$$

the modulus $k_{\prime}$ being greater than k, because the complement $k_{\prime}'$ is less than k'. The like operations being continued, we shall at length arrive at a modulus k_n, as near the limit 1 as may be required.

Another use of the second theorem, when combined with the first, is to find any multiple of an elliptic function, or any aliquot part of it. By the first theorem, we have

$$\mathrm{F}(h, \psi) = \beta \, \mathrm{F}(k, \phi)\,;$$

and by the second, making $\sigma = \psi$ in the equations (17),

$$\mathrm{F}(k, \tau) = \beta' \, \mathrm{F}(h, \psi)\,;$$

and, by combining the two equations, observing that $\beta\beta' = p$, we get

$$\mathrm{F}(k, \tau) = p\,\mathrm{F}(k, \phi).$$

If p be an odd number, the amplitudes are obtained by the formulas (2) and (17), viz.

$$\sin\psi = \beta\sin\phi \times \frac{1 - \dfrac{\sin^2\phi}{\sin^2\lambda_2}}{1 - k^2\sin^2\phi\sin^2\lambda_2} \cdots\cdots \frac{1 - \dfrac{\sin^2\phi}{\sin^2\lambda_{p-1}}}{1 - k^2\sin^2\phi\sin^2\lambda_{p-1}},$$

$$\sin\tau = \beta'\sin\psi \times \frac{1 + \dfrac{\sin^2\psi}{\tan^2\mu_2}}{1 + \dfrac{\sin^2\psi}{\tan^2\mu_1}} \cdots\cdots\cdots\cdots \frac{1 + \dfrac{\sin^2\psi}{\tan^2\mu_{p-1}}}{1 + \dfrac{\sin^2\psi}{\tan^2\mu_{p-2}}}.$$

When a multiple is required, we pass directly, by means of the two equations, from the given amplitude ϕ to τ which is sought. In the case of an aliquot part, the amplitude τ being given, the solution of the second equation, of which p is the dimensions, will determine $\sin\psi$; and the amplitude ϕ which is sought, will then be found by solving the first equation, which is also of p dimensions. From the nature of the second equation, it has only one real root, and $p - 1$ impossible roots, for every real value of $\sin\tau$; and therefore it follows from the first equation, that the amplitude ϕ of the function $\frac{1}{p}\mathrm{F}(k, \tau)$ admits in all of p^2 values, of which only p values are real quantities, and the rest impossible.

If p be an even number, the expression of $\sin\psi$ will contain radical quantities, but instead of it we may take the value of $\tan\psi$ in the formula (8); and the two equations for the amplitudes will be,

$$\tan\psi = \frac{\beta\tan\phi}{1 - \dfrac{\tan^2\phi}{\tan^2\lambda_1}} \cdot \frac{1 - \dfrac{\tan^2\phi}{\tan^2\lambda_2}}{1 - \dfrac{\tan^2\phi}{\tan^2\lambda_3}} \cdots\cdots \frac{1 - \dfrac{\tan^2\phi}{\tan^2\lambda_{p-2}}}{1 - \dfrac{\tan^2\phi}{\tan^2\lambda_{p-1}}},$$

$$\sin\tau = \frac{\beta'\sin\psi}{1 + \dfrac{\sin^2\psi}{\tan^2\mu_1}} \cdot \frac{1 + \dfrac{\sin^2\psi}{\tan^2\mu_2}}{1 + \dfrac{\sin^2\psi}{\tan^2\mu_3}} \cdots\cdots \frac{1 + \dfrac{\sin^2\psi}{\tan^2\mu_{p-2}}}{1 + \dfrac{\sin^2\psi}{\tan^2\mu_{p-1}}};$$

from which the same general properties may be deduced, as when p is an odd number.

11. We have now demonstrated, as was proposed, the principal and leading points of this theory, for which we are indebted to M. JACOBI. For the subordinate details, and for many curious and important collateral researches that have been suggested by the new views laid open in this branch of analysis, we must refer to M. JACOBI's own work, to the papers of M. ABEL, and to the writings of LEGENDRE. We shall conclude this paper by applying the formulas that have been investigated to two particular instances, taking for p the most simple values, namely 2 and 3.

Example 1. Supposing $p = 2$.

By the formulas (3) and (6) we have these equations between the amplitudes ψ and ϕ, z being $= \sin\phi$,

$$\sin\psi = \frac{\beta z \sqrt{1-z^2}}{\sqrt{1-k^2 z^2}}; \quad \cos\psi = \frac{1 - \frac{z^2}{\sin^2\lambda_1}}{\sqrt{1-k^2 z^2}}:$$

wherefore

$$1 - k^2 z^2 = \beta^2 z^2 (1 - z^2) + \left(1 - \frac{z^2}{\sin^2\lambda_1}\right)^2:$$

and

$$\beta^2 = \frac{1}{\sin^4\lambda_1}; \quad \frac{2}{\sin^2\lambda_1} - \beta^2 = k^2.$$

We now get

$$k'^2 = 1 - k^2; \quad \beta = 1 + k'; \quad \sin^2\lambda_1 = \frac{1}{1+k'}.$$

Also, by the formula (9),

$$h = k^2 \sin^4\lambda_1 = \frac{k^2}{(1+k')^2} = \frac{1-k'}{1+k'};$$

from which we deduce

$$(1 + h)(1 + k') = 2.$$

The equation (1), viz.

$$F(h, \psi) = \beta F(k, \phi),$$

may now be put in one or other of these two forms,

$$F(k, \phi) = \frac{1+h}{2} F(h, \psi),$$

$$F(h, \psi) = (1 + k') F(k, \phi):$$

by the first we pass from the greater modulus k to the less h; and by the second, from the less modulus h to the greater k.

In the first of the two cases we must derive the amplitude ψ from φ: and, for this purpose we immediately obtain from the formula (11),

$$\tan(\psi - \varphi) = k' \tan \varphi.$$

Wherefore, if

$$k, h, h_1, h_2, \&c.$$

represent a series of decreasing moduli, of which the complements are,

$$k', h', h_1', h_2', \&c.$$

the successive quantities being derived from one another by these formulas,

$$h = \frac{1 - k'}{1 + k'}, \quad h_1 = \frac{1 - h'}{1 + h'}, \quad h_2 = \frac{1 - h_1'}{1 + h_1'}, \&c. :$$

and, if we likewise deduce a series of amplitudes in this manner,

$$\tan(\psi - \varphi) = k' \tan \varphi$$
$$\tan(\psi_1 - \psi) = h' \tan \psi$$
$$\tan(\psi_2 - \psi_1) = h_1' \tan . \psi_1, \&c.$$

we shall have these successive transformations, by which the value of the given function $F(k, \varphi)$ is made to approach indefinitely to the arc of a circle,

$$F(k,\varphi) = \frac{1+h}{2} F(h, \psi)$$

$$F(k,\varphi) = \frac{1+h}{2} \cdot \frac{1+h_1}{2} F(h_1, \psi_1)$$

$$F(k,\varphi) = \frac{1+h}{2} \cdot \frac{1+h_1}{2} \cdot \frac{1+h_2}{2} \cdot F(h_2, \psi_2), \&c.$$

In the second case, when we would pass from the less modulus h to the greater k, the amplitude φ must be deduced from ψ. For this purpose we have

$$\frac{\sin\psi}{\cos\psi} = \frac{\beta z \sqrt{1 - z^2}}{1 - \frac{z^2}{\sin^2 \lambda_1}} = \frac{\beta \sin\varphi \cos\varphi}{1 - \beta \sin^2 \varphi}:$$

but $\beta = \frac{2}{1+h}$; $\cos\phi \sin\phi = \frac{\sin 2\phi}{2}$; and $1 - 2\sin^2\phi = \cos 2\phi$: wherefore,

$$\frac{\sin\psi}{\cos\psi} = \frac{\sin 2\phi}{h + \cos 2\phi}; \text{ and, } \sin(2\phi - \psi) = h\sin\psi.$$

Wherefore if the quantities,

$$h, k, k_1, k_2, \&c.$$

represent a series of increasing moduli derived from one another by these equations,

$$k' = \frac{1-h}{1+h}, \qquad k_1' = \frac{1-k}{1+k}, \qquad k_2' = \frac{1-k_1}{1+k_1}, \&c.;$$

and further, if the amplitudes $\psi, \phi, \phi_1, \phi_2$, &c. be deduced from the formulas,

$$\sin(2\phi - \psi) = h\sin\psi,$$
$$\sin(2\phi_1 - \phi) = k\sin\phi,$$
$$\sin(2\phi_2 - \phi_1) = k_1\sin\phi_1: \&c.$$

we shall have these transformations in which the successive moduli tend to the limit 1,

$$F(h, \psi) = (1 + k')\, F(k, \phi),$$
$$F(h, \psi) = (1 + k')(1 + k_1')\, F(k_1, \phi_1),$$
$$F(h, \psi) = (1 + k')(1 + k_1')(1 + k_2')\, F(k_2, \phi_2), \&c.$$

Example 2. Supposing $p = 3$.

By the formulas (2) and (5) we have these equations between the amplitudes ψ and ϕ,

$$\sin\psi = \frac{\beta z\left(1 - \frac{z^2}{\sin^2\lambda_2}\right)}{1 - k^2 z^2 \sin^2\lambda_2}; \quad \cos\psi = \sqrt{1 - z^2} \cdot \frac{1 - \frac{z^2}{\sin^2\lambda_1}}{1 - k^2 z^2 \sin^2\lambda_2}:$$

wherefore,

$$(1 - k^2 z^2 \sin^2\lambda_2)^2 = \beta^2 z^2 \left(1 - \frac{z^2}{\sin^2\lambda_2}\right)^2 + (1 - z^2)\left(1 - \frac{z^2}{\sin^2\lambda_1}\right)^2:$$

from which we get

$$2k^2 \sin^2\lambda_2 = \frac{2}{\sin^2\lambda_1} + 1 - \beta^2,$$

$$k^4 \sin^4 \lambda_2 = \frac{1}{\sin^4 \lambda_1} + \frac{2}{\sin^2 \lambda_1} - \frac{2\beta^2}{\sin^2 \lambda_2},$$

$$\beta = \frac{\sin^2 \lambda_2}{\sin^2 \lambda_1}.$$

Now, observing that $\frac{\beta^2}{\sin^2 \lambda_2} = \frac{\beta}{\sin^2 \lambda_1}$, we obtain by equating the values of $k^4 \sin^4 \lambda_2$,

$$\left(\frac{1}{\sin^2 \lambda_1} + \frac{1-\beta^2}{2}\right)^2 = \frac{1}{\sin^4 \lambda_1} + \frac{2(1-\beta)}{\sin^2 \lambda_1}:$$

from which we deduce

$$\beta = \frac{2}{\sin \lambda_1} - 1.$$

In order to simplify the formulas I shall put $\frac{1}{\sin \lambda_1} = 1 + \varepsilon$: then $\beta = 1 + 2\varepsilon$, and $\sin^2 \lambda_2 = \beta \sin^2 \lambda_1 = \frac{1 + 2\varepsilon}{(1+\varepsilon)^2}$: and, having substituted these values in the first of the foregoing equations, we shall get,

$$\varepsilon^4 + 2\varepsilon^3 - 2k'^2\varepsilon - k'^2 = 0.$$

This equation may be resolved by the usual method into the two following quadratic factors,

$$\varrho^3 = 4k^2 k'^2,$$

$$\varepsilon^2 + (1 + \sqrt{1-\varrho})\,\varepsilon - \tfrac{1}{2}\sqrt{\varrho^2 + 4k'^2} - \tfrac{1}{2}\varrho = 0,$$

$$\varepsilon^2 + (1 - \sqrt{1-\varrho})\,\varepsilon + \tfrac{1}{2}\sqrt{\varrho^2 + 4k'^2} - \tfrac{1}{2}\varrho = 0.$$

The second of these equations has two impossible roots: the first has two real roots, one being negative and foreign to the question, and the other positive, which solves the problem. Thus ε has only one value, which may be constructed geometrically, but the algebraic expression of it need not be written down.

We have next to derive the amplitude ψ from ϕ. We readily obtain from the foregoing biquadratic equation,

$$k^2 = \frac{(1+\varepsilon)^3(1-\varepsilon)}{1 + 2\varepsilon};$$

and hence,

$$1 - k^2 \sin^2 \lambda_2 = c_2^2 = \varepsilon^2.$$

And the formula (10) will determine ψ when ϕ is given.

Finally, therefore, we have these determinations,

$$\beta = 1 + 2\,\varepsilon, h = \frac{k^3}{(1 + \varepsilon)^4}, \tan\left(\frac{\psi - \phi}{2}\right) = \varepsilon \tan \phi\,;$$

$$\mathrm{F}\,(k, \phi) = \frac{1}{1 + 2\,\varepsilon} \times \mathrm{F}\,(h, \psi),$$

the modulus h being less than k^3. By repeating the like operations, a succession of moduli rapidly decreasing may be formed, by means of which the given elliptic function will be reduced to a circular arc as near as may be required.

XXI. *On the Tides in the Port of London.* *By* J. W. Lubbock, *Esq., V.P. and Treas. R.S.*

Read June 16, 1831.

I HAVE the honour to present to the Society a discussion of observations of the tides made at the London Docks, in the form of various Tables, which show the time and height of high water, not only at different points of the moon's age, but also for the different months of the year, for every minute of the moon's parallax, and for every three degrees of her declination. This work has been accomplished by Mr. Dessiou of the Admiralty; but for the arrangement of the Tables, and the methods employed, I alone am responsible.

The tides take place in the river Thames with extreme regularity, and, as the rise is considerable*, the observations are made with facility. Those upon which the annexed Tables are founded are made at the entrance to the London Docks; the time of high water, or the time when the water has just made its mark, is there noted on a slate by the watchman on the pier-head, generally only to the nearest five minutes; this is afterwards copied in a book kept for the purpose by Mr. Peirse. I am enabled, through the kindness of Mr. Solly, the worthy Chairman of the London Dock Company, to present to the Society the books containing the observations which serve as the foundation of Mr. Dessiou's Tables. These observations are not made with sufficient care; but they are valuable from the extent of time during which they have been carried on, as they were instituted soon after the opening of the Docks in 1804, and have been continued without interruption to the present time. I am not aware of any series of observations of the tides so extensive, except that made at Brest by order of the French Government. Mr. Peirse informs me that the observations of the night are generally more correct than those of

* I believe about nineteen feet. I am not however able to speak with precision, not having yet been able to examine the observations of low water.

the day, (which I was not aware of until lately,) because the persons in attendance at night have nothing else to do than to look out for high water; but in the day, owing to the press of business at the top of the tide, the water has fallen an inch or two at times before it has been noticed. The water remains stationary for some little time at high water, and therefore it is difficult to fix the time of that phenomenon precisely, without the aid of some mechanical contrivance; this will however I hope soon be supplied both at the London and St. Katherine Docks.

I have also to present to the Society some observations of the tides made during one year at the East India Docks, under the superintendence of Captain Eastfield, and which were kindly undertaken at my suggestion: these observations were made with great care, and may, I believe, generally be depended upon to the minute.

When the variations in the time and height of high water due to changes in the parallax and declination of the luminaries are neglected, the theories of Bernoulli and of Laplace lead to the same results, and these results agree most remarkably with observation. The same agreement is manifested in the circumstance that the variations of the interval between the time of the moon's transit and the time of high water, and the variations of the height of high water from the mean height, are greatest at the equinoxes, and least at the solstices, or soon after; and also that the height of high water increases as the distance of the moon decreases, that is, as her parallax increases. The difference between the heights when her parallax is greatest and least may be considered nearly a foot at the London Docks.

The changes in the declinations of the sun and moon, and in the parallax of the moon, have a sensible effect both on the time and on the height of high water; the law however of these changes is so complicated, and the observations are so imperfect, that for the present I think it will be necessary to have recourse to empirical tables for their calculation. They are not in conformity with the theory of Bernoulli, which has, I believe, never before been compared with observation; nor can I reconcile them to the theory of Laplace. The constants which might be supposed the same for this port and for Brest, determined by means of these observations, do not agree with those determined by Laplace from the observations at Brest. The comparison would be ex-

tremely interesting, if the results of the observations at Brest were arranged in Tables similar to those which accompany this paper. In a classification of this kind it is necessary that the epoch of the phenomena should be defined by the minute of the moon's transit; the day of the moon's age is quite insufficient. A complete classification of this kind is desirable, because erroneous conclusions may be drawn from isolated portions of the curve; and it is necessary, in order to ascertain the agreement between theory and the fact, to compare them throughout the whole period or extent of the inequality whose effect is considered.

According to the theory of BERNOULLI, and the Tables founded upon it, which are given in various works on navigation *, the variation in the interval between the time of the moon's transit and the time of high water, due to changes in the moon's parallax, vanishes or is equal to zero when the moon passes the meridian at 2 o'clock or 8 o'clock (or when $\theta - \theta_{,} - \lambda + \lambda_{,} = 0$ or $180°$). This is directly contrary to our observations, according to which the variation is the greatest at this age of the moon. See Table XV.

According to our observations, the time and the height of high water are the same, whether the moon's declination be north or south; if any difference obtain, it must be determined by more delicate means: nor is there any sensible difference whether the moon's transit be superior or inferior, as may be seen by Table XIII., where the observations for one month are separated, those corresponding to the superior and those corresponding to the inferior transits being kept distinct.

The Tables XV, XVI, XVII, XVIII, XIX and XX are intended to furnish the means of calculating empirically the time and height of high water, taking into account the changes in the different months of the year, and also those due to changes in the moon's parallax and declination. By means of these Tables, Mr. DESSIOU has calculated the times and heights of high water for the year 1826; and, having classed all the observations made at the London Docks in that year according to the different winds, has found for each a correction, which is given in Table XXI. The variation or correction due to the wind depends, of course, also on its rapidity or force; it appears however

* I have had occasion in the Preface to my "Account of the Traité sur le Flux et Réflux de la Mer," of DANIEL BERNOULLI, to point out an error which exists in several Tables of this kind, from the heading being reversed; but this error does not remedy the discrepancy in question.

generally to have but little effect on the tide. North-westerly gales raise the tide; south-westerly winds depress it.

Two maps are annexed to this paper, drawn by Mr. Walker; the one showing the time of high water on the coast of Great Britain, the other throughout the world, or at least in as many places as it has been ascertained.

The following are the authorities from which these maps have been compiled.

The Chart of the British Isles is taken from the observations and surveys of Messrs. M'Kenzie, Spence and Murray on the South coast of England; from those of Captain Hewett and Mr. Thomas on the East coast; France and the North Sea have been copied from the Admiralty Charts; the coast of Scotland is from M'Kenzie, Captain Huddart and the Admiralty Charts; and Ireland has been drawn from the surveys of Captain White, Captain Huddart, Mr. Nimmo, &c. &c.

In the Chart of the World, the Coast of France, Spain and Portugal is copied from the Admiralty Charts and the surveys of Tofino; Africa from those of Captain Owen; Newfoundland is from Captain Bullock; Nova Scotia and the coast of the United States are from the Admiralty Charts; South America and the west coast of North America have been copied from the Spanish Charts, and from the surveys of Captains King, Beechey, Vancouver, &c. &c. The coasts of Persia, India and the Indian Seas have been taken chiefly from Captain Horsburgh's Charts and Book of Directions; Australia is from the surveys of Captain Flinders, King, &c. &c.

It will be seen that the continents alter the direction of the *cotidal* lines, (I mean the series of points at which it is high water at the same instant,) and that the progress of the tide is not always from east to west: in the Atlantic it is from south to north; so that it is high water at nearly the same time on the coast of Portugal and on the opposite coast of America. This remarkable circumstance is noticed by Bacon*.

The map of the world is offered as a mere sketch, for our information on this subject is at present lamentably deficient. The map of Great Britain presents

* De Fluxu et Refluxu Maris. Bacon's Works, vol. ii. p. 81. "Idque non fortuito notatum, sed de industria inquisitum atque repertum, aquas ad littora adversa Europæ et Floridæ iisdem horis ab utroque littore refluere, neque deserere littus Europæ cum advolvantur ad littora Floridæ, more aquæ (ut supra diximus) agitatæ in pelvi, sed plane simul ad utrumque littus attolli et demitti."

more details, and is, I trust, more accurate; the time of high water being generally well ascertained (as a first approximation) and easily observed on these coasts. Lines are drawn round the coast to mark the depth of the water: it is to be regretted that this method of indicating the soundings, which, according to Captain ALEXANDER *, is adopted by the Russians in their charts, is not more general. The tide which reaches the port of London is principally due to the tide which descends along the eastern coast of Scotland and England: this branch of the tide meets the tide which comes up the Channel off the sand called the Kentish Knock.

If when the tide is single, the height of the water is represented by a series of cosines, affected with constant coefficients; when the tide results from the union of two branches, each of this kind, the height of the water will still be represented by a series of cosines of angles, affected with other constant coefficients, the periods of the inequalities being the same as before, but the epochs, or the times when they arrive at their maxima and minima, different. This is shown by LAPLACE in the Méc. Cel. vol. ii. p. 225. FRESNEL has applied the same method (Mémoire sur la Diffraction de la Lumière, p. 279.) to finding the resultant of any number of luminous waves of the same length. This method rests upon the superposition or coexistence of small oscillations, which may not be rigorously true; but it may be stated generally, that when the tide results from the union of any number of partial tides, the resultant of the whole may be considered as one tide, affected with inequalities, of which the periods are the same as the periods of the inequalities of the partial tides of which it is composed, but of which the magnitudes and the epochs are different †. It is owing to this cause that the high water in some places takes place only once in twenty-four hours, which is a case of interference.

I shall now proceed to explain the manner in which the annexed Tables were formed.

Table I. was made by classifying all the transits of the moon which took place in the years 1808 to 1826 inclusive, and in each half hour; thus, all the transits were found from the Nautical Almanac which took place between twelve o'clock (A.M.) and half-past twelve (A.M.), the moon being above the

* See ALEXANDER's Travels to the Seat of War in the East, vol. ii. p. 101.

† If $A \cos(\theta - \lambda)$ represent any inequality, the constant A determines its magnitude, the variable θ its period, and the constant λ its epoch.

horizon; the inferior transits of the moon between the same limits were interpolated, and added to the rest. The mean of these transits for the month of January, with the equation of time, was found to be twenty-four minutes, and the mean corresponding time of high water $2^h\ 9^m$; the transits between half-past twelve and one, and in every succeeding half-hour, were treated in the same way. The columns underneath are given to explain more fully the method employed, which was the same for all the Tables.

All the details of these calculations are deposited in the library of the Society.

Table I.

	Time of Moon's Transit from Nautical Almanac.				Time of corresponding High Water from the Dock Books.			
	January. 0 to 30′.		January. 30′ to 1 hour.					
	January.	h. m.	January.	h. m.	January.	h. m.	January.	h. m.
1808	28	0 10 intd.	28	0 33	28	2 15	28	2 40
——	....	....	29	0 56 intd.	....	....	29	2 45
1809	16	0 21	17	0 48 intd.	16	2 0	17	2 30
1810	6	0 19 int1.	6	0 49	6	2 0	6	2 30
1811	25	0 11 intd.	25	0 41	25	2 0	25	2 30
1812	14	0 6	15	0 35 intd.	14	2 0	15	2 35
1813	3	0 13 intd.	3	0 39	3	2 30	3	2 25
1814	22	0 20 intd.	22	0 44	22	2 30	22	2 45
1815	11	0 20 intd.	11	0 44	11	2 15	11	2 40
1816	29	0 10	29	0 34 intd.	29	2 35	29	2 45
——	....	....	30	0 57	....	....	30	3 0
1817	17	0 2	....	....	17	1 45	....	....
——	18	0 30 intd.	18	0 56	18	2 15	18	2 25
1818	7	0 5 intd.	7	0 37	7	1 45	7	2 15
1819	26	0 2 intd.	26	0 33	26	1 45	26	2 0
1820	16	0 23 intd.	16	0 55	16	2 15	16	2 30
1821	4	0 17	5	0 47 intd.	4	2 10	5	2 20
1822	23	0 16	24	0 41 intd.	23	2 20	24	2 35
1823	12	0 7	13	0 31 intd.	12	2 15	13	2 30
——	....	....	13	0 54	....	....	13	2 45
1824	1	0 8	2	0 32 intd.	1	2 10	2	2 25
——	31	0 12	2	0 55	31	2 35	2	2 30
1825	19	0 14	20	0 36 intd.	19	1 50	20	2 30
——	....	....	20	0 58	....	....	20	3 0
1826	8	0 4	9	0 56	8	2 0	9	2 40
——	9	0 30 intd.	....	....	9	2 20	....	....
	22)	300	24)	1071	22)	47 30	24)	61 30
Equation of Time*....		$13\frac{3}{4}$ 10	......	$44\frac{1}{2}$ 10	Mean corresponding Time of High Water.	2 9	......	2 34
Mean time of transit...		24	......	$54\frac{1}{2}$				

* The equation of time was generally taken for the 15th of the month.

Hence I infer that neglecting the influence of the moon's parallax and declination, when the moon passes the meridian at twenty-four minutes past twelve mean solar time in January, the time of high water at the London Docks is nine minutes past two.

The other Tables, which result immediately from the observations, were formed in the same manner, and are, I trust, sufficiently explained by the heading which accompanies each.

			In the notation of the Méc. Cel.
Let m be the mass		of the sun.	L
δ	. . declination		v
θ	. . hour angle		$nt + \varpi - \psi$
r	. . distance from the centre of the earth		r
Π	. . mean horizontal parallax		
l	. . longitude		φ
n	. . mean motion in its orbit		m
ω	. . obliquity of the ecliptic		ε

The same letters accented at foot refer to the moon, $\omega_{,}$ being the inclination of her orbit to the equator; let also φ denote the geographical latitude of the port, in the notation of the Méc. Cel. $90° - \theta$.

Considering only the terms which are multiplied by the cube of the parallax, the forces which produce the phenomena of the tides are the partial differences of the function

$$\frac{3m}{2r^3}\left\{(\sin\varphi\sin\delta + \cos\varphi\cos\delta\cos\theta)^2 - \frac{1}{3}\right\}$$ See the Méc. Cel. vol. v. p. 168.

According to the theory of LAPLACE, this function being equal to $\Sigma A \cos(\theta - \lambda)$, θ being any variable angle depending on the time, and A and λ constants, the height of the water at any given time is equal to $\Sigma (A' \cos(\theta - \lambda')$, θ being the same angle as before, and A' and λ' other constants.

$$\left\{\sin\varphi\sin\delta + \cos\varphi\cos\delta\cos\theta\right\}^2 = \frac{\sin^2\varphi}{2} + \frac{\cos^2\varphi}{4} + \frac{\cos^2\varphi}{4}\cos 2\theta - \left\{\frac{\sin^2\varphi}{2} - \frac{\cos^2\varphi}{4}\right\}\cos 2\delta$$

$$+ \frac{\cos^2\varphi}{8}\left\{\cos(2\theta - 2\delta) + \cos(2\theta + 2\delta)\right\} + \frac{\sin 2\varphi}{2}\sin 2\delta\cos\theta.$$

If the longitude of the sun be introduced by putting for δ its value from the equation

$$\sin\delta = \sin\omega \sin l$$

l being reckoned from the first point of Aries,

$$\left\{\sin\varphi\sin\delta + \cos\varphi\cos\delta\cos\theta\right\}^2 = \frac{\cos^2\varphi}{2}\left(1 - \frac{\sin^2\omega}{2}\right) + \frac{\sin^2\varphi\sin^2\omega}{2}$$

$$+ \frac{\cos^2\varphi}{2}\left(1 - \frac{\sin^2\omega}{2}\right)\cos 2\theta$$

$$+ \frac{\cos^2\varphi\sin^2\omega}{8}\left\{\cos(2\theta - 2l) + \cos(2\theta + 2l)\right\}$$

$$+ \sin^2\omega\left\{\frac{\cos^2\varphi}{4} - \frac{\sin^2\varphi}{2}\right\}\cos 2l$$

$$+ \frac{\sin 2\varphi}{2}\sin 2\delta\cos\theta$$

Considering the results of many years so as to destroy the effects of changes in the moon's parallax and declination, and taking the mean of the times of high water when the moon passes the meridian at any given time, and twelve hours later, the tides, owing to the united action of the sun and moon, depend on the terms

$$\frac{3m\Pi^3\cos^2\varphi}{4}\left(1 - \frac{\sin^2\omega}{2}\right)\cos 2\theta$$

$$+ \frac{3m\Pi^3\cos^2\varphi\sin^2\omega}{16}\left\{\cos(2\theta - 2l) + \cos(2\theta + 2l)\right\}$$

$$+ \frac{3m\Pi^3\sin^2\omega}{2}\left\{\frac{\cos^2\varphi}{4} - \frac{\sin^2\varphi}{2}\right\}\cos 2l$$

$$+ \frac{3m_{\prime}\Pi_{\prime}^3\cos^2\varphi}{4}\left(1 - \frac{\sin^2\omega_{\prime}}{2}\right)\cos 2\theta_{\prime}$$

and the height of the water will be represented by

$$A\,m\,\Pi^3\left(1 - \frac{\sin^2\omega}{2}\right)\cos(2\theta - 2\lambda)$$

$$+ \frac{B\,m\,\Pi^3}{4}\sin^2\omega\left\{\cos(2\theta - 2l - 2\lambda) + \cos(2\theta + 2l - 2\lambda)\right.$$

$$+ C\,\Pi^3\sin^2\omega\cos 2l$$

$$+A_{\prime} m_{\prime} \Pi_{\prime}^{3}\left(1-\frac{\sin^2 \omega_{\prime}}{2}\right) \cos (2\,\theta_{\prime}-2\,\lambda_{\prime})$$

This expression coincides with that given by LAPLACE, (Méc. Cel. vol. v. p. 169, at foot,) when the terms multiplied by $\sin^4 \frac{\omega}{2}$ are neglected, $n\,t+\varpi-m\,t$ (in the notation of the Méc. Cel.) being equal to the hour angle θ at the equinoxes and solstices, that is, when $l=0$, $90°$, $180°$ or $270°$; but LAPLACE neglects the term $\cos 2\,l$.

LAPLACE supposes that

$$A=(1+n\,x)\,B \qquad A_{\prime}=(1+n_{\prime}x)\,B$$

n being the mean motion of the luminary in its orbit, and x an indeterminate quantity, to be determined by the observations. Making these substitutions, differentiating the expression for the height to find the time of high water, and supposing $\mathrm{d}\,\theta=\mathrm{d}\,\theta_{\prime}$ and $\omega=\omega_{\prime}$,

$$\tan(2\,\theta_{\prime}-2\,\lambda_{\prime})=\frac{\frac{m\,\Pi^3}{m_{\prime}\Pi_{\prime}^3}\frac{(1+n\,x)}{(1+n_{\prime}x)}\sin(2\,\theta_{\prime}-2\,\theta-2\,\lambda_{\prime}+2\,\lambda)+\frac{m\,\Pi_3}{2\,m_{\prime}\Pi_{\prime}^3}\frac{\sin^2\omega\cos 2\,l}{(1+n_{\prime}x)\left(1\quad\frac{\sin^2\omega)}{2}\right.}\sin(2\,\theta_{\prime}-2\,\theta-2\,\lambda_{\prime}+2\,\lambda)+C'\sin(2\,\theta_{\prime}-2\,\lambda_{\prime}-2\,l)}{1+\frac{m\,\Pi^3}{m_{\prime}\Pi_{\prime}^3}\frac{(1+n\,x)}{(1+n_{\prime}x)}\cos(2\,\theta_{\prime}-2\,\theta-2\,\lambda_{\prime}+2\,\lambda)+\frac{m\,\Pi^3}{2\,m_{\prime}\Pi_{\prime}^3}\frac{\sin^2\omega\cos 2\,l}{(1+n_{\prime}x)\left(1-\frac{\sin^2\omega)}{2}\right.}\cos(2\,\theta_{\prime}-2\,\theta-2\,\lambda_{\prime}+2\,\lambda)+C'\cos(2\,\theta_{\prime}-2\,\lambda_{\prime}-2\,l)}$$

C' being a constant different from C, θ and $\theta_{\prime}$ being the values of those variables at the instant of high water.

If we consider the mean of all the months of the year, column A, Table III.

$$\tan(2\,\theta_{\prime}-2\,\lambda_{\prime})=\frac{\frac{m\,\Pi^3}{m_{\prime}\Pi_{\prime}^3}\frac{(1+n\,x)}{(1+n_{\prime}x)}\sin(2\,\theta_{\prime}-2\,\theta-2\,\lambda_{\prime}+2\,\lambda)}{1+\frac{m\,\Pi^3}{m_{\prime}\Pi_{\prime}^3}\frac{(1+n\,x)}{(1+n_{\prime}x)}\cos(2\,\theta_{\prime}-2\,\theta-2\,\lambda_{\prime}+2\,\lambda)}$$

The constants which enter into this expression may be determined by means of column A. The mean of this column is $1^{\mathrm{h}}\ 25^{\mathrm{m}}$; I therefore take

$$\lambda_{\prime}=1^{\mathrm{h}}\ 25^{\mathrm{m}}, \qquad \lambda-\lambda_{\prime}=2^{\mathrm{h}}, \qquad \lambda=3^{\mathrm{h}}\ 25^{\mathrm{m}}$$

$\frac{m\,\Pi^3(1+n\,x)}{m_{\prime}\Pi_{\prime}^3(1+n_{\prime}x)}=$ tangent of twice the difference of the interval when the moon passes the meridian at 2^{h} and at 5^{h}. I take

$$\log.\frac{m\,\Pi^3(1+n\,x)}{m_{\prime}\Pi_{\prime}^3(1+n_{\prime}x)}=9{\cdot}5784858$$

When the moon passes the meridian at three o'clock, $\theta - \theta_{,} - \lambda + \lambda_{,} = 15°$.

$$\begin{array}{rl}
\text{log. sin } 30° = & 9.6989700 \\
 & 9.5784858 \\ \hline
 & 9.2774558 \\
\text{log. } 1.3281 = & .1232308 \\ \hline
 & 9.1542250 = \text{log. tan } 8°\ 7' \text{ or } 32'\ 38'' \text{ in time}
\end{array}
\qquad
\begin{array}{rl}
\text{log. cos } 30° = & 9.9375806 \\
 & 9.5784858 \\ \hline
 & 9.5160164 = \text{log. } \cdot 3281
\end{array}$$

$$2\,\theta_{,} - 2\,\lambda_{,} = -32'\ 38'',\ \theta_{,} - \lambda_{,} = 16\tfrac{1}{4}',\ \theta_{,} = 1^{h}\ 25^{m} - 16^{m} = 1^{h}\ 9^{m}$$

In this way the following Table was calculated.

Time of Moon's Transit.	Interval between the Moon's Transit and the Time of High Water.		Error of Calculation.
	Observed.	Calculated.	
h	h. m.	h. m.	m.
0	1 57	1 56	— 1
1	1 42	1 41	— 1
2	1 26	1 25	— 1
3	1 11	1 9	— 2
4	56	54	— 2
5	45	44.1	— 0.9
6	42	41	— 0
7	52	53.7	+ 1.7
8	1 23	1 25	+ 2
9	1 56	1 56.3	+ 0.3
10	2 10	2 9	— 1
11	2 8	2 6.5	1·5

According to Table III. the establishment * of the London Docks is $1^{h}\ 57^{m}$. Adding ten minutes, the establishment of London Bridge is $2^{h}\ 7^{m}$. The establishment of the London Docks according to Mr. Bulpit, who has calculated the times and heights of high water in the river for many years, is 2^{h}. Mr. Bulpit's calculations are founded on tables constructed by the late Captain Huddart, which have not been published.

In the Philosophical Transactions, vol. xiii. p. 10, Flamsteed gives "A correct Tide-table, showing the true times of the high waters at London Bridge to every day in the year 1683." It appears from his remarks that the earliest Tide-tables which were calculated for the Port of London were made on the supposition that the tide always followed three hours after the moon's transit.

* By the establishment of any port is meant the time of high water at new and full moon.

The Annuaire du Bureau des Longitudes and various works on navigation give $2^h\ 45^m$ for the establishment of the Port of London; it seems therefore probable that the high water takes place now much earlier than it did formerly. It is generally admitted that the constant $\lambda - \lambda_{\prime}$ is the same at different ports; this, however, requires to be confirmed by accurate determinations, and is one of the most interesting questions in the theory of the tides. BERNOULLI makes this constant 20° or $1^h\ 20^m$ only in time; LAPLACE adopts the same value, though not expressly. BERNOULLI's Table for finding the time of high water, which is given in the Annuaire du Bureau des Longitudes, and in works on navigation, is quite inapplicable to the Port of London on this account, even in the mean distances of the moon, as the following comparison will show:

Time of Moon's Transit.	Interval between the Moon's Transit and the Time of High Water.		Error of Calculation.
	Observed.	Calculated.	
h	h m	h m	m
0	1 57	1 57	0
2	1 26	1 23½	− 2½
4	56	55	− 1
6	42	54½	+ 12½
8	1 23	2 0	+ 37
10	2 10	2 20	+ 10

The Table in the Annuaire for the year 1829 was made use of; the Table in that for 1831 differs from that only in form.

If $\lambda = \lambda_{\prime}$ the greatest tide takes place at new and full moon.

The quantity λ is called by LAPLACE the fundamental hour of the port.—Exposition du Système du Monde, p. 289.

I shall now compare the heights of high water, calculated by means of the same constants with those given by observation column B, Table V.

According to the preceding theory, the height of high water

$$= D + E\left\{\cos 2\,(\theta_{\prime} - \lambda_{\prime}) + .3788 \cos 2\,(\theta - \lambda)\right\}$$

D and E being constants to be determined by observation.

When the moon passes the meridian at two o'clock, $2\theta_{\prime} - 2\lambda_{\prime} = 0$, $2\theta - 2\lambda = 0$

———————————— eight o'clock, $2\theta_{\prime} - 2\lambda_{\prime} = 0$, $2\theta - 2\lambda = 180°$

Hence by column B. Table V.

$$22.80 = D + 1.3788\,E$$
$$19.43 = D + \ .6212\,E$$

Whence $D = 16.68$ and log. $E = .64819$, $E = 4{\cdot}448$

When the moon passes the meridian at

h. h.			
2	the height of high water		$= 16.68 + 4{\cdot}448\{1 + .3788\} = 22.80$
3 or 1	do.	do.	$= 16.68 + 4.448\{\cos 8^\circ\ 7' + .3788 \cos 22^\circ\} = 22.64$
4 or 0	do.	do.	$= 16.68 + 4.448\{\cos 15^\circ\ 25' + .3788 \cos 44^\circ\ 30'\} = 22.17$
5 or 11	do.	do.	$= 16.68 + 4.448\{\cos 20^\circ\ 45' + .3788 \cos 69^\circ\ 30'\} = 21.42$
6 or 10	do.	do.	$= 16.68 + 4.448\{\cos 22^\circ\ 2' - .3788 \sin 8^\circ\} = 20.56$
7 or 9	do.	do.	$= 16.68 + 4.448\{\cos 15^\circ\ 45' - .3788 \sin 44^\circ\} = 19.79$
8	do.	do.	$= 16.68 + 4.448\{1 - .3788\} = 19.43$

The following Table gives a comparison between theory and observation.

Time of Moon's Transit.	Height of High Water.		Error of Calculation.
	Observed.	Calculated.	
h.	Ft.	Ft.	Ft.
0	22.46	22.17	− .29
1	22.72	22.64	− .08
2	22.80	22.80	0
3	22.59	22.64	+ .05
4	22.10	22.17	+ .07
5	21.28	21.42	+ .14
6	20.37	20.56	+ .19
7	19.56	19.79	+ .23
8	19.43	19.43	0
9	20.10	19.79	− .31
10	20.92	20.56	− .36
11	21.85	21.42	− .43

I shall now endeavour to show that the expression for tan $(2\,\theta_{\prime} - 2\,\lambda_{\prime})$, p. 387, line 13, does not satisfy the observations.

When the moon passes the meridian at 2 o'clock $\theta - \theta_{\prime} - \lambda + \lambda_{\prime} = 0$.

$$\tan(2\,\theta - 2\,\lambda_{\prime}) = \frac{-C' \sin 2\,l}{1 + C' \cos 2\,l} \text{ nearly.}$$

When the moon passes the meridian at 8 o'clock, $\theta - \theta_{\prime} - \lambda + \lambda_{\prime} = 90^\circ$

$$\tan(2\,\theta_{\prime} - 2\,\lambda_{\prime}) = \frac{-C' \sin 2\,l}{1 - \frac{m\,\Pi^3}{m_{\prime}\Pi_{\prime}^3} + C' \cos 2\,l} \text{ nearly.}$$

At the end of May $l = 45°$ nearly, and then according to Table III. when the moon passes the meridian at 2, the interval between her transit and H.W. is $1^h\ 16^m$ nearly; when the moon passes the meridian at 8, the interval between her transit and H.W. is $1^h\ 38^m$ nearly. The variation or difference being in the one case -5^m, $(1^h\ 20^m - 1^h\ 25^m = -5^m)$, and in the other $+13^m$, $(1^h\ 38^m - 1^h\ 25^m = +13^m)$.

The formula gives in the one case

$$\tan(2\theta_{,} - 2l_{,}) = -C'$$

and in the other

$$\tan(2\theta_{,} - 2l_{,}) = \frac{-C'}{1 - \frac{m\Pi^3}{m_{,}\Pi_{,}^3}}$$

according to which the variation or difference should be in both cases of the same sign: hence the variations or differences -5^m and $+13^m$ are not owing to this inequality.

It is possible that this discrepancy may be owing to the terms which have $r_{,}^4$ in the denominator; but the observations are made so carelessly that I shall not attempt to take these terms into consideration.

The discrepancy which I have pointed out is sufficient to show that Table III. cannot be represented by the expression of p. 387, line 13, whatever value be given to the constants employed: this discrepancy is not confined to the epoch I have noticed, but extends throughout the year.

Neglecting this discrepancy which not only prevents the expression above from representing the observations, but must also I think vitiate any conclusions to be drawn respecting the mass of the moon; in April when $l = 0$, and

when the moon passes the meridian at 5^h, the interval is $0^h\ 34^m$
. . . . 11 . . . . . . 2 14

$$2^h\ 14^m - 34^m = 1^h\ 40^m$$

I may suppose

$$\tan 1^h\ 40^m = \tan 25° = \frac{m\Pi^3(1 + nx)}{m_{,}\Pi_{,}^3(1 + n_{,}x)} + \frac{2m_{,}\Pi_{,}^3}{m\Pi^3}\ \frac{\sin^2\omega}{(1 + n_{,}x)\left(1 - \frac{\sin^2\omega}{2}\right)}$$

$$\cdot 4663 = \cdot 3788 + \frac{m\Pi^3}{2m_{,}\Pi_{,}^3}\ \frac{\sin^2\omega}{(1 + n_{,}x)\left(1 - \frac{\sin^2\omega}{2}\right)}$$

from which equation combined with the equation

$$\frac{m\,\Pi^3}{m_{\prime}\Pi_{\prime}^3}\frac{(1+n\,x)}{(1+n_{\prime}x)} = .3788$$

$1 + n\,x$ may be determined.

The theory of LAPLACE differs essentially from that of BERNOULLI, in supposing the constants to be modified by local circumstances, which consideration introduces the factor $\frac{1+n\,x}{1+n_{\prime}\,x}$.

According to the observations at Brest, $1 + n\,x = 1.01891, 1 + n_{\prime}\,x = 1.25291$

$$\log. \frac{m\,\Pi^3}{m_{\prime}\Pi_{\prime}^3}\frac{(1+nx)}{(1+n_{\prime}x)} = 9.5385031 ? \qquad \frac{m\,\Pi^3}{m_{\prime}\,\Pi_{\prime}^3}\frac{(1+n\,x)}{(1+n_{\prime}x)} = .3455.$$

Whence

$$\log. \frac{m\,\Pi^3}{m_{\prime}\Pi_{\prime}^3} = 9.6283227$$

which gives the mass of the moon equal to that of the earth divided by 74.946.

If $m_{\prime\prime}$ be the mass of the earth, by an extension of the third law of KEPLER, $\frac{(m+m_{\prime\prime})\,\Pi^3}{(m_{\prime}+m_{\prime\prime})\,\Pi_{\prime}^3}$ is nearly equal to the ratio of the squares of the periodic times of the moon about the earth, and of the earth about the sun, or of their mean motions. This ratio is known very accurately to be .0748013. Hence neglecting $m_{\prime\prime}$ with regard to m, we have

$$\frac{m\,\Pi^3}{(m_{\prime}+m_{\prime\prime})\,\Pi_{\prime}^3} = (.0748013)^2.$$

If we neglect the factor $\frac{1+n\,x}{1+n_{\prime}x}$ which is nearly unity;

$$\log. \frac{m\,\Pi^3}{m_{\prime}\,\Pi_{\prime}^3} = 9.57846$$

$$\log. \frac{m\,\Pi^3}{(m_{\prime}+m_{\prime\prime})\,\Pi_{\prime}^3} = 7.74782 \qquad \log. \frac{m_{\prime}+m_{\prime\prime}}{m_{\prime}} = 1.83064$$

$$\frac{m_{\prime}}{m_{\prime}+m_{\prime\prime}} = \frac{1}{67.7},$$

which gives the mass of the moon equal to that of the earth, divided by 66.7.

Astronomers will, no doubt, concur in following the opinion of LAPLACE. "En considérant la pétitesse des quantités qui m'ont servi à déterminer l'ac-

croissement de l'action lunaire et en réflechissant que ces quantités sont du même ordre que les pétites erreurs dont l'application du principe de la coéxistence des ondulations très pétites aux phénomènes des marées est susceptible ; je n'ose garantir l'exactitude de cette valeur de la masse lunaire, et j'incline à penser que les phénomènes astronomiques sont plus propres à le fixer."

A very slight change in the interval employed produces a considerable alteration in the mass of the moon.

After differentiation $d\,\theta$ was supposed $= d\,\theta_{\prime}$, but since the time of the moon's synodic revolution is 29.530 days, and that in this time the hour angle of the moon is less by one circumference than that of the sun,

$$d\,\theta = \frac{29.530}{28.530}\,d\,\theta_{\prime}$$

The observations of the times and height of high water in different months of the year may be nearly represented by neglecting in the expression for the height, the term of which the argument is $2\,\theta + 2\,l - 2\lambda$, but it is not easy to see why the coefficient of this inequality differs so much from that of the inequality of which the argument is $2\,\theta - 2\,l - 2\,\lambda$, the cause of this difference must be clearly established before the theory can be considered complete.

If $l_{\prime}$ be the longitude of the moon reckoned from her perigee, and increased by a constant, considering the terms depending on the changes of the moon's parallax, the height of the water may be represented by

$$m\,\Pi^3 \cos\,(2\,\theta - 2\,\lambda)$$
$$+ m_{\prime}\Pi_{\prime}^3 \left\{\cos\,(2\,\theta_{\prime} - 2\,\lambda_{\prime}) + C\left(\cos\,(2\,\theta_{\prime} - 2\,\lambda_{\prime} - l_{\prime}) + \cos\,(2\,\theta_{\prime} - 2\,\lambda_{\prime} + l_{\prime})\right)\right\}$$

or

$$m\,\Pi^3 \cos(2\,\theta - 2\,\lambda) + m_{\prime}\,\Pi_{\prime}^3 \{1 + C\cos l_{\prime}\} \cos\,(2\,\theta_{\prime} - 2\,\lambda_{\prime})$$

$$\tan\,(2\,\theta_{\prime} - 2\,\lambda_{\prime}) = \frac{\dfrac{m\,\Pi^3}{m\,\Pi_{\prime}^3}\,\dfrac{\sin\,(2\,\theta_{\prime} - 2\,\theta - 2\,\lambda_{\prime} + 2\,\lambda)}{(1 + C\cos\,l_{\prime})}}{1 - \dfrac{m\,\Pi^3}{m_{\prime}\Pi_{\prime}^3}\,\dfrac{\cos\,(2\,\theta_{\prime} - 2\,\theta - 2\,\lambda_{\prime} + 2\,\lambda)}{(1 + C\cos\,l_{\prime})}}$$

According to this formula, the variation in the interval which elapses between the time of the moon's transit, and the term of high water, is equal to zero when the moon passes the meridian at 2 o'clock or 8 o'clock; and this is the case whatever value be given to the constant C, or the constant which I have sup-

posed to be included in the angle $l_{\prime}$: this result is directly contrary to observation, as may be seen by Table VIII. This difficulty may be got over by supposing the coefficients (C) of the angles $2\theta_{\prime}-2\lambda_{\prime}-l_{\prime}$ and $2\theta_{\prime}-2\lambda_{\prime}+l_{\prime}$ to be different; it remains however to assign the cause of this difference.

Not having been able to satisfy myself with respect to this and other discrepancies, I shall not attempt to form any Tables from theory for the purpose of calculating the effects due to changes in the declination of the luminaries, and in the parallax of the moon; but I hope that no great length of time will elapse before the results contained in M. DESSIOU's Tables will be confirmed by more accurate observations here and elsewhere, and that the problem of the tides will meet with the attention it deserves, no less practically than in connection with Physical Astronomy, and particularly with the determination, even if imperfectly, of the mass of the moon. "Les marées ne sont pas moins interessantes à connoître, que les inégalités des mouvemens célestes. On a négligé pendant long-temps de les suivre avec une éxactitude convenable, à cause des irrégularités qu'elles présentent; mais ces irrégularités disparaissent en multipliant les observations."—Exp. Syst. du Monde, p. 289.

The Tables XV. XVI. &c. present irregularities, owing to the comparative paucity and the imperfection of the observations. M. DESSIOU has therefore formed Tables from these, in which these irregularities are arbitrarily removed: these Tables are published in the Companion to the British Almanac for 1832, and no doubt represent the phenomena rather better than the former. As, however, these alterations are arbitrary, I have preferred giving here the results of the observations without any change. The observations employed were taken in all cases indiscriminately from the Dock books.

I have now only to express my acknowledgements to the Committee of the Society for the Diffusion of Useful Knowledge, and to Mr. POND the late superintendent of the Nautical Almanac, who have given me M. DESSIOU's assistance in forming the annexed Tables.

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XXI. Showing the influence of the direction of the Wind on the time and height of High Water.

Table I.

Showing the time of High Water at the London Docks corresponding to the mean time
vations made between the 1st of January

January.			February.			March.			April.			May.			June.		
Moon's Transit. P.M.	Corre-sponding H. W.	No. of Obs.	Moon's Transit. P.M.	Corre-sponding H. W.	No. of Obs.	Moon's Transit. P.M.	Corre-sponding H. W.	No. of Obs.	Moon's Transit. P.M.	Corre-sponding H. W.	No. of Obs.	Moon's Transit. P.M.	Corre-sponding H. W.	No. of Obs.	Moon's Transit. P.M.	Corre-sponding H. W.	No. of Obs.
h m	h m		h m	h m		h m	h m		h m	h m		h m	h m		h m	h m	
0 24	2 9	22	0 26	2 17.8	19	0 23.5	2 25.8	26	0 17.5	2 17.8	23	0 12	2 5.5	23	0 15.8	2 3	19
0 54.5	2 34	24	0 58	2 45	25	0 53.2	2 48	26	0 47	2 38.2	20	0 43.5	2 26.5	21	0 44.2	2 25	20
1 24	3 1	20	1 27	3 7.5	25	1 23.5	3 5	22	1 15	2 57.2	23	1 12.5	2 46.8	21	1 14.2	2 46.5	21
1 55	3 20	24	1 59	3 31.5	21	1 53.5	3 32	28	1 46	3 21	23	1 42.2	3 11.5	22	1 46	3 10.5	23
2 24	3 47	27	2 28.5	3 51.5	25	2 24.2	3 52.5	23	2 16.5	3 42	22	2 11.2	3 29.5	22	2 16	3 32.2	20
2 55	4 8	22	3 0	4 18.2	25	2 55	4 14.2	25	2 46.8	4 4.5	20	2 42.2	3 51.5	21	2 45	3 54.5	22
3 24	4 33	27	3 30	4 43	24	3 25	4 32.5	24	3 14	4 23	19	3 10.5	4 9.5	20	3 14	4 18.2	22
3 54	4 59	27	3 59	5 7.2	20	3 55.5	4 54	22	3 43.2	4 39	19	3 42	4 35.8	26	3 45	4 42	24
4 26	5 26	24	4 27.8	5 26.5	23	4 26.2	5 19	24	4 13	5 3	23	4 11	4 56	22	4 15	5 8	23
4 55	5 50	25	5 3.2	5 54.2	25	4 53.8	5 37.2	20	4 43	5 20.2	19	4 42	5 23	21	4 46	5 37.2	26
5 25	6 25	26	5 31.5	6 18.5	22	5 23.8	6 0.8	20	5 11.5	5 45.2	21	5 11	5 44.8	25	5 17	6 8	24
5 56	6 54	28	6 0.2	6 39.8	19	5 56	6 23.2	24	5 43.8	6 11	22	5 43	6 19.5	24	5 46	6 39	23
6 26	7 19	23	6 29	7 15	21	6 24	6 50.8	21	6 15.8	6 43	25	6 9.8	6 45.5	20	6 16	7 14	25
6 55	7 57	24	7 0.5	7 44.2	19	6 55	7 24.3	23	6 47	7 17.8	20	6 39	7 22.5	26	6 46	7 52	25
7 24	8 30	24	7 29	8 22	19	7 25.5	7 55.2	21	7 16.5	8 4.5	24	7 11	8 16	28	7 15	8 34	22
7 56	9 13	25	7 58	8 57	18	7 55	8 48.2	23	7 49	9 2.2	23	7 41.5	9 8.8	24	7 46	9 19	27
8 26	10 1	21	8 28.2	9 44.5	22	8 24.5	9 42	23	8 15.5	9 49.8	21	8 12.5	9 54	26	8 17	9 58.5	23
8 56	10 41	23	8 58	10 34.5	16	8 55.2	10 37.8	22	8 41.5	10 34.2	23	8 43.8	10 48.5	27	8 46	10 42	23
9 27	11 22	20	9 28.5	11 20.2	23	9 24.2	11 21.5	22	9 14.8	11 27	26	9 12.5	11 22.5	21	9 16	11 17.5	23
9 54	11 56	23	9 59.5	12 4.5	18	9 54	12 3.5	25	9 45.2	12 1.5	24	9 40.5	11 56.5	24	9 46	11 51	19
10 25	12 29	18	10 29.2	12 34.8	21	10 24.2	12 38.5	22	10 15.8	12 32	24	10 10.8	12 24.5	25	10 15	12 21	23
10 53	12 56	21	11 0.5	13 11	22	10 53.8	13 4.5	25	10 45	13 0	25	10 40.5	12 50.5	22	10 45	12 48	20
11 24	13 23	24	11 30	13 36.5	18	11 25.2	13 31.2	27	11 15.5	13 23.5	22	11 11	13 18.2	25	11 14	13 12	19
11 56	13 48	21	11 59	14 0	21	11 54.2	13 54	21	11 45.8	13 53	26	11 41.2	13 42.5	21	11 45	13 39	22
12 26	14 13	23	12 28.2	14 20	21	12 24	14 21	28	12 16.5	14 12.2	22	12 10	14 1.5	21	12 15	14 0.5	20
12 56	14 35	21	12 58.8	14 42.8	23	12 54.2	14 38.2	22	12 46.5	14 34.2	22	12 40.2	14 26.8	23	12 45	14 22	20
13 26	14 58	22	13 28.5	15 6.2	23	13 24	15 6	25	13 14.8	14 56	22	13 11.2	14 46.5	22	13 14	14 45	20
13 54	15 19	21	13 59	15 26.5	22	13 55	15 29.5	27	13 44.5	15 14.5	21	13 41.5	15 9	20	13 45.5	15 9	23
14 26	15 47	28	14 29.5	15 52.2	27	14 25	15 50.5	20	14 14.5	15 37	21	14 10	15 25	21	14 17	15 32	22
14 57	16 11	23	15 0.2	16 16.5	20	14 54	16 10.5	25	14 44.5	15 59	22	14 41	15 46.2	23	14 47	15 55	20
15 26	16 33	23	15 29.5	16 39	24	15 23	16 32	22	15 16.5	16 19	21	15 10.5	16 10	20	15 15	16 17	22
15 54	17 3	25	16 1	17 1.2	25	15 52	16 52	21	15 47	16 38	21	15 38.2	16 30.2	20	15 46	16 41	25
16 26	17 31	27	16 31.5	17 27.8	20	16 23	17 16	26	16 15.5	17 2.2	19	16 9.2	16 54.8	25	16 16	17 10	22
16 55	17 57	24	16 59	17 45	20	16 55.5	17 38.5	22	16 45	17 22	22	16 44	17 23	26	16 45	17 36	24
17 26	18 25	27	17 32	18 12	23	17 24	17 57.2	20	17 16.2	17 45	22	17 12.2	17 50	22	17 15.5	18 6	26
17 56	18 55	25	17 59	18 39.2	19	17 50.5	18 23.8	20	17 46	18 13.5	20	17 40.5	18 16.8	23	17 46	18 41	25
18 24	19 21	24	18 26.5	18 58	18	18 23	18 53.8	24	18 13.2	18 41	19	18 10.5	18 50.5	24	18 15	19 13	22
18 56	20 2	24	18 58.8	19 36.8	25	18 54	19 22.5	20	18 44	19 16	25	18 40	19 34	26	18 45	19 50	30
19 27	20 40	23	19 29.5	20 8.5	17	19 23	20 1.5	23	19 15.5	20 5	24	19 10.5	20 15.5	24	19 17.5	20 38	24
19 56	21 22	23	19 59	20 50.8	20	19 53.5	20 46	22	19 45	20 54.5	21	19 40	21 5	26	19 46	21 26	21
20 26	21 57	22	20 28	21 39.2	19	20 23.3	21 44	23	20 14.5	21 39	24	20 11.5	21 58.5	27	20 14	21 57	24
20 55	22 44	20	20 59.2	22 29.2	20	20 54	22 40.5	25	20 45.5	22 37.2	26	20 41.8	22 46	24	20 45	22 42	25
21 24	23 23	22	21 33	23 14.2	19	21 24.3	23 24.8	22	21 16	23 22.2	23	21 11.2	23 23	26	21 17	23 23.5	24
21 54	23 56	21	21 57.5	23 56.2	19	21 53	24 4.5	23	21 45.2	24 3.5	25	21 42.5	24 0	25	21 46	23 55.5	20
22 26	24 29	23	22 28.8	24 29.5	21	22 23.5	24 43	26	22 13.8	24 29	22	22 11.2	24 30	24	22 15	24 23.5	22
22 57	25 0	21	23 0	25 4	23	22 55.5	25 7.2	27	22 45.8	25 4	30	22 40.2	24 58	21	22 47	24 51	22
23 26	25 26	21	23 31	25 31.8	20	23 26	25 35.5	23	23 17	25 31.2	20	23 8	25 23.5	26	23 17	25 17	21
23 55	25 49	20	24 0	25 57	21	23 54	25 57.6	22	23 45.8	25 55	22	23 41.2	25 44	22	23 46	25 40	20

TABLE I.

(reckoned from noon) of the Moon's Transit in each month of the year; from 13,073 obser-
1808 and the 31st of December 1826.

July.			August.			September.			October.			November.			December.		
Moon's Transit. P.M.	Corre-sponding H. W.	No. of Obs.	Moon's Transit. P.M.	Corre-sponding H. W.	No. of Obs.	Moon's Transit. P.M.	Corre-sponding H. W.	No. of Obs.	Moon's Transit. P.M.	Corre-sponding H. W.	No. of Obs.	Moon's Transit. P.M.	Corre-sponding H. W.	No. of Obs.	Moon's Transit. P.M.	Corre-sponding H. W.	No. of Obs.
h m	h m		h m	h m		h m	h m		h m	h m		h m	h m		h m	h m	
0 21	2 11	19	0 19	2 17	22	0 9	2 14	26	0 0	2 6	24	23 59	1 51	22	0 13.2	1 58	22
0 49	2 32	22	0 51.5	2 45	25	0 40	2 36	25	0 30	2 26	23	0 29	2 9	20	0 42	2 21.2	19
1 20	2 57	23	1 19	3 3	20	1 10.5	2 59	24	1 1	2 48	25	0 59	2 33	22	1 9.2	2 38.5	20
1 50	3 18	23	1 48	3 27	26	1 41	3 24	24	1 41	3 7	23	1 31	2 55	21	1 40.2	3 6	22
2 20	3 41	25	2 19	3 49	25	2 10	3 43	20	2 2	3 30	21	2 1	3 16	20	2 11	3 23	23
2 51	4 5	25	2 48	4 14	27	2 39.5	4 3	27	2 32	3 50	25	2 31	3 38	22	2 41	3 46	22
3 21	4 33	24	3 20.5	4 35	24	3 10	4 26.5	20	3 3	4 8	20	3 1	4 0	20	3 11	4 11	23
3 51	4 57	27	3 49.5	5 0	23	3 39	4 44	22	3 31	4 28	20	3 31	4 20	22	3 41	4 34	24
4 21	5 23	23	4 19	5 25	25	4 9	5 3	21	4 0	4 46	21	4 3	4 44	23	4 11.5	5 0	26
4 50	5 50	26	4 50	5 47	25	4 40	5 26	25	4 29	5 8	22	4 33	5 12	21	4 42	5 24	25
5 21	6 19	28	5 20.5	6 14	22	5 12.5	5 56	21	5 1	5 33	24	5 0	5 35.5	20	5 11.5	5 57	24
5 52	6 51	25	5 49	6 36	22	5 41.5	6 17	19	5 31	5 57.5	20	5 29	6 4.5	23	5 41	6 31	28
6 21	7 24	22	6 18	7 6	22	6 10.5	6 40	21	6 1	6 23	20	5 59	6 35	23	6 12	7 13	26
6 50	7 50	27	6 49	7 33	24	6 41	7 10	22	6 32	7 1	25	6 31	7 13	26	6 42	7 46	24
7 21	8 36	25	7 19	8 12	21	7 11.5	7 45	21	7 4	7 42	23	7 1	8 3	24	7 11	8 23	27
7 53	9 17	24	7 48	8 51.5	22	7 42	8 34	22	7 31.5	8 39	21	7 29	8 51	23	7 41	9 15	23
8 23	9 58	21	8 19	9 39	24	8 13	9 27	21	7 59.5	9 25	22	8 1	9 44	26	8 10	9 54	25
8 51	10 33	20	8 50	10 29	23	8 41	10 22	22	8 29.3	10 13	25	8 31	10 29	24	8 43	10 35	26
9 23	11 14.5	23	9 20.5	11 12	22	9 11.5	11 14	23	8 59.5	11 4	27	9 0	11 13	25	9 11	11 16	22
9 53	11 49	21	9 50	11 55.5	21	9 42	11 48	23	9 31	11 52	24	9 30	11 50	25	9 41	11 51	23
10 23	12 23	22	10 18	12 23.5	20	10 12	12 30	23	10 0	12 21	26	10 0	12 14	23	10 12	12 19	23
10 53	12 55	19	10 48	12 55.5	22	10 41.5	12 59	23	10 30	12 49	26	10 31	12 47	25	10 41.5	12 55	20
11 23	13 19	23	11 18	13 22	23	11 9	13 19	21	11 1	13 15	25	11 1	13 12	22	11 10	13 17	21
11 53	13 47.5	21	11 48.5	13 47.5	23	11 40	13 49	27	11 31	13 43	25	11 30	13 33	21	11 39	13 34	20
12 23	14 6.5	21	12 20	14 18	26	12 10	14 14	23	12 1	14 3	23	12 0	13 56	22	12 9	13 58	23
12 53	14 35	22	12 51	14 41	24	12 39	14 34	23	12 29.5	14 25	25	12 29	14 18	20	12 40	14 22	22
13 20	14 58	23	13 18.5	15 4	21	13 10	14 56	27	13 0	14 44	22	12 58	14 32	21	13 11	14 45	21
13 51	15 16	21	13 48	15 27.5	26	13 41	15 18.5	25	13 30	15 8.5	24	13 30	14 57	23	13 41.5	15 10	25
14 20	15 41.5	25	14 19	15 50	26	14 12	15 40.5	23	14 0	15 28	22	14 1	15 17	21	14 14	15 31	21
14 51	16 6.5	25	14 50	16 12	23	14 41	16 3	22	14 29	15 45	23	14 30	15 38	20	14 43	15 52	21
15 21	16 31.5	25	15 19	16 35	22	15 10	16 22	23	15 0	16 8	22	15 0	15 59.5	21	15 11	16 12	23
15 52	16 56.5	25	15 50	17 0	27	15 39.5	16 43	21	15 30	16 23	22	15 30	16 20	22	15 42	16 40	25
16 21	17 26	27	16 22	17 23	22	16 9	17 5	23	16 1	16 48.5	23	15 59	16 40	20	16 12	17 5	23
16 52	17 55	26	16 51.5	17 47.5	23	16 40	17 30	22	16 31	17 11	20	16 30	17 12	24	16 40.5	17 32	24
17 22	18 21.5	23	17 19	18 11.5	23	17 11	17 48	23	17 0.5	17 30	23	17 1	17 34	24	17 11	18 2	29
17 50	18 51	26	17 48	18 35.5	22	17 41	18 15	19	17 31	17 56	23	17 30	18 4	20	17 42	18 33	25
18 22	19 21	28	18 19.5	19 2	25	18 10.5	18 40	20	18 2	18 29	21	18 0	18 39	26	18 12	19 6	26
18 51	19 54	24	18 50	19 36	22	18 41	19 11	19	18 30.5	19 1	24	18 31.5	19 13	25	18 41	19 44	24
19 20	20 33	23	19 20	20 10	23	19 10.5	19 44	21	19 0	19 39	20	19 2	20 1.5	23	19 9.5	20 33	24
19 49	21 15	22	19 50	20 58	21	19 41.5	20 38	21	19 39.5	20 34	26	19 30	20 52	23	19 40	21 3	28
20 20	21 54	24	20 19.5	21 43	24	20 11	21 37	22	20 1	21 35	25	19 59.5	21 41	26	20 12	21 52	25
20 50	22 39	22	20 49	22 32	21	20 41	22 27	20	20 31	22 20	28	20 29	22 22	23	20 40	22 40	23
21 21	23 16.5	22	21 19	23 9	23	21 10	23 9	22	21 1	23 15	25	21 0.5	23 9	26	21 13.5	23 18.5	25
21 51	23 53	21	21 51	23 53.5	23	21 40	23 53	25	21 30	23 50	23	21 31	23 47	24	21 42.5	23 45	19
22 20	24 22	20	22 21	24 27	21	22 10	24 26	23	22 0.5	24 28	29	22 1.5	24 17	23	22 11	24 14	22
22 50	24 51.5	21	22 51.5	24 55	22	22 40	24 58	21	22 32	24 53	25	22 31	24 43	22	22 41.5	24 45	22
23 20	25 22	22	23 19	25 23	21	23 9.5	25 24	26	23 2	25 20	24	22 59	25 10	19	23 10.5	25 10	20
23 51	25 46	24	23 49	25 44	22	23 39	25 50	22	23 31	25 40	22	23 28	25 30	23	23 41	25 34	21

Table II. (Interpolated from Table I.)

Showing the Interval between the Moon's Transit and the time of High Water at the London Docks for every month in the year.

Moon's Transit. Mean Solar Time.	Jan. High Water.	Feb. High Water.	March. High Water.	April. High Water.	May. High Water.	June. High Water.	July. High Water.	Aug. High Water.	Sept. High Water.	Oct. High Water.	Nov. High Water.	Dec. High Water.	Mean. High Water.
P.M.													
h m	h m	h m	h m	h m	h m	h m	h m	h m	h m	h m	h m	h m	h m
0 0	1 53	1 57	2 3	2 5	1 57	1 51	1 54	1 56	2 7	2 6	1 52	1 50	1 57
0 30	1 44	1 51	2 1	1 56	1 48	1 44	1 48	1 57	1 59	1 56	1 40	1 43	1 50
1 0	1 39	1 47	1 52	1 47	1 39	1 36	1 41	1 51	1 52	1 47	1 34	1 33	1 43
1 30	1 35	1 39	1 41	1 39	1 31	1 28	1 34	1 42	1 45	1 36	1 24	1 27	1 35
2 0	1 25	1 32	1 37	1 30	1 22	1 21	1 26	1 36	1 37	1 28	1 15	1 16	1 26
2 30	1 21	1 26	1 26	1 22	1 12	1 13	1 19	1 29	1 27	1 19	1 7	1 8	1 19
3 0	1 12	1 18	1 17	1 14	1 2	1 7	1 13	1 22	1 19	1 6	0 59	1 3	1 11
3 30	1 8	1 13	1 6	1 1	0 56	1 0	1 10	1 14	1 9	0 57	0 49	0 55	1 3
4 0	1 4	1 7	0 57	0 53	0 48	0 55	1 6	1 9	1 2	0 46	0 42	0 50	0 57
4 30	0 59	0 58	0 52	0 43	0 43	0 52	1 1	1 3	0 50	0 39	0 39	0 46	0 50
5 0	0 56	0 52	0 42	0 35	0 36	0 51	1 0	0 55	0 45	0 32	0 36	0 44	0 45
5 30	1 0	0 47	0 35	0 30	0 35	0 52	0 58	0 50	0 39	0 26	0 36	0 49	0 43
6 0	0 57	0 39	0 27	0 27	0 36	0 55	1 0	0 47	0 32	0 22	0 36	0 57	0 41
6 30	0 54	0 46	0 27	0 29	0 41	1 2	1 3	0 46	0 29	0 29	0 42	1 3	0 44
7 0	1 3	0 44	0 29	0 39	0 58	1 12	1 5	0 47	0 32	0 37	1 1	1 9	0 51
7 30	1 8	0 53	0 34	0 58	1 19	1 26	1 18	0 57	0 46	1 6	1 23	1 24	1 6
8 0	1 19	1 0	0 57	1 21	1 36	1 37	1 27	1 10	1 5	1 25	1 42	1 41	1 22
8 30	1 36	1 17	1 23	1 45	1 55	1 48	1 36	1 26	1 30	1 43	1 58	1 49	1 39
9 0	1 46	1 37	1 45	2 3	2 8	1 58	1 45	1 44	1 55	2 4	2 13	2 0	1 55
9 30	1 56	1 53	2 0	2 14	2 15	2 3	1 53	1 57	2 5	2 20	2 20	2 8	2 5
10 0	2 3	2 5	2 10	2 16	2 15	2 5	1 57	2 6	2 13	2 21	2 14	2 8	2 9
10 30	2 4	2 5	2 13	2 16	2 11	2 5	2 1	2 7	2 18	2 19	2 16	2 11	2 10
11 0	2 2	2 11	2 9	2 12	2 8	2 0	2 0	2 7	2 13	2 14	2 11	2 9	2 8
11 30	1 56	2 6	2 5	2 8	2 3	1 56	1 55	2 2	2 9	2 12	2 3	1 59	2 3
A.M.													
0 0	1 51	2 1	1 59	2 2	1 54	1 49	1 52	1 59	2 6	2 2	1 56	1 51	1 57
0 30	1 46	1 51	1 54	1 52	1 48	1 41	1 43	1 55	1 58	1 55	1 49	1 44	1 50
1 0	1 38	1 44	1 44	1 45	1 39	1 34	1 41	1 49	1 49	1 44	1 34	1 37	1 41
1 30	1 31	1 38	1 40	1 36	1 31	1 28	1 34	1 44	1 41	1 38	1 27	1 31	1 35
2 0	1 24	1 27	1 32	1 27	1 19	1 20	1 24	1 37	1 32	1 28	1 16	1 22	1 27
2 30	1 20	1 23	1 24	1 19	1 9	1 12	1 20	1 28	1 24	1 16	1 8	1 10	1 18
3 0	1 13	1 16	1 15	1 9	1 2	1 5	1 14	1 20	1 17	1 8	1 0	1 3	1 10
3 30	1 8	1 9	1 7	0 57	0 54	0 59	1 9	1 14	1 7	0 53	0 50	0 59	1 2
4 0	1 8	1 0	0 58	0 47	0 47	0 55	1 5	1 7	0 58	0 47	0 41	0 55	0 56
4 30	1 5	0 56	0 51	0 42	0 42	0 53	1 4	1 0	0 52	0 40	0 42	0 52	0 52
5 0	1 1	0 46	0 41	0 33	0 38	0 51	1 2	0 55	0 42	0 29	0 33	0 51	0 45
5 30	0 58	0 40	0 33	0 28	0 37	0 53	1 0	0 51	0 35	0 25	0 34	0 51	0 42
6 0	0 59	0 40	0 32	0 28	0 39	0 56	1 0	0 46	0 31	0 27	0 39	0 53	0 43
6 30	0 58	0 33	0 30	0 30	0 49	1 1	1 1	0 44	0 30	0 30	0 42	1 0	0 44
7 0	1 8	0 38	0 30	0 40	1 1	1 13	1 6	0 47	0 33	0 39	0 59	1 6	0 52
7 30	1 14	0 39	0 40	0 59	1 12	1 31	1 17	0 56	0 49	1 4	1 22	1 18	1 5
8 0	1 27	0 52	0 57	1 17	1 39	1 40	1 29	1 13	1 15	1 34	1 41	1 34	1 23
8 30	1 33	1 12	1 27	1 37	1 57	1 50	1 39	1 31	1 39	1 49	1 53	1 53	1 40
9 0	1 51	1 30	1 50	1 59	2 9	2 1	1 51	1 45	1 55	2 14	2 9	2 4	1 57
9 30	2 0	1 40	2 0	2 12	2 16	2 8	1 58	1 54	2 8	2 20	2 16	2 4	2 5
10 0	2 2	1 59	2 13	2 16	2 19	2 8	2 2	2 4	2 15	2 27	2 16	2 4	2 10
10 30	2 3	2 1	2 18	2 16	2 18	2 6	2 2	2 5	2 17	2 21	2 12	2 3	2 10
11 0	2 2	2 4	2 12	2 16	2 16	2 2	2 2	2 4	2 16	2 18	2 11	2 1	2 9
11 30	1 59	2 1	2 9	2 11	2 8	1 57	2 0	2 1	2 12	2 9	2 2	1 55	2 4

Table III.

Formed from the preceding by taking the mean of the interval when the Moon passes the meridian at any given time, and twelve hours later.

Moon's Transit. Mean Solar Time.	Jan. High Water.	Feb. High Water.	March. High Water.	April. High Water.	May. High Water.	June. High Water.	July. High Water.	Aug. High Water.	Sept. High Water.	Oct. High Water.	Nov. High Water.	Dec. High Water.	Mean. A.
h m	h m	h m	h m	h m	h m	h m	h m	h m	h m	h m	h m	h m	h m
0 0	1 52	1 59	2 1	2 4	1 55	1 50	1 53	1 57	2 6	2 4	1 54	1 50	1 57
0 30	1 45	1 51	1 57	1 54	1 48	1 43	1 45	1 56	1 58	1 55	1 44	1 44	1 50
1 0	1 39	1 45	1 48	1 46	1 39	1 35	1 41	1 50	1 50	1 45	1 34	1 35	1 42
1 30	1 33	1 38	1 40	1 38	1 31	1 28	1 34	1 43	1 43	1 37	1 26	1 29	1 35
2 0	1 25	1 29	1 34	1 29	1 20	1 20	1 25	1 37	1 34	1 28	1 16	1 19	1 26
2 30	1 20	1 24	1 25	1 20	1 10	1 12	1 19	1 29	1 25	1 18	1 8	1 9	1 18
3 0	1 13	1 17	1 16	1 12	1 2	1 6	1 13	1 21	1 18	1 7	1 0	1 3	1 11
3 30	1 8	1 11	1 7	0 59	0 55	1 0	1 9	1 14	1 8	0 56	0 50	0 57	1 3
4 0	1 6	1 3	0 58	0 50	0 48	0 55	1 6	1 8	1 0	0 47	0 42	0 53	0 56
4 30	1 2	0 57	0 52	0 43	0 42	0 53	1 3	1 1	0 51	0 40	0 40	0 49	0 51
5 0	0 59	0 49	0 42	0 34	0 37	0 51	1 1	0 55	0 43	0 31	0 35	0 47	0 45
5 30	0 59	0 43	0 34	0 29	0 36	0 53	0 59	0 51	0 37	0 26	0 35	0 50	0 43
6 0	0 58	0 40	0 30	0 28	0 38	0 56	1 0	0 47	0 32	0 24	0 38	0 55	0 42
6 30	0 56	0 40	0 29	0 30	0 45	1 2	1 2	0 45	0 30	0 29	0 42	1 2	0 44
7 0	1 5	0 41	0 30	0 40	0 59	1 13	1 6	0 47	0 33	0 38	1 0	1 8	0 52
7 30	1 11	0 46	0 37	0 58	1 16	1 28	1 18	0 56	0 47	1 5	1 22	1 21	1 5
8 0	1 23	0 56	0 57	1 19	1 37	1 38	1 28	1 12	1 10	1 29	1 41	1 37	1 23
8 30	1 35	1 14	1 25	1 41	1 56	1 49	1 38	1 29	1 34	1 46	1 55	1 51	1 39
9 0	1 49	1 33	1 47	2 1	2 9	1 59	1 48	1 45	1 55	2 9	2 11	2 2	1 56
9 30	1 58	1 46	2 1	2 13	2 16	2 5	1 55	1 56	2 6	2 20	2 18	2 6	2 5
10 0	2 2	2 2	2 12	2 16	2 17	2 7	1 59	2 5	2 14	2 24	2 15	2 6	2 10
10 30	2 3	2 3	2 15	2 16	2 14	2 6	2 2	2 6	2 17	2 20	2 14	2 7	2 10
11 0	2 2	2 7	2 11	2 14	2 12	2 1	2 1	2 6	2 15	2 16	2 11	2 5	2 8
11 30	1 57	2 3	2 7	2 10	2 5	1 57	1 58	2 2	2 10	2 10	2 3	1 57	2 3

Table IV.

Showing the Height of High Water at the London Docks corresponding to the mean time of the Moon's Transit in each month of the year; from 6638 observations made between the 1st of January 1808 and the 31st of December 1826.

January.			February.			March.			April.			May.			June.		
Moon's Transit. P.M.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	Height of Tide.	No. of Obs.
h m	Feet.		h m	Feet.		h m	Feet.		h m	Feet.		h m	Feet.		h m	Feet.	
0 24	22	22	0 26	22.73	19	0 23.5	22.87	26	0 17.5	22.8	23	0 12	22.75	23	0 16	22.47	19
0 54.5	22.33	24	0 58	22.69	25	0 53	22.8	26	0 47	23.03	20	0 43.5	22.76	21	0 44	22.4	20
1 24	22.42	20	1 27	22.75	25	1 23.5	23	22	1 15	22.97	23	1 12.5	22.95	21	1 14	22.64	21
1 55	22.79	24	1 59	22.7	21	1 53.5	22.77	28	1 46	22.85	23	1 42	22.89	22	1 46	22.5	23
2 24	22.6	27	2 28.5	22.98	25	2 24	22.79	23	2 16.5	23	22	2 11	22.98	22	2 16	22.58	20
2 55	22.71	22	3 0	22.72	25	2 55	22.42	25	2 47	22.83	20	2 42	22.52	21	2 45	22.5	22
3 24	22.46	27	3 30	22.39	24	3 25	22.5	24	3 14	22.42	19	3 10.5	22.45	20	3 14	22.18	22
3 54	22.19	27	3 59	22.23	20	3 55.5	22.4	22	3 43	22.17	19	3 42	22.09	26	3 45	22.1	24
4 26	21.92	24	4 28	21.81	23	4 26	21.83	24	4 13	21.83	23	4 11	21.78	22	4 15	22	23
4 55	21.7	25	5 3	21.3	25	4 56	21.44	20	4 43	21.5	19	4 42	21.33	21	4 46	21.36	26
5 25	21.06	26	5 31.5	21.17	22	5 24	21.17	20	5 11.5	21.02	21	5 11	20.7	25	5 17	21.02	24
5 56	21	28	6 0	20.46	19	5 26	20.56	24	5 44	20.28	22	5 43	20.13	24	5 46	20.68	23
6 26	20.15	23	6 29	19.6	21	6 24	19.92	21	6 16	19.88	25	6 10	19.98	20	6 16	20.47	25
6 55	19.85	24	7 0.5	19.37	19	6 55	19.3	23	6 47	19.08	20	6 39	19.83	26	6 46	20.2	25
7 24	19.73	24	7 29	18.5	19	7 25.5	19.3	21	7 16.5	18.92	24	7 11	19.74	28	7 15	19.73	22
7 56	19.25	25	7 58	19.1	18	7 55	18.81	23	7 49	18.94	23	7 41.5	19.74	14	7 46	19.86	27
8 26	19.85	21	8 28	18.94	22	8 24.5	19.25	23	8 15.5	19.25	21	8 12.5	20.15	26	8 17	20.06	23
8 56	19.89	23	8 58	19.3	16	8 55	19.3	22	8 41.5	19.48	23	8 44	20.28	27	8 46	20.56	23
9 27	20.23	20	9 28.5	19.85	23	9 24	20.06	22	9 15	20.4	26	9 12.5	20.99	21	9 16	20.7	23
9 54	20.37	23	9 59.5	20.2	18	9 54	20.46	25	9 45	21	24	9 40.5	21.19	24	9 46	21.08	19
10 25	21.08	18	10 29	20.97	21	10 24	21.04	22	10 16	21.51	24	10 11	21.72	25	10 15	21.28	23
10 53	21.44	21	11 0.5	21.15	22	10 54	21.52	25	10 45	22	25	10 40.5	22.28	22	10 45	21.75	20
11 24	21.5	24	11 30	21.21	18	11 25	22.12	27	11 15.5	22.69	22	11 11	22.39	25	11 14	22.12	19
11 56	22.35	21	11 59	22.22	21	11 54.5	22.34	21	11 46	22.56	26	11 41	22.75	11	11 45	22.14	22

July.			August.			September.			October.			November.			December.		
Moon's Transit. P.M.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	Height of Tide.	No. of Obs.
h m	Feet.		h m	Feet.		h m	Feet.		h m	Feet.		h m	Feet.		h m	Feet.	
0 21	22.3	19	0 19	22.28	22	0 9	22.67	26	0 0	22.97	24	23 59	22.67	22	0 13	22.75	22
0 49	22.43	22	0 51.5	22.5	25	0 40	22.77	25	0 30	23	23	0 29	23.08	20	0 42	22.11	19
1 20	22.57	23	1 19	22.73	20	1 10.5	22.92	24	1 1	23.12	25	0 59	23.04	22	1 9	22.26	20
1 50	22.72	23	1 48	22.57	26	1 41	22.83	24	1 31	23.11	23	1 31	23	21	1 40	22.46	22
2 20	22.69	25	2 19	22.78	25	2 10	23.13	20	2 2	22.97	21	2 1	22.92	20	2 11	22.86	23
2 51	22.6	25	2 48	22.75	27	2 39.5	22.91	27	2 32	22.58	25	2 31	22.53	22	2 41	22.46	22
3 21	22.5	24	3 20.5	22.59	24	3 10	22.82	20	3 3	22.58	20	3 1	22.8	20	3 11	22.29	23
3 51	22.44	27	3 49.5	22.57	23	3 39	22.69	22	3 31	22.62	20	3 31	22.35	22	3 41	22.01	24
4 21	22.22	23	4 19	21.6	25	4 9	22.13	21	4 0	21.94	21	4 3	21.9	23	4 11.5	21.95	26
4 50	21.75	26	4 50	21.76	25	4 40	21.7	25	4 29	21.61	22	4 33	21.52	21	4 42	21.43	25
5 21	21.4	28	5 20.5	21.29	22	5 12.5	21.08	21	5 1	20.84	24	5 0	20.9	20	5 11.5	21.23	24
5 52	21	25	5 49	20.99	22	5 41.5	20.7	19	5 31	20.64	22	5 29	20.6	23	5 41	20.78	28
6 21	20.67	22	6 18	20.28	22	6 10.5	20.08	21	6 1	19.69	20	5 59	19.9	23	6 12	20.28	26
6 50	20.29	27	6 49	19.92	24	6 41	19.62	22	6 32	19.85	25	6 31	19.52	26	6 42	20.04	24
7 21	19.97	25	7 19	19.66	21	7 11.5	18.94	21	7 4	19.22	23	7 1	19.17	24	7 11	19.68	26
7 53	20.04	24	7 48	19.39	22	7 42	19.23	22	7 31.5	19.04	21	7 29	19.57	23	7 41	19.54	23
8 23	20.02	21	8 19	19.31	24	8 13	18.84	21	7 59.5	19.14	22	8 1	19.63	26	8 10	19.74	25
8 51	20.23	20	8 50	19.63	23	8 41	19.58	22	8 29	19.82	25	8 31	19.65	24	8 43	20.5	26
9 23	20.28	23	9 20.5	19.89	22	9 11.5	19.94	23	8 59.5	20.18	27	9 0	20.65	25	9 11	20.79	22
9 53	20.6	21	9 50	20.55	21	9 42	20.42	23	9 31	21.19	24	9 30	20.73	25	9 41	21.3	23
10 23	20.9	21	10 18	20.73	20	10 12	21	23	10 0	21.38	26	10 0	21.17	23	10 12	21.28	23
10 53	21.28	20	10 48	21.14	22	10 41.5	21.64	23	10 30	22.08	26	10 31	21.83	25	10 41.5	21.62	20
11 23	21.67	23	11 18	21.73	23	11 9	22.08	21	11 1	22.19	25	11 1	22.47	22	11 10	21.77	21
11 53	21.84	21	11 48.5	22.03	23	11 40	22.31	17	11 31	22.48	25	11 30	22.58	21	11 39	22.06	20

TABLE V. (Interpolated from Table IV.)
Showing the Height of High Water at the London Docks.

	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Mean.
Moon's Transit.	Height of Tide.	Height of Tide.	Height of Tide.	Height of Tide.	Height of Tide.	Height of Tide.	Height of Tide.	Height of Tide.	Height of Tide.	Height of Tide.	Height of Tide.	Height of Tide.	B.
h m	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.
0 0	22.30	22.23	22.44	22.67	22.75	22.30	21.95	22.13	22.56	22.97	22.68	22.49	22.46
0 30	22.07	22.73	22.85	22.90	22.75	22.43	22.34	22.35	22.74	23.00	23.08	22.37	22.63
1 0	22.36	22.70	22.85	23.00	22.87	22.53	22.48	22.56	22.87	23.12	23.04	22.21	22.72
1 30	22.47	22.75	22.95	22.91	22.91	22.57	22.62	22.67	22.86	23.11	23.00	22.39	22.76
2 0	22.76	22.71	22.78	22.92	22.95	22.54	22.71	22.65	23.03	22.98	22.92	22.72	22.80
2 30	22.62	22.97	22.72	22.93	22.70	22.54	22.66	22.77	22.96	22.60	22.54	22.61	22.72
3 0	22.67	22.72	22.43	22.63	22.47	22.34	22.57	22.69	22.85	22.58	22.79	22.35	22.59
3 30	22.41	22.39	22.48	22.28	22.23	22.14	22.48	22.49	22.73	22.62	22.36	22.11	22.39
4 0	22.14	22.22	22.29	21.98	21.89	22.05	22.37	22.08	22.30	21.94	21.94	21.97	22.10
4 30	21.89	21.79	21.78	21.64	21.49	21.69	22.07	21.66	21.84	21.61	21.56	21.64	21.72
5 0	21.59	21.34	21.40	21.22	20.94	21.21	21.63	21.60	21.32	20.86	20.90	21.31	21.28
5 30	21.05	21.18	21.11	20.59	20.37	20.86	21.28	21.19	20.85	20.65	20.58	20.95	20.88
6 0	20.89	20.46	20.46	20.08	20.03	20.58	20.91	20.72	20.30	19.72	19.89	20.47	20.37
6 30	20.11	19.59	19.80	19.52	19.87	20.34	20.55	20.14	19.79	19.84	19.53	20.14	19.93
7 0	19.83	19.37	19.30	19.01	19.79	19.98	20.19	19.82	19.20	19.30	19.18	19.82	19.56
7 30	19.64	18.68	19.23	18.93	19.74	19.79	19.99	19.56	19.11	19.05	19.57	19.59	19.40
8 0	19.33	19.09	18.88	19.07	19.98	19.95	20.03	19.36	19.00	19.15	19.63	19.67	19.43
8 30	19.86	18.96	19.26	19.38	20.22	20.28	20.07	19.42	19.29	19.83	19.65	20.20	19.70
9 0	19.93	19.32	19.43	19.98	20.68	20.62	20.24	19.72	19.80	20.20	20.65	20.68	20.10
9 30	20.24	19.86	20.14	20.70	21.12	20.88	20.35	20.04	20.23	21.16	20.73	21.11	20.54
10 0	20.51	20.20	20.58	21.25	21.53	21.18	20.67	20.48	20.81	21.38	21.17	21.29	20.92
10 30	21.14	20.98	21.14	21.75	22.08	21.51	20.99	20.89	21.39	22.08	21.81	21.49	21.43
11 0	21.45	21.45	21.64	22.34	22.35	21.94	21.37	21.37	21.94	22.19	22.45	21.72	21.85
11 30	21.66	21.21	22.16	22.63	22.63	22.11	21.71	21.85	22.24	22.47	22.58	21.97	22.10
	21.29	21.12	21.25	21.34	21.51	21.43	21.43	21.26	21.33	21.43	21.42	21.39	21.34

Table VI.

Showing the Time and the Height of High Water at the London Docks, corresponding to the mean time of the Moon's Transit for every minute of Horizontal Parallax, from 5414 observations made between the 1st of January 1808 and 31st of December 1826.

Hor. Par. 54′.				Hor. Par. 55′.				Hor. Par. 56′.				Hor. Par. 57′.			
Moon's Transit. P.M.	High Water.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	High Water.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	High Water.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	High Water.	Height of Tide.	No. of Obs.
h m	h m	ft. in.		h m	h m	ft. in.		h m	h m	ft. in.		h m	h m	ft. in.	
0 34	2 34.9	22 1	95	0 31.5	2 29.8	22 3.4	51	0 34	2 24.9	22 4.3	38	0 33.8	2 25.1	22 7.7	43
1 19.7	3 6.4	22 2.6	89	1 21.2	3 3.8	22 3.6	54	1 22.4	3 1.8	22 9.8	50	1 26	3 7	22 8.3	38
2 35.8	3 59.2	22 1.4	80	2 37.4	3 58.1	22 6.5	71	2 31.8	3 51	22 5.2	38	2 33.3	3 51.2	22 10	45
3 22.3	4 33.2	21 8.5	73	3 23.1	4 31.2	22 1.3	65	3 23	4 30	22 2.3	56	3 20.7	4 27	22 5.1	46
4 29.3	5 20.2	21 1	72	4 31.5	5 23.8	21 3.5	74	4 32.2	5 26.2	21 4.8	50	4 33	5 22.1	21 9.3	59
5 34.3	6 18	20 2.2	55	5 35.4	6 18	20 3	90	5 31.2	6 13.2	20 4	58	5 33.6	6 18	21 0.7	57
6 34.8	7 22	19 3.1	68	6 38	7 27.3	19 6.8	71	6 36.5	7 18.6	19 8.3	66	6 33.7	7 17.7	19 10.2	57
7 25.8	8 35	18 8.5	72	7 31.1	8 34.5	19 0.1	73	7 26.3	8 27.7	19 2.2	58	7 24.8	8 23.5	19 6	56
8 29.7	10 30	19 3.2	86	8 37.6	10 29	19 8	54	8 35	10 20	19 7.2	65	8 35.8	10 19.2	19 10.2	44
9 38.5	12 2.4	20 2.2	82	9 38.5	11 56.3	20 5.5	66	9 35.5	11 49.2	20 5.7	42	9 35.5	11 43	20 4.3	44
10 54.7	13 5	21 3	84	10 39.2	13 1.9	21 0.7	56	10 35.7	12 53.6	21 4	50	10 33.6	12 43	21 5	39
11 36.5	13 51.6	21 8	83	11 35.3	13 46.8	21 10.5	60	11 33	13 41.7	21 9.7	35	11 38	13 18.6	22 4.5	43

Hor. Par. 58′.				Hor. Par. 59′.				Hor. Par. 60′.				Hor. Par. 61′.			
Moon's Transit. P.M.	High Water.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	High Water.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	High Water.	Height of Tide.	No. of Obs.	Moon's Transit. P.M.	High Water.	Height of Tide.	No. of Obs.
h m	h m	ft. in.		h m	h m	ft. in.		h m	h m	ft. in.		h m	h m	ft. in.	
0 36.2	2 21.8	22 9.3	42	0 37.6	2 20.9	23 0	33	0 31.8	2 12.5	22 11.1	52	0 32.2	2 10.4	23 1.5	92
1 27.3	3 2	23 0	37	1 28	3 1	23 0	42	1 30.8	2 56	23 2.1	66	1 28.2	2 53.2	23 6.5	78
2 32.6	3 51.8	22 9.7	50	2 30	3 46	23 0.9	48	2 34	3 45.4	23 5.2	72	2 28.3	3 39.3	23 6.3	43
3 24	4 30.8	22 5.2	46	3 24.6	4 26.3	22 10.4	73	3 28.1	4 26.4	23 3.4	90	3 29	4 32.5	21 2.5	2
4 32.3	5 22.3	22 2.6	55	4 32.3	5 20.6	22 3.7	114	4 33.7	5 8	22 5.4	27				
5 31.7	6 13	21 2.4	73	5 31.3	6 12.8	21 6.8	119								
6 30.4	7 11	20 3	57	6 34	7 15.6	20 5.2	121	6 33.2	7 22	20 3.2	8				
7 28.2	8 25.4	19 7.8	64	7 26.6	8 24	19 10.1	88	7 36	8 36.4	20 0.6	40				
8 31.2	10 4	19 7.2	49	8 32	10 6.7	20 2.8	56	8 34.8	10 4.2	20 3.5	99				
9 40	11 40	20 7.5	46	9 35.5	11 30	20 11	51	9 34.6	11 23.2	21 0.7	98	9 26.3	11 27	21 5	22
10 33.2	12 41	21 8.2	43	10 37.2	12 37	21 9	42	10 33.8	12 29.8	21 10.3	63	10 32.3	12 25.2	22 1.8	73
11 38.8	13 38	22 2.8	39	11 31.6	13 30	22 2.2	43	11 35.9	13 22	22 6.5	50	11 34.7	13 20.7	22 9	97

Table VII. (Interpolated from Table VI.)

Showing the Interval between the Moon's Transit and the Time of High Water at the London Docks, for every minute of her Horizontal Parallax.

Moon's Transit.	H. P. 54′.	H. P. 55′.	H. P. 56′.	H. P. 57′.	H. P. 58′.	H. P. 59′.	H. P. 60′.	H. P. 61′.
h m	h m	h m	h m	h m	h m	h m	h m	h m
0 0	2 9	2 5.8	2 1	1 56.6	1 53.9	1 53	1 43.5	1 42.6
0 30	2 1.9	1 58.8	1 51.6	1 52	1 47	1 44.9	1 41.1	1 38.6
1 0	1 52.6	1 49.7	1 45	1 46.8	1 40.8	1 38.6	1 33.4	1 31.5
1 30	1 43.4	1 39.7	1 37.6	1 40	1 34.1	1 32.5	1 25.4	1 24.4
2 0	1 34.2	1 31.8	1 28.6	1 29.6	1 27	1 25	1 18.6	1 17
2 30	1 25	1 23	1 19.7	1 19	1 19.8	1 16	1 12.4	1 10.7
3 0	1 16.9	1 14.8	1 12.3	1 10.5	1 12.6	1 8.3	1 5.2	
3 30	1 8.8	1 6.8	1 6	1 4	1 6.3	1 0.4	0 57.4	
4 0	0 59.8	0 59.2	0 58.6	0 55.6	0 58	0 54	0 50	
4 30	0 50.8	0 52.4	0 54.2	0 49.6	0 50.1	0 48.6	0 42.7	
5 0	0 46.3	0 46.6	0 47.6	0 46	0 45	0 44		
5 30	0 44	0 42.7	0 42	0 44.5	0 41.4	0 41.8		
6 0	0 43	0 43.4	0 39.5	0 43.4	0 39.8	0 40.4		
6 30	0 46.2	0 47.9	0 41	0 43.6	0 41	0 41.2		
7 0	0 56.2	0 53.7	0 49	0 49.6	0 47	0 47		
7 30	1 14.6	1 3.1	1 3.3	1 1.5	0 58	0 58.8	0 58.4	
8 0	1 38.2	1 22	1 22	1 18.8	1 14.8	1 16.6	1 11.6	
8 30	2 0.5	1 47	1 42.6	1 40.2	1 32	1 33.8	1 27.6	
9 0	2 14.6	2 4.8	2 0	1 56.6	1 46.4	1 45.2	1 39.2	
9 30	2 22.3	2 15.7	2 12.5	2 6.3	1 57.2	1 53.5	1 47.5	
10 0	2 26	2 21.5	2 17.6	2 10	2 4.8	1 58	1 53.2	
10 30	2 25.4	2 23.2	2 18.2	2 9.6	2 7.8	2 0	1 56	1 53.2
11 0	2 21.8	2 20	2 15.5	2 6.2	2 6	2 0.5	1 54.3	1 50
11 30	2 16.2	2 12.7	2 9.4	2 1.8	2 0.8	1 58.8	1 47	1 46.7

Table VIII. (Interpolated from Table VI.)

Showing the Height of High Water at the London Docks for every minute of the Moon's Horizontal Parallax.

Moon's Transit.	H. P. 54′.	H. P. 55′.	H. P. 56′.	H. P. 57′.	H. P. 58′.	H. P. 59′.	H. P. 60′.	H. P. 61′.
h m	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.
0 0	21.85	22.04	22.07	22.47	22.43	22.60	22.70	22.90
0 30	22.06	22.28	22.36	22.64	22.73	22.97	22.92	23.09
1 0	22.16	22.30	22.65	22.67	22.90	22.98	23.05	23.35
1 30	22.22	22.31	22.80	22.70	23.00	23.00	23.18	23.54
2 0	22.24	22.29	22.66	22.76	22.93	23.05	23.31	23.56
2 30	22.19	22.28	22.45	22.83	22.81	23.07	23.42	23.52
3 0	21.87	22.18	22.30	22.61	22.60	22.97	23.38	
3 30	21.64	22.04	22.14	22.35	22.40	22.83	23.27	
4 0	21.37	21.66	21.82	22.08	22.14	22.60	22.92	
4 30	21.08	21.28	21.44	21.80	22.24	22.33	22.40	
5 0	20.70	20.80	20.90	21.45	21.73	22.00		
5 30	20.25	20.32	20.35	21.09	21.22	21.58		
6 0	19.78	19.95	20.04	20.50	20.73	21.00		
6 30	19.31	19.64	19.76	19.90	20.25	20.47	20.28	
7 0	18.95	19.30	19.48	19.64	19.90	20.13	20.12	
7 30	18.75	19.02	19.22	19.51	19.65	19.83	20.08	
8 0	18.90	19.28	19.33	19.63	19.60	19.50	20.12	
8 30	19.27	19.54	19.55	19.82	19.61	20.23	20.25	
9 0	19.90	19.93	19.95	20.05	19.93	19.40	20.55	
9 30	20.58	20.31	20.40	20.29	20.16	20.85	21.00	21.42
10 0	20.85	20.64	20.83	20.80	21.08	21.30	21.43	21.63
10 30	21.13	20.96	21.26	21.36	21.65	21.69	21.81	21.88
11 0	21.38	21.38	21.55	21.90	21.95	21.95	22.20	22.30
11 30	21.60	21.81	21.78	22.30	22.16	22.17	22.49	22.71

Table IX.

Showing the Time and Height of High Water at the London Docks corresponding to the mean time of the Moon's Transit for every three degrees of her declination north and south; from 5372 observations made between the 1st of January 1808 and the 31st of December 1826.

1° 30′ S. to 1° 30′ N. Decl.

Moon's Transit.	High Water.	Height of Tide.	No. of Obs.
h m	h m	ft. in.	
0 35	2 32.5	22 9.5	22
1 26	3 7	23 1	21
2 42	4 4	23 1	17
3 26	4 40.5	22 7.8	22
4 35	5 33	21 10	18
5 36	6 36.7	21 0	21
6 36.5	7 43.5	19 9	17
7 22	8 40.5	19 11	17
8 32	10 27	19 11	23
9 39	11 59	21 4	13
10 31	12 52	21 9	21
11 36	13 46	22 3	21

1° 30′ to 4° 30′ North Dec.

Moon's Transit.	High Water.	Height of Tide.	No. of Obs.
h m	h m	ft. in.	
0 38	2 35	22 8.5	19
1 27	3 12	22 9	18
2 41	4 2	22 8	12
3 28	4 41	22 6	22
4 37	5 37	21 8.5	19
5 32	6 31	20 5	10
6 36.5	7 46	20 3.5	20
7 19	8 32.5	20 1	18
8 28	10 17	20 4	17
9 29	11 37	21 1	18
10 35	12 56	21 1	15
11 28	13 32	22 4	14

4° 30′ to 7° 30′ North Dec.

Moon's Transit.	High Water.	Height of Tide.	No. of Obs.
h m	h m	ft. in.	
0 36	2 37.5	22 8	20
1 30	3 11	22 11	23
2 42	4 3	22 8	20
3 26	4 42	22 2	15
4 42	5 38	22	19
5 34	6 28	21 4.5	15
6 31	7 27.5	20 2	14
7 20.2	8 34	19 9	19
8 31	10 21	19 11	21
9 35	11 52	20 8	22
10 38	12 53	22	23
11 38	13 44	22 4	18

7° 30′ to 10° 30′ North Dec.

Moon's Transit.	High Water.	Height of Tide.	No. of Obs.
h m	h m	ft. in.	
0 41	2 36	22 6	21
1 27	3 6	22 11	19
2 41	4 3	23 2	26
3 32	4 41	22 4.7	16
4 37	5 37	21 9.3	20
5 41	6 40	20 7.5	12
6 34.5	7 34.5	20	19
7 22.5	8 43	19 11.5	18
8 26	10 15	19 9	20
9 32	11 46	20 8	25
10 35	12 54	22	31
11 33	13 41	22 5	15

1° 30′ to 4° 30′ South Dec.

Moon's Transit.	High Water.	Height of Tide.	No. of Obs.
0 36	2 29	22 7	18
1 28	3 10	22 6	18
2 39	4 4.7	22 3	25
3 30	4 46	22 6.5	16
4 35	5 33	21 5.5	18
5 29	6 22.7	21 1.7	15
6 37	7 35	20 2.5	18
7 18.5	8 43.5	19 6.5	17
8 24	10 21	20 5	13
9 37	11 53	21 2	18
10 45	12 59	22 1	17
11 39	13 46	22 7	20

4° 30′ to 7° 30′ South Dec.

Moon's Transit.	High Water.	Height of Tide.	No. of Obs.
0 34	2 28	22 9	17
1 30	3 13	22 11	20
2 39	4 0	22 11	16
3 21	4 32	22 8	15
4 39	5 39	21 8	15
5 40	6 36	21 4	15
6 40.5	7 40	20 2	17
7 19	8 42	19 7	17
8 37	10 39	20 5	24
9 30	11 45	20 10	14
10 31	12 42	21 10	20
11 31	13 39	22 5	26

7° 30′ to 10° 30′ South Dec.

Moon's Transit.	High Water.	Height of Tide.	No. of Obs.
0 33	2 28	22 8.5	21
1 30	3 6	22 9	23
2 36	4 0	22 10	17
3 20	4 33	22 4	23
4 34	5 31	21 9	14
5 32	6 24	21 0.3	20
6 36	7 34	20 4	19
7 20	8 38	20 2	20
8 36	10 31	20 1	19
9 36	11 47	21 1	24
10 35	12 49	21 10	18
11 42	13 45	22 5	19

Table IX. (Continued).

10° 30′ to 13° 30′ North Dec.				13° 30′ to 16° 30′ North Dec.				16° 30′ to 19° 30′ North Dec.			
Moon's Transit.	High Water.	Height of Tide.	No. of Obs.	Moon's Transit.	High Water.	Height of Tide.	No. of Obs.	Moon's Transit.	High Water.	Height of Tide.	No. of Obs.
h m	h m	ft. in.		h m	h m	ft. in.		h m	h m	ft. in.	
0 38	2 25	22 10	17	0 37	2 27	22 6.7	28	0 32	2 21	22 7.2	37
1 30	3 8	22 8	27	1 28	3 8	22 9	23	1 23	2 58	22 11	33
2 39	3 56	22 11	25	2 37	3 57	22 7	21	2 31	3 52	23 0	39
3 28	4 37	22 8	24	3 28	4 35	22 5.5	24	3 27	4 31	22 5.5	45
4 37	5 34	21 5.5	21	4 34	5 23	21 7.5	30	4 36	5 28	21 9.5	36
5 35.5	6 25	20 9	26	5 38	6 18	21 2	24	5 37.5	6 13.2	20 8.5	31
6 34.5	7 24	20 0.5	26	6 36.5	7 20.5	20 1	20	6 42	7 22.5	19 7.7	34
7 30	8 41	19 6.5	21	7 28	8 37	19 3.3	27	7 37	8 38	19 1.3	40
8 32	10 19	20 7	26	8 35	10 20	20 1	22	8 41	10 22	19 7	38
9 37	11 54	21 1	19	9 37	11 42	20 7	31	9 42	11 45	20 5	40
10 34	12 46	21 9	19	10 40	12 53.5	21 8	29	10 38	12 44	21 4	44
11 37	13 43	22 8	29	11 35	13 43	21 10	28	11 37	13 39	22 2	36

10° 30′ to 13° 30′ South Dec.				13° 30′ to 16° 30′ South Dec.				16° 30′ to 19° 30′ South Dec.			
0 40.6	2 26.4	22 8.7	25	0 32	2 20	23 0	25	0 30	2 17	22 7.5	39
1 21	3 2	22 9	20	1 23	3 2	22 11	26	1 20	2 52	22 10	38
2 32	3 53	23 0	24	2 26	3 45	22 10.5	28	2 29	3 42	22 11	34
3 20	4 28	22 6.8	20	3 25.5	4 27	22 7	29	3 21	4 22	22 9.3	35
4 31	5 31	21 8.2	22	4 27	5 17	21 11.2	28	4 25	5 15	21 11	39
5 39	6 26.5	20 11	21	5 24	6 9	20 9	28	5 28	6 10.5	21 0.5	39
6 38	7 35	20 2.3	23	6 32	7 19	19 10.3	26	6 28	7 5	20 0.2	38
7 20	8 35	19 7	22	7 23	8 30	19 3	26	7 24	8 19	19 6	42
8 32	10 22	20 4	24	8 36	10 18	20 1	25	8 32	10 12	19 9	36
9 33	11 43	20 10	29	9 35	11 38	20 9	22	9 36	11 35	20 8	45
10 35	12 50	21 8	18	10 34	12 41	21 4.5	25	10 36	12 38	21 5	39
11 34	13 40	22 10	14	11 38	13 38	22 4	25	11 32	13 30	22 2	36

19° 30′ to 22° 30′ North Dec.				22° 30′ to 25° 30′ North Dec.				Above 25° 30′ North Dec.			
h m	h m	ft. in.		h m	h m	ft. in.		h m	h m	ft. in.	
0 28.5	2 19	22 7	26	0 31	2 16	22 6.5	25	0 31	2 13	22 5.5	17
1 28	2 56	22 9	30	1 24	2 57	22 9	26	1 25	2 51	22 8	18
2 33	3 49	22 6	23	2 33	3 49	22 7	24	2 29	3 40	23 0	22
3 25	4 25	22 5.2	24	3 23	4 19	22 3	22	3 20	4 15	22 4.5	23
4 32	5 14	21 9.5	28	4 35	5 14	21 4.7	26	4 27	5 5	21 8.3	23
5 35	6 15	20 5	30	5 35	6 4.5	20 8.8	25	5 40.7	6 10.2	20 8	23
6 37	7 10	19 9	32	6 43.5	7 9.7	19 3	22	6 35	7 1	19 1	23
7 34.5	8 33.3	18 5	27	7 38	8 23	19 0.5	23	7 41	8 15	18 10	22
8 36	10 10	19 4	19	8 41	10 3	19 5	29	8 40	10 0	18 10	20
9 41	11 34	20 3	26	9 40	11 28.7	19 10	21	9 47	11 42	20 4	19
10 38	12 43	21 2	20	10 39	12 42	21 4	24	10 37	12 38	20 11	18
11 33	13 32	22 3	30	11 37	13 26	21 8	17	11 33	13 25	21 9	21

19° 30′ to 22° 30′ South Dec.				22° 30′ to 25° 30′ South Dec.				Above 25° 30′ South Dec.			
0 31	2 13	22 4.7	27	0 34	2 15	22 6.8	16	0 26	2 4	22 5	22
1 24	2 54	23 1	23	1 17	2 47	22 8	30	1 26	2 48	21 11	16
2 25	3 37	22 7	31	2 29	3 38	22 6	21	2 21	3 32	22 6	21
3 22	4 20.5	22 7.8	28	3 19	4 13.5	21 11.3	24	3 20	4 11	22 2	21
4 26	5 14	21 8.7	31	4 19	5 0	21 9.5	22	4 15	4 47	21 9.5	24
5 23.5	6 0.5	20 9.5	28	5 24.4	6 0.4	20 11.2	28	5 23.8	5 50.3	20 6.7	26
6 30.5	7 8	20 3	28	6 30	6 59	19 8	22	6 23	6 43	19 9	26
7 20	8 15	19 4.5	23	7 24	8 11	19 4.5	24	7 25	7 58	18 5.3	23
8 31	10 8	19 7	28	8 39	10 2	19 4	27	8 33	9 55	19 4	18
9 37	11 44	20 5	26	9 34	11 33	20 3	20	9 33	11 29	20 1	21
10 39	12 43	21 3.5	28	10 34	12 40	21 3.5	27	10 39	12 32	20 10.5	19
11 34.5	13 25	22 4	27	11 35	13 22	21 11	25	11 35	13 25	21 8	21

Table X. (Interpolated from Table IX.)

Showing the Interval between the Moon's Transit and the Time of High Water, and the Height of High Water at the London Docks for every three degrees of her Declination, North and South.

Time of Moon's Transit.	0° Dec.		3° Dec. N.		6° Dec. N.		9° Dec. N.		12° Dec. N.	
	Interval.	Height.	Interval.	Height.	Interval.	Height.	Interval.	Height.	Interval.	Height.
h m	h m	ft. in.	h m	ft. in.	h m	ft. in.	h m	ft. in.	h m	ft. in.
0 30	1 58.9	22 8.6	1 58.4	22 7.8	2 2.8	22 7.7	1 58	22 5.8	1 49	22 9.7
1 30	1 40	23 1.2	1 44	22 9	1 41	22 11	1 38.5	22 11.2	1 38	22 8
2 30	1 24.5	23 1.3	1 24	22 8	1 23.7	22 8.3	1 24.7	23 1.5	1 19.3	22 10.5
3 30	1 10.7	22 7.3	1 13	22 6	1 15.2	22 2.4	1 9.4	22 5	1 8.7	22 7.8
4 30	0 59	21 10.7	1 0	21 9.8	0 58	22 0.2	1 0.5	21 10.4	0 58	21 6.8
5 30	1 0.2	21 1.2	0 59	20 5.3	0 54	21 4.8	1 0	20 9.7	0 50	20 9.8
6 30	1 5.1	19 10.1	1 10.4	20 3.7	0 56.1	20 2	0 59.1	20 0.3	0 49	20 0.1
7 30	1 10.5	19 11	1 11.3	20 0.1	1 9.3	19 9	1 24	19 11.3	1 11	19 6.5
8 30	1 55	19 11	1 49	20 4	2 0.3	19 11	1 51	19 9.5	1 46	20 6.7
9 30	2 18.2	21 2.4	2 8	21 1	2 16	20 7	2 13	20 7.8	2 15.6	21 0.2
10 30	2 21	21 9	2 21.2	22 0.4	2 16	21 10.7	2 19	21 11	2 12.3	21 8.4
11 30	2 11	22 2.4	2 4.6	22 4	2 7	22 3.5	2 8	22 4.9	2 7.3	21 7.3
			3° Dec. S.		6° Dec. S.		9° Dec. S.		12° Dec. S.	
0 30			1 54.4	22 7	1 55	22 8.8	1 56	22 8.3	1 47.5	22 8.6
1 30			1 41.4	22 6	1 43	22 11	1 36	22 9	1 39.2	22 9.2
2 30			1 18	22 3.4	1 23.4	22 11	1 25.2	22 9.8	1 21.5	23 0
3 30			1 16	22 6.5	1 9.5	22 7	1 10.6	22 11.9	1 6.3	22 5.5
4 30			0 59	21 6.2	1 1	22 9	0 57.6	21 9.4	1 0	21 8.1
5 30			0 53.7	21 1.7	0 56	21 5	0 55	21 0.2	0 47.5	21 0.3
6 30			0 56	20 3.7	0 56.5	20 0.8	0 56.3	20 4.3	0 55	20 3.5
7 30			1 31·5	19 6.7	1 29	19 8.5	1 22.7	20 1.8	1 19.3	19 7
8 30			2 0.3	20 5.8	1 59.3	20 4.2	1 52	20 1.2	1 49	20 3.8
9 30			2 15	21 1	2 15	20 10	2 9	21 0	2 9.4	20 9.6
10 30			2 15	21 11	2 11	21 10	2 14	21 9.3	2 15	21 7
11 30			2 8.6	22 6.5	2 8	22 5	2 5	22 4	2 7	22 10

Time of Moon's Transit.	15° Dec. N.		18° Dec. N.		21° Dec. N.		24° Dec. N.		27° Dec. N.	
	Interval.	Height.	Interval.	Height.	Interval.	Height.	Interval.	Height.	Interval.	Height.
h m	h m	ft. in.	h m	ft. in.	h m	ft. in.	h m	ft. in.	h m	ft. in.
0 30	1 51.8	22 5.5	1 49.5	22 7	1 50	22 7	1 45	22 6.4	1 42	22 5.4
1 30	1 39.5	22 9	1 33.4	22 11	1 28	22 9	1 31.6	22 8.9	1 24.7	22 8.2
2 30	1 22	22 7.2	1 21	23 0	1 17	22 6	1 17	22 7.1	1 11	23 0
3 30	1 6.6	22 5.4	1 3.3	22 5.3	0 51.4	22 4.9	0 54	22 2.4	0 52.3	22 3.2
4 30	0 50	21 8	0 53.2	21 10.5	0 42	21 9.9	0 40	21 5.4	0 37.5	21 7.8
5 30	0 40	21 3.2	0 36.7	20 10	0 40.3	20 6	0 29.1	20 10	0 30.7	20 10.6
6 30	0 42.5	20 2.3	0 38.3	19 9.6	0 32	19 10.4	0 24.2	19 3.6	0 26	19 3
7 30	1 10	19 3.3	0 57.5	19 1.5	0 56.6	18 5.4	0 41	19 0.6	0 29	18 10.2
8 30	1 43	20 0.4	1 38.6	19 5.6	1 31.4	19 3	1 13	19 4	1 13.7	18 10.6
9 30	2 3.3	20 6	2 0.5	20 3	1 50.3	20 1	1 58.7	19 10	2 2.6	20 2.6
10 30	2 13.5	21 6.8	2 3.7	21 2.5	2 5	21 1.2	2 4.6	21 3.3	2 1	20 10
11 30	2 9	21 9.6	2 3	22 1	1 59.4	22 3	1 50	21 7.2	1 52.5	21 8.6
	15° Dec. S.		18° Dec. S.		21° Dec. S.		24° Dec. S.		27° Dec. S.	
0 30	1 48	23 0	1 48	22 7.5	1 42	22 4.7	1 42	22 6.4	1 37	22 4.8
1 30	1 37	22 10.9	1 29	22 10.3	1 28.4	23 0.3	1 26.6	22 7.9	1 21	21 11.4
2 30	1 17.8	22 10.4	1 13	22 11	1 10.6	22 7	1 9	22 6	1 8.6	22 5.6
3 30	1 2.6	22 6.4	0 59	22 8.4	0 56.9	22 6.6	0 51.7	21 10.8	0 47.7	22 1.2
4 30	0 49.5	21 10.7	0 49.2	21 10.2	0 47.3	21 8	0 40	21 8.4	0 28.3	21 7.4
5 30	0 45	20 7.8	0 42.3	21 0.1	0 35.5	20 8.5	0 43	20 9.2	0 26	20 5.5
6 30	0 47	19 10.6	0 37	20 0	0 38	20 3	0 29	19 8	0 20.5	19 7.5
7 30	1 10	19 3.1	0 58	19 6.3	1 1.7	19 5.1	0 49.5	19 4.3	0 33	18 6.4
8 30	1 40	20 0.2	1 39	19 8.8	1 37	19 7	1 17	19 4.1	1 22	19 3
9 30	2 2	20 8.4	1 58	20 7	2 5.6	20 4.6	1 57.7	20 1.7	1 44	20 0
10 30	2 7.6	21 3.9	2 2	21 4.1	2 5	21 1.9	2 6.4	21 2.5	1 53	20 9
11 30	2 1.3	22 2.7	1 58	22 1.8	1 51.4	22 3.6	1 48.5	21 10.4	1 51	21 7.2

Table XI. (Interpolated from Table IX.)

Showing the Interval between the Moon's Transit and the Time of High Water at the London Docks, for every three degrees of her Declination, North or South.

Moon's Transit.	0	3° Dec.	6° Dec.	9° Dec.	12° Dec.	15° Dec.	18° Dec.	21° Dec.	24° Dec.	27° Dec.	Mean.
h m	h m	h m	h m	h m	h m	h m	h m	h m	h m	h m	h m
0 0	2 5	2 1.5	2 3.6	2 1.8	1 59	1 59	1 54.3	1 50.7	1 46.3	1 46.6	1 56.8
0 30	1 58.9	1 56.4	1 58.9	1 57	1 48.2	1 49.9	1 48.2	1 46	1 43.5	1 39.5	1 50.6
1 0	1 49.8	1 50.5	1 51.7	1 46.8	1 44	1 44.2	1 39.7	1 37.2	1 36.2	1 31	1 42.8
1 30	1 40	1 42.7	1 42	1 37.1	1 38.6	1 38.3	1 31.2	1 28.2	1 29	1 22.8	1 35
2 0	1 32	1 34	1 32.7	1 30.6	1 29.6	1 29.3	1 24	1 21.3	1 21.2	1 16.3	1 27.1
2 30	1 24.5	1 26	1 23.5	1 25	1 20.4	1 19.9	1 17	1 14.4	1 13	1 9.8	1 19.3
3 0	1 19.3	1 20	1 17	1 17.3	1 14.5	1 12.3	1 9	1 4.2	1 3.3	1 0	1 11.7
3 30	1 13.7	1 14	1 12.3	1 10	1 7.5	1 4.6	1 1.1	0 54.1	0 52.9	0 50	1 4
4 0	1 5.3	1 6.7	1 6	1 3.5	1 3	0 56.8	0 56.1	0 49.3	0 46	0 40.5	0 57.3
4 30	0 59	0 59.5	0 59.5	0 59	0 59	0 49.7	0 51.2	0 44.6	0 40	0 32.9	0 51.4
5 0	0 58	0 57.2	0 56.8	0 56.8	0 53.8	0 45	0 45.3	0 41.2	0 36	0 29.5	0 48
5 30	0 59.8	0 56.4	0 55	0 55.5	0 48.7	0 42.5	0 39.5	0 37.9	0 32.8	0 28.3	0 45.6
6 0	1 2	0 58.4	0 54.8	0 55.5	0 48	0 42.8	0 37	0 35.3	0 29.5	0 25.5	0 44.9
6 30	1 5.1	1 3.2	0 56.3	0 57.7	0 52	0 44.7	0 37.6	0 35	0 26.6	0 23.3	0 46.1
7 0	1 12.3	1 10.2	1 4	1 8.6	1 1.5	0 56	0 45.8	0 44	0 33	0 24.6	0 54
7 30	1 21.3	1 21.4	1 19.1	1 23.3	1 15.1	1 10	0 57.8	0 59.1	0 45.2	0 31	1 6.3
8 0	1 36	1 40	1 38	1 38	1 30	1 26.4	1 16	1 16	0 59	0 52	1 23.1
8 30	1 55	1 54.6	1 54.6	1 51.5	1 47.5	1 41.5	1 38.8	1 34.2	1 15	1 17.8	1 41
9 0	2 9	2 4.5	2 8	2 2.7	2 3	1 53.5	1 51	1 48	1 38	1 39	1 55.7
9 30	2 18.2	2 11.5	2 15.5	2 11	2 12.5	2 2.6	1 59.2	1 58	1 58.2	1 53.3	2 6
10 0	2 21.3	2 16	2 16	2 15.5	2 14.2	2 8	2 2.1	2 3.3	2 6.8	1 57	2 9.8
10 30	2 21	2 18.2	2 13.5	2 16.5	2 13.6	2 10.5	2 2.8	2 5	2 5.5	1 57	2 10.4
11 0	2 17.6	2 15	2 11	2 12.8	2 11.3	2 9.5	2 2.5	2 2.5	1 59.5	1 55	2 7.7
11 30	2 11	2 6.6	2 7.5	2 6.5	2 7.1	2 6.4	2 0.5	1 55.4	1 49.2	1 51.7	2 2.2

Table XII. (Interpolated from Table IX.)

Showing the Height of High Water at the London Docks for every three degrees of the Moon's Declination, North or South.

Moon's Transit.	0	3° Dec.	6° Dec.	9° Dec.	12° Dec.	15° Dec.	18° Dec.	21° Dec.	24° Dec.	27° Dec.	Mean.
h m	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.
0 0	22.36	22.53	22.52	22.48	22.50	22.40	22.36	22.38	22.12	22.04	22.37
0 30	22.70	22.62	22.69	22.58	22.77	22.73	22.60	22.48	22.53	22.43	22.61
1 0	22.95	22.62	22.83	22.71	22.70	22.79	22.74	22.75	22.65	22.37	22.71
1 30	23.10	22.63	22.92	22.84	22.72	22.83	22.88	22.89	22.70	22.31	22.78
2 0	23.20	22.55	22.90	22.95	22.83	22.79	22.95	22.77	22.65	22.55	22.81
2 30	23.11	22.47	22.80	22.97	22.94	22.73	22.96	22.54	22.54	22.73	22.78
3 0	22.86	22.49	22.62	22.86	22.80	22.63	22.80	22.50	22.31	22.50	22.64
3 30	22.61	22.52	22.39	22.70	22.55	22.49	22.56	22.48	22.05	22.19	22.45
4 0	22.24	22.09	22.64	22.30	22.10	22.18	22.22	22.13	21.78	21.92	22.16
4 30	21.89	21.67	21.88	21.82	21.62	21.78	21.86	21.75	21.49	21.64	21.74
5 0	21.49	21.20	21.65	21.35	21.27	21.37	21.40	20.20	21.16	21.20	21.23
5 30	21.10	20.79	21.41	20.92	20.92	20.96	20.92	20.60	20.80	20.67	20.91
6 0	20.50	20.52	20.88	20.55	20.53	20.50	20.40	20.32	20.18	20.00	20.44
6 30	19.84	20.30	20.12	20.20	20.15	20.04	19.90	20.04	19.48	19.44	19.95
7 0	19.75	19.54	19.87	20.12	19.82	19.60	19.56	19.50	19.32	19.03	19.61
7 30	19.92	19.78	19.72	20.04	19.55	19.27	19.32	18.94	19.21	18.70	19.44
8 0	19.91	20.03	19.87	20.00	19.90	19.60	19.36	19.08	19.23	18.82	19.58
8 30	19.92	20.41	20.13	19.95	20.44	20.06	19.60	19.42	19.33	19.06	19.83
9 0	20.53	20.75	20.42	20.25	20.68	20.35	20.00	19.82	19.60	19.59	19.20
9 30	21.20	21.08	20.71	20.82	20.91	20.60	20.42	20.23	19.98	20.11	20.61
10 0	21.50	21.50	21.29	21.36	21.28	20.98	20.86	20.66	20.55	20.48	21.05
10 30	21.75	21.97	21.87	21.85	21.64	21.45	21.28	21.14	21.24	20.79	21.50
11 0	22.00	22.26	22.16	22.16	21.98	21.90	21.70	21.71	21.55	21.24	21.97
11 30	22.22	22.44	22.36	22.38	22.22	22.02	22.12	22.28	21.72	21.66	22.14

Table XIII.

Showing the Time of High Water at the London Docks corresponding to the Mean Time of the Moon's superior and inferior Transits, and also the Heights of High Water, in the Month of June, from 1090 Observations, made between the 1st of January 1808 and 31st of December 1826.

Superior.				Inferior.			
Moon's Transit.	Time of High Water.	Height of Tide.	No. of Obs.	Moon's Transit.	Time of High Water.	Height of Tide.	No. of Obs.
h m	h m	ft. in.		h m	h m	ft. in.	
0 16	2 3.2	22 5	20	0 15	2 0	22 3.4	19
0 44	2 22.5	22 5	20	0 45	2 24.2	22 5	20
1 11.8	2 45	22 8	22	1 14.9	2 44.5	22 7.2	19
1 41.9	3 10.2	22 7.8	23	1 45	3 9.8	22 6.5	23
2 15.2	3 30.5	22 6.5	20	2 17.7	3 33.6	22 7.6	22
2 46.8	3 55.6	22 7.2	26	2 44	3 53.1	22 6.4	16
3 17.2	4 22.6	22 6.4	17	3 13	4 14.3	22 4.8	27
3 44.9	4 40.3	22 3.6	29	3 46.7	4 43	21 10.1	28
4 16.9	5 11.3	21 10.9	19	4 12.7	5 7.1	21 11.1	26
4 44.6	5 35.9	21 4.8	29	4 46.2	5 37.4	21 4.4	21
5 17.6	6 10.2	21 3.3	23	5 14.4	6 5.2	21 1.7	27
5 44.5	6 37.7	20 7.8	26	5 47.4	6 42.7	20 8.1	22
6 15.7	7 16.6	20 2.7	25	6 15.4	7 10.2	20 5.6	22
6 45.5	7 51.5	20 0.8	26	6 45.3	7 50.7	20 2.2	29
7 16	8 35.2	19 8.7	24	7 16.5	8 37	19 9.8	22
7 45.3	9 16	19 9.2	25	7 46.5	9 21.5	19 11	23
8 15.6	9 58.5	20 1.9	24	8 15.3	9 57.2	19 10.4	23
8 45.4	10 44.3	20 2.6	23	8 45.3	10 39.6	20 7.4	25
9 16	11 21	20 9.2	25	9 17.4	11 19.5	20 8.3	22
9 46.1	11 56	21 1.1	20	9 45.5	11 50.8	20 11.9	19
10 16.2	12 21.7	21 4.3	23	10 14	12 22.7	21 2.4	22
10 43	12 52.7	21 8.1	20	10 48.9	12 46.8	21 7.5	22
11 16.1	13 11.1	22 1.4	22	11 15.5	13 18.3	21 9.6	18
11 45.6	13 40.7	22 0.5	20	11 45.3	13 38.2	22 2.6	22

Table XIV. (Interpolated from Table XIII.)

Showing the Interval between the Moon's Transit and the Times of High Water, and the Height of High Water in the month of June; the Interval and Heights corresponding to the Moon's superior and inferior Transits being distinguished.

	Superior.		Inferior.		Difference.	
Moon's Transit.	High Water.	Height.	High Water.	Height.		
h m	h m	Feet.	h m	Feet.	m	
0 0	1 52	22.21	1 49	22.25	+3	−.04
0 30	1 43.8	22.42	1 42.1	22.35	+1.7	+.07
1 0	1 35.5	22.56	1 34.4	22.51	+1.1	+.05
1 30	1 30.3	22.80	1 27.2	22.57	+3.1	+.23
2 0	1 21.3	22.59	1 20.9	22.58	+0.4	+.01
2 30	1 12.2	22.56	1 12.8	22.58	−0.6	+.02
3 0	1 7.4	22.56	1 4.6	22.46	+2.8	+.1
3 30	1 1.4	22.39	0 58.7	22.12	+2.7	+.27
4 0	0 55	22.11	0 55.3	21.90	−0.3	+.21
4 30	0 52.9	21.66	0 52.7	21.64	+0.2	+.02
5 0	0 51.9	21.34	0 51	21.26	+0.9	+.08
5 30	0 52.9	20.99	0 53	20.92	−0.1	+.07
6 0	0 56.9	20.45	0 55	20.58	+1.9	−.13
6 30	1 3.3	20.15	1 0.1	20.33	+3.2	−.18
7 0	1 15.5	19.90	1 12.2	20.01	+3.3	−.11
7 30	1 29.9	19.74	1 27.2	19.87	+2.7	−.13
8 0	1 36.8	19.96	1 38.5	19.89	−1.7	+.07
8 30	1 50.8	20.19	1 48.1	20.25	+2.7	−.06
9 0	2 2	20.47	1 57.9	20.64	+4.1	−.17
9 30	2 7.5	20.92	2 3.7	20.84	+3.8	+.08
10 0	2 7.8	21.22	2 7	21.10	+0.8	+.12
10 30	2 7.7	21.53	2 3.8	21.39	+3.9	+.14
11 0	2 2.3	21.91	2 0.1	21.70	+2.2	+.21
11 30	1 55	22.08	1 57.8	22.00	−2.8	+.08

Table XV.

Showing the Difference in the Interval between the Time of the Moon's Transit and the Time of High Water, and the Mean Interval (Column A. Table III.) in different Months of the Year.

Moon's Transit.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
h m	m	m	m	m	m	m	m	m	m	m	m	m
0 0	− 5	+ 2	+ 4	+ 7	− 2	− 7	− 4	......	+ 9	+ 7	− 3	− 7
0 30	− 5	+ 1	+ 7	+ 4	− 2	− 8	− 5	+ 6	+ 8	+ 5	− 6	− 6
1 0	− 3	+ 3	+ 6	+ 4	− 3	− 7	− 1	+ 8	+ 8	+ 3	− 8	− 7
1 30	− 2	+ 3	+ 5	+ 3	− 4	− 7	− 1	+ 8	+ 8	+ 2	− 9	− 6
2 0	− 1	+ 3	+ 8	+ 3	− 6	− 6	− 1	+11	+ 8	+ 2	−10	− 7
2 30	+ 2	+ 6	+ 7	+ 2	− 8	− 6	0	+11	+ 7	0	−10	− 9
3 0	+ 2	+ 6	+ 5	+ 1	− 9	− 5	+ 2	+10	+ 7	− 4	−11	− 8
3 30	+ 5	+ 8	+ 4	− 4	− 8	− 4	+ 6	+11	+ 5	− 7	−13	− 6
4 0	+10	+ 7	+ 2	− 6	− 8	− 1	+10	+12	+ 4	− 9	−14	− 3
4 30	+11	+ 6	+ 1	− 8	− 9	+ 2	+12	+10	0	−11	−11	− 2
5 0	+13	+ 3	− 4	−12	− 9	+ 5	+15	+ 9	− 3	−15	−11	+ 1
5 30	+16	......	− 9	−14	− 7	+ 9	+16	+ 8	− 6	−17	− 8	+ 7
6 0	+16	− 2	−12	−14	− 4	+14	+18	+ 5	−10	−18	− 4	+13
6 30	+12	− 4	−15	−14	+ 1	+18	+18	+ 1	−14	−15	− 2	+18
7 0	+13	−11	−22	−12	+ 7	+21	+14	− 5	−19	−14	+ 8	+16
7 30	+ 5	−20	−29	− 8	+10	+22	+12	−10	−19	− 1	+16	+15
8 0	0	−27	−26	− 4	+14	+16	+ 5	−11	−13	+ 6	+18	+14
8 30	− 4	−25	−14	+ 2	+17	+10	− 1	−10	− 5	+ 7	+16	+12
9 0	− 7	−23	− 9	+ 5	+13	+ 3	− 8	−11	− 1	+13	+15	+ 6
9 30	− 7	−19	− 4	+ 8	+11	0	−10	− 9	+ 1	+15	+13	+ 1
10 0	− 8	− 8	+ 2	+ 6	+ 7	− 3	−11	− 5	+ 4	+14	+ 5	− 4
10 30	− 7	− 7	+ 5	+ 6	+ 4	− 4	− 8	− 4	+ 7	+10	+ 4	− 3
11 0	− 6	− 1	+ 3	+ 6	+ 4	− 7	− 7	− 2	+ 7	+ 8	+ 3	− 3
11 30	− 6	......	+ 4	+ 7	+ 2	− 6	− 5	− 1	+ 7	+ 7	......	− 3

Table XVI.

Showing the Difference in the Height of High Water and the Mean Height (Column B. Table V.) in different Months of the Year.

Moon's Transit.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
h m	m	m	m	m	m	m	m	m	m	m	m	m
0 0	−.16	−.23	−.02	+.21	+.29	−.16	−.51	−.33	+.10	+.51	+.22	+.03
0 30	−.56	+.10	+.22	+.27	+.12	−.20	−.29	−.28	+.11	+.37	+.45	−.26
1 0	−.36	−.02	+.13	+.28	+.15	−.19	−.24	−.16	+.15	+.40	+.32	−.51
1 30	−.29	−.01	+.19	+.15	+.15	−.19	−.14	−.09	+.10	+.35	+.24	−.37
2 0	−.04	−.09	−.02	+.12	+.15	−.26	−.09	−.15	+.23	+.18	+.12	−.08
2 30	−.10	+.25	0	+.21	−.02	−.18	−.06	+.05	+.26	−.12	−.18	−.11
3 0	+.08	+.13	−.16	+.04	−.12	−.25	−.02	+.10	+.26	−.01	+.20	−.24
3 30	+.02	......	+.09	−.11	−.16	−.25	+.09	+.10	+.34	+.23	−.03	−.28
4 0	+.04	+.12	+.19	−.12	−.21	−.05	+.27	−.02	+.20	−.16	−.16	−.13
4 30	+.17	+.07	+.06	−.08	−.23	−.03	+.35	−.06	+.12	−.11	−.16	−.08
5 0	+.31	+.06	+.12	−.06	−.34	−.07	+.35	+.32	+.04	−.42	−.38	+.03
5 30	+.17	+.30	+.23	−.29	−.51	−.02	+.40	+.31	−.03	−.23	−.30	+.07
6 0	+.52	+.09	+.09	−.29	−.34	+.21	+.54	+.35	−.07	−.65	−.48	+.10
6 30	+.18	−.34	+.07	−.41	−.06	+.41	+.62	+.21	−.14	−.09	−.40	+.21
7 0	+.27	−.19	−.26	−.55	+.23	+.42	+.63	+.26	−.36	−.26	−.38	+.26
7 30	+.24	−.72	−.17	−.47	+.34	+.39	+.59	+.16	−.29	−.35	+.17	+.19
8 0	−.10	−.34	−.55	−.36	+.55	+.52	+.60	−.07	−.43	−.28	+.20	+.24
8 30	−.16	−.74	−.44	−.32	+.52	+.58	+.37	−.28	−.41	+.13	−.05	+.50
9 0	−.17	−.78	−.67	−.12	+.58	+.52	+.14	−.38	−.30	+.10	+.55	+.58
9 30	−.30	−.68	−.40	+.16	+.58	+.34	−.19	−.50	−.31	+.62	+.19	+.57
10 0	−.41	−.72	−.34	+.33	+.61	+.26	−.25	−.44	−.11	+.46	+.25	+.37
10 30	−.29	−.45	−.29	+.32	+.65	+.08	−.44	−.54	−.04	+.65	+.38	+.06
11 0	−.40	−.40	−.21	+.49	+.50	+.09	−.48	−.48	+.09	+.34	+.60	−.13
11 30	−.44	−.89	+.06	+.53	+.53	+.01	−.39	−.25	+.14	+.37	+.48	−.13

Table XVII.

Showing the Difference in the Interval between the Time of the Moon's Transit and the Time of High Water, and the Mean Interval (Column A. Table III.) for every Minute of the Moon's Horizontal Parallax.

Moon's Transit.	H. P. 54′.	H. P. 55′.	H. P. 56′.	H. P. 57′.	H. P. 58′.	H. P. 59′.	H. P. 60′.	H. P. 61′.
h m	m	m	m	m	m	m	m	m
0 0	+12	+ 9	+ 4	......	− 3	− 4	−13	−14
0 30	+12	+ 9	+ 2	+ 2	− 3	− 5	− 9	−11
1 0	+10	+ 8	+ 3	+ 5	− 1	− 4	− 9	−11
1 30	+ 8	+ 5	+ 3	+ 5	− 1	− 3	−10	−11
2 0	+ 8	+ 6	+ 2	+ 3	+ 1	− 1	− 8	− 9
2 30	+ 7	+ 5	+ 1	+ 1	+ 2	− 2	− 6	− 8
3 0	+ 6	+ 4	+ 2	......	+ 2	− 2	− 6	
3 30	+ 6	+ 4	+ 3	+ 1	+ 3	− 2	− 5	
4 0	+ 4	+ 3	+ 2	− 1	+ 2	− 2	− 6	
4 30	......	+ 1	+ 3	− 1	− 1	− 2	− 8	
5 0	+ 1	+ 1	+ 3	+ 1	0	− 1		
5 30	+ 1	0	− 1	+ 2	− 1	− 1		
6 0	+ 1	+ 1	− 3	+ 1	− 2	− 2		
6 30	+ 2	+ 4	− 3	− 1	− 3	− 3		
7 0	+ 4	+ 2	− 3	− 2	− 5	− 4		
7 30	+ 9	− 2	− 2	− 4	− 7	− 7	− 7	
8 0	+16	0	0	− 3	− 8	− 6	−11	
8 30	+21	+ 8	+ 3	+ 1	− 7	− 6	−12	
9 0	+19	+ 9	+ 4	+ 1	− 9	−10	−16	
9 30	+17	+11	+ 7	+ 1	− 8	−11	−18	
10 0	+16	+12	+ 8	0	− 5	−12	−17	
10 30	+15	+13	+ 8	− 1	− 2	−10	−14	−17
11 0	+13	+12	+ 7	− 2	− 2	− 8	−14	−18
11 30	+13	+10	+ 6	− 1	− 2	− 4	−16	−16

Table XVIII.

Showing the Difference in the Height of High Water, and the Mean Height (Column B. Table V.) for every Minute of the Moon's Horizontal Parallax.

Moon's Transit.	H. P. 54′.	H. P. 55′.	H. P. 56′.	H. P. 57′.	H. P. 58′.	H. P. 59′.	H. P. 60′.	H. P. 61′.
h m	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.
0 0	−0.52	−.33	−.30	+.10	+.06	+.23	+ .33	+.53
0 30	− .50	−.28	−.20	+.08	+.17	+.41	+ .36	+.53
1 0	− .58	−.44	−.09	−.07	+.26	+.24	+ .31	+.61
1 30	− .55	−.46	+.03	−.07	+.23	+.23	+ .41	+.77
2 0	− .57	−.52	−.15	−.05	+.12	+.24	+ .50	+.75
2 30	− .54	−.45	−.28	+.10	+.08	+.34	+ .69	+.79
3 0	− .68	−.37	−.25	+.06	+.05	+.42	+ .83	
3 30	− .63	−.23	−.13	+.08	+.13	+.56	+1.00	
4 0	− .57	−.28	−.12	+.14	+.20	+.66	+ .98	
4 30	− .49	−.29	−.13	+.23	+.67	+.76	+ .83	
5 0	− .50	−.40	−.30	+.25	+.53	+.80		
5 30	− .47	−.40	−.37	+.37	+.50	+.86		
6 0	− .45	−.28	−.19	+.27	+.50	+.77		
6 30	− .54	−.21	−.09	+.05	+.40	+.62	+ .43	
7 0	− .64	−.29	−.11	+.05	+.31	+.54	+ .53	
7 30	− .75	−.48	−.28	+.01	+.15	+.33	+ .58	
8 0	− .68	−.30	−.25	+.05	+.02	−.08	+ .54	
8 30	− .54	−.27	−.26	+.01	−.20	+.42	+ .44	
9 0	− .26	−.23	−.21	−.11	−.23	−.76	+ .39	
9 30	+ .03	−.24	−.15	−.26	−.09	+.30	+ .45	+.87
10 0	− .09	−.30	−.11	−.14	+.14	+.36	+ .49	+.69
10 30	− .20	−.37	−.07	+.03	+.32	+.36	+ .48	+.55
11 0	− .31	−.31	−.14	+.21	+.26	+.26	+ .51	+.61
11 30	− .43	−.22	−.25	+.27	+.13	+.14	+ .46	+.68

Table XIX.

Showing the Difference in the Interval between the Time of the Moon's Transit and the Time of High Water, and the Mean Interval (Column A. Table III.) for every three degrees of the Moon's Declination.

Moon's Transit.	0	3° Dec.	6° Dec.	9° Dec.	12° Dec.	15° Dec.	18° Dec.	21° Dec.	24° Dec.	27° Dec.
h m	m	m	m	m	m	m	m	m	m	m
0 0	+8	+5	+7	+5	+2	+2	−3	−6	−11	−10
0 30	+9	+6	+9	+7	−2	......	−2	−4	−7	−11
1 0	+8	+8	+10	+5	+2	+2	−3	−5	−6	−11
1 30	+5	+8	+7	+2	+4	+3	−4	−7	−6	−12
2 0	+6	+8	+6	+4	+3	+3	−2	−5	−5	−10
2 30	+6	+8	+5	+7	+2	+2	−1	−4	−5	−9
3 0	+9	+9	+6	+7	+4	+2	−2	−7	−7	−11
3 30	+11	+11	+9	+7	+5	+2	−2	−9	−10	−13
4 0	+9	+10	+10	+7	+7	+1	0	−7	−10	−16
4 30	+8	+8	+8	+8	+8	−1	0	−6	−11	−18
5 0	+13	+12	+11	+12	+9	0	0	−4	−9	−16
5 30	+17	+14	+12	+13	+6	0	−3	−5	−10	−14
6 0	+20	+16	+13	+13	+6	+1	−5	−7	−13	−17
6 30	+21	+19	+12	+13	+8	......	−7	−9	−18	−21
7 0	+21	+19	+12	+17	+10	+4	−6	−8	−19	−27
7 30	+16	+16	+14	+18	+10	+5	−8	−6	−20	−34
8 0	+14	+18	+16	+16	+8	+4	−6	−6	−23	−30
8 30	+16	+15	+15	+12	+8	+2	−1	−5	−24	−22
9 0	+13	+9	+12	+7	+7	−2	−5	−8	−18	−17
9 30	+13	+7	+11	+6	+7	−2	−6	−7	−7	−12
10 0	+11	+6	+6	+6	+4	−2	−8	−7	−5	−13
10 30	+11	+8	+3	+6	+3	0	−7	−5	−5	−13
11 0	+9	+7	+3	+4	+3	+1	−6	−6	−9	−13
11 30	+8	+3	+4	+3	+4	+3	−3	−8	−14	−12

Table XX.

Showing the Difference in the Height of High Water, and the Mean Height (Column B. Table V.) for every three degrees of the Moon's Declination.

Moon's Transit.	0	3° Dec.	6° Dec.	9° Dec.	12° Dec.	15° Dec.	18° Dec.	21° Dec.	24° Dec.	27° Dec.
h m	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.
0 0	−.10	+.07	+.06	+.02	+.04	−.06	−.10	−.08	−.34	−.42
0 30	+.09	−.01	+.06	−.05	+.14	+.10	−.03	−.15	−.10	−.20
1 0	+.23	−.10	+.11	−.01	−.02	+.07	+.02	+.03	−.07	−.35
1 30	+.34	−.13	+.16	+.08	−.04	+.07	+.12	+.13	−.06	−.45
2 0	+.40	−.25	+.10	+.15	+.03	−.01	+.15	−.03	−.15	−.25
2 30	+.39	−.25	+.08	+.25	+.22	+.01	+.24	−.18	−.18	+.01
3 0	+.27	−.10	+.03	+.27	+.21	+.04	+.21	−.09	−.28	−.09
3 30	+.22	+.13	......	+.31	+.16	+.10	+.17	+.01	−.34	−.20
4 0	+.14	−.01	+.54	+.20	......	+.08	+.12	+.03	−.32	−.18
4 30	+.17	−.05	+.16	+.10	−.10	+.06	+.14	−.08	−.13	−.08
5 0	+.21	−.08	+.37	+.07	−.01	+.09	+.12	−.28	−.12	−.08
5 30	+.22	−.09	+.53	+.04	+.04	+.08	+.04	−.05	−.08	−.21
6 0	+.13	+.15	+.51	+.18	+.16	+.13	+.03	+.11	−.19	−.37
6 30	−.09	+.37	+.19	+.27	+.22	+.11	−.03	−.06	−.45	−.49
7 0	+.19	−.02	+.31	+.56	+.26	+.04	0	−.46	−.24	−.53
7 30	+.52	+.38	+.32	+.64	+.15	−.13	−.08	−.35	−.19	−.70
8 0	+.48	+.60	+.44	+.57	+.47	+.17	−.07	−.28	−.20	−.61
8 30	+.22	+.71	+.43	+.25	+.74	+.36	−.10	−.28	−.37	−.64
9 0	+.43	+.65	+.32	+.15	+.58	+.25	−.10	−.28	−.50	−.41
9 30	+.66	+.54	+.17	+.28	+.37	+.06	−.12	−.31	−.56	−.43
10 0	+.58	+.58	+.37	+.44	+.36	+.06	−.06	−.26	−.37	−.44
10 30	+.32	+.54	+.44	+.42	+.21	+.02	−.15	−.29	−.19	−.64
11 0	+.15	+.41	+.31	+.31	+.13	+.05	−.15	−.14	−.30	−.61
11 30	+.12	+.34	+.26	+.28	+.12	−.08	+.02	+.18	−.38	−.44

Table XXI.

Showing the Influence of the Direction of the Wind on the Time and Height of High Water.

Direction of Wind.	Number of Tides observed.	Sum of Times of High Water. Calculated.	Sum of Times of High Water. Observed.	Mean Error.	Sum of Heights of High Water. Calculated.	Sum of Heights of High Water. Observed.	Mean Error.
		h m	h m	m	ft. in.	ft. in.	inches.
N	38	202 43	201 43	−2	812 4	813 4	......
NNE	53	271 20	272 45	+2	1150 2	1148 4	......
NE	81	359 50	362 25	+2	1766 11	1772 9	+1
ENE	43	230 54	229 55	−1	938 3	951 4	+4
E	38	126 8	127 0	+1	829 7	834 9	+2
ESE	16	96 47	95 25	−5	348 5	344 5	−3
SE	23	125 0	125 5	......	485 6	485 8	......
SSE	18	117 34	117 55	+1	372 8	369 3	−2
S	29	148 58	149 30	+1	608 8	603 6	−2
SSW	36	262 25	264 40	+4	734 3	736 1	......
SW	148	877 29	874 0	−1	3129 2	3123 2	......
WSW	55	286 52	283 35	−4	1189 8	1179 8	−2
W	36	221 57	220 0	−3	771 9	771 1	......
WNW	10	70 40	69 55	−4	210 7	209 10	......
NW	37	189 51	188 35	−2	810 4	810 11	......
NNW	36	197 41	197 10	−1	755 2	754 3	......

XXII. *On the extensive Atmosphere of Mars. By Sir* James South, *F.R.S. Communicated by His Royal Highness the President.*

Read June 16, 1831.

That several of the planets as well as that which we inhabit are surrounded by atmosphere, astronomical observations have long since established; the extent, however, to which in particular planets such atmospheres are diffused, is as yet not satisfactorily determined. The former rests principally upon phenomena observed on the planets' discs, whilst the latter derives its support chiefly from those detected at or near their respective limbs. Every night, nay almost every hour, may give us indication of the one, whilst years are sometimes necessary, as in the case of planets unattended by satellites, to help us to the other; thus the hypothesis of the *extensive* atmosphere of Mars derives its origin from the observations of Cassini and Roëmer, and has stood more than a century and a half without refutation or support.

The observations to which I allude formed part of a series undertaken for the determination of the parallax of Mars, and are recorded in the Mémoires de l'Académie des Sciences. Cassini's were made at Briare and at La Charité sur Loire; whilst Roëmer's was obtained at the Royal Observatory of Paris.

Of the Briare observation Cassini says, "Le premier Octobre 1672 à 2^h 45^m du matin à Briare, Mars vû par une lunette de trois pieds, sembloit toucher par son bord septentrional, la ligne droite tirée par la première et par la seconde de l'eau d'Aquarius marquée ψ, d'où il n'étoit plus éloigné que de six minutes. Cette étoile paroissoit si diminuée et si affoiblie de lumière, qu'on ne la pouvoit plus distinguer ni à la vue simple, ni par une lunette un peu plus foible."—Mém. de l'Acad. tome vii. p. 357.

The La Charité and Paris observations entitled "Eclipse de la Moyenne ψ dans l'eau d'Aquarius" are thus narrated: "Quoique le ciel fut alors assez beau de part et d'autre, et que l'on vit Mars pendant un assez long espace de temps,

on ne vit point l'étoile moyenne ψ, qui devoit être cachée par son disque. Le diamètre de Mars étoit alors de 25 secondes.—Mém. de l'Acad. tom. vii. p. 358.

"Les nuages qui survinrent ne permirent pas d'en voir la sortie ; et l'on ne sçait pas même si on l'auroit pû voir immédiatement, car trois quarts d'heure après le ciel s'étant découvert à Paris, M. Roëmer la chercha attentivement autour de Mars, et il ne la trouva qu'après l'attention de deux minutes, quand elle étoit déjà éloignée bu bord oriental de Mars de deux tiers de son diamètre. C'étoit alors 11^{h} 15^{m}, et le parallèle de l'étoile coupoit le diamètre de Mars en raison de 2 à 3. Il commença de la voir sans difficulté, quand elle étoit éloignée de Mars de trois quarts de son diamètre."—Mém. de l'Acad. tom. vii. p. 359.

Hence we learn that a star of the fifth magnitude at the distance of six minutes from the planet Mars became invisible to Cassini ; and that after occultation by the planet, the same star could not be detected by Roëmer, till the planet's limb had receded from it, almost seventeen seconds of a degree. Experience, however, has long shown us that stars of the same magnitude are visible even when in actual contact with the moon's enlightened limb ; to what cause then is the invisibility of the star when in the vicinity of Mars referable ? Cassini attributed it to the atmosphere of Mars ; and although it seems difficult to imagine one of such *enormous extent* as the Briare observation would require, still, as any other hypothesis would involve us in greater difficulty, I shall adopt it, and shall present it, as also the comments which precede it, in Cassini's own words : "Cette difficulté de voir cette étoile de la cinquième grandeur très proche de Mars est considérable, d'autant qu'il n'y a point de difficulté à voir des étoiles de la même grandeur jusqu'au bord de la lune. Ce qui pourroit fair juger que Mars est environné de quelque atmosphère."—Mém. de l'Acad. tom. vii. p. 359.

Admitting, then, that an *extensively* diffused atmosphere of Mars is indicated by the observations above quoted, let us see if modern observations can confirm it.

On the 27th October 1783, Sir William Herschel, with a new 20-feet Reflector of 18.7 inches aperture, saw a star of the 13th or 14th magnitude at a distance of two minutes and fifty-six seconds from the planet, "not otherwise

affected by the approach of Mars, than what the brightness of its superior light might account for."—Vide Phil. Trans. vol. lxxiv. p. 272.

On the 19th of February 1822, in Blackman-street, a star of the 9th or 10th magnitude was for several hours seen in the field of the 5-feet Equatorial with the planet Mars. At a distance of one minute and forty-three seconds of a degree from the planet, (which took place at 11^h 15^m sidereal time,) its splendour suffered no sensible diminution.

On the following night a star 42 Leonis, of the 6th magnitude, was in the field of the same instrument with Mars, and the planet's progress towards the star was observed micrometrically for several hours; nor did the star suffer any loss of its brilliancy as its distance from the planet diminished.

Fatigued by previous watchings, at about two in the morning I retired to rest; but thinking it probable that the star would undergo occultation, accompanied by my brother Mr. HENRY SOUTH I returned to the instrument about 4 o'clock, and found Mars about half his own diameter from the star. The planet had about twenty-four degrees of altitude; its limb was at times well defined and steady, at other times extremely unsteady; the star was comparatively steady*, could be kept tolerably well upon the micrometer-wire, and was of a beautiful blue colour. At 15^h 3^m $23^s.3$ sidereal time, it was seen admirably defined, and was distant from the limb of the planet a diameter of one of the micrometer-wires, equal nearly to one second of a degree; from which time till 15^h 3^m 53^s the planet's limb was so extremely unsteady and ill defined, that the precise moment of occultation could not be obtained.

After emersion, at 15^h 20^m $38^s.3$ the star was seen when it was about $1\frac{1}{2}$ diameter of the wire, or one second and a half from the limb; it was almost indigo blue; and the contrast between it and the planet, which was of a deep red, was exquisitely beautiful. By reference to the double star 48 ι Cancri, some idea of it may be entertained, if we regard the larger of its stars as Mars, and the smaller as 42 Leonis. At the time of observation the planet had passed his opposition only forty-seven hours, and his apparent diameter as measured with the micrometer, was sixteen seconds and six tenths.

* This steadiness of one sidereal object, as contrasted with the extreme unsteadiness of another, seen under similar circumstances of atmosphere and altitude, has *long* been familiar to me, and is a phenomenon to which, on some future occasion, I shall probably invite the Society's attention.

The phenomena were also witnessed by Mr. HENRY SOUTH, and his observations strictly accorded with my own. To render our results as independent as possible of optical misrepresentation, his instrument was my Gregorian reflector, by WATSON, of six inches aperture and thirty inches focus; the figure of its metal was exquisitely perfect, whilst twelve years of constant use, had not occasioned the slightest tarnish of its almost colourless surface.

Accustomed as the first business of the night, to point a telescope to such of the principal planets as are above the horizon, on the 17th of March of the present year I had the satisfaction of seeing in the field of the 5-feet Equatorial, with the planet Mars, several stars, some of them minute, but one of the 5th and another of the 6th magnitude. Convinced that the planet would pass close by, or perhaps occult the larger, I took its place, and found it to be 37 Tauri. The covering under which the dome for my large equatorial was being built, unfortunately rendered the 5-feet equatorial useless, when the planet had approached within forty seconds of the star. The observations therefore were continued with the 12-feet Achromatic of $7\frac{3}{4}$ inches aperture, and also with the 42-inch of $2\frac{3}{4}$ aperture, till the star was one diameter and a half of the planet from his nearest limb; when, fearing lest the trees to the north-west of my grounds, might intercept the planet from my view, at the instant of nearest appulse or occultation, the 8-feet Achromatic of six inches aperture previously placed on the top of the house, was recurred to. The star, from being a full diameter distant from the planet when first observed with this telescope, was watched most unremittingly till the planet, having been in contact with it, had receded from it a quantity equal to its own semi-diameter. The star suffered not the least change of colour, nor the least diminution of its lustre, except what of the latter might fairly be attributed to the splendour of the planet; its rays were certainly in contact with the planet's limb, but only at their circumference; at times the planet and star were very steady, at other times far otherwise; but at no period was there such contrast between the steadiness of the star and the unsteadiness of the planet, as occurred at the occultation of 42 Leonis. As the star's distance from the planet diminished, the former seemed to undergo not the slightest alteration; and when in actual contact, both the star and the planet were red, but the planet had the deeper tint. The night was remark-

ably fine, and although Mars had not more than seven or eight degrees of altitude, a power of 320 was used with advantage.

The diameter of Mars, as taken with the 5-feet Equatorial, was about ten seconds. The observations were commenced at about seven hours sidereal time, whilst the nearest appulse was perhaps at about eleven hours; they were not made for determining the place of Mars, but for noticing any phenomena which the star might exhibit.

The facts being now before us, the inferences may be comprised in a few lines.

Sir W. Herschel's observation of the 27th of October 1783, and mine of the small star on the 19th of February 1822, are at variance with Cassini's observation, but impugn not the accuracy of Roemer's; whilst my observations of 42 Leonis and of 37 Tauri, being apparently subversive of the observations both of Cassini and of Roëmer, point out the "*extensive* atmosphere of Mars" as a subject meriting further investigation.

Such are all the observations relative to the *extensive* atmosphere of Mars which my observatory can furnish. One of these, viz. that of the 17th of last March, demands further consideration, lest, having served to invalidate one hypothesis, it might be brought forward to support another; namely, that "the red colour of the planet Mars is dependent upon the physical properties of his atmosphere."* Moreover, it seems inconsistent with a previous observation.

The star 37 Tauri had "nearly the colour of Mars" whilst in contact with the planet; whereas, the star 42 Leonis was "beautifully blue" previously and subsequently to occultation by the planet. The facts are different;—are they reconcileable? The following then are extracts from the Observatory Journal. On the night following the observation of 37 Tauri, namely, the 18th of March, "the five-feet Equatorial was placed upon Mars, in order to compare its colour with that of 37 Tauri; the star and planet were still in the field together, though nearly at opposite points of its circumference. "I can have no hesitation in saying, that the star is red, but not of so deep a tint as the planet."

"Mars being placed out of the field, I requested an attendant (accustomed to use a telescope) to look at the star and to tell me its colour; entirely un-

* Vide Brewster's Encyclopædia, vol. ii. pages 636 & 637 (article Mars).

acquainted with the bearing of the question, and ignorant of the colour assigned to it by me, he replied, 'Certainly a light red.' Mars was now brought into the field with the star, and being asked what colour he now considered the star to have, he answered, 'Certainly red, but not so deep a red as Mars.' Looking at it again, he said, 'Mars is the darker, but there is not a great deal of difference.'"

"I now applied several other eye-pieces, magnifying from 70 to 548 times, and with all of them I felt convinced that the star was red, but not so deep a red as Mars."

"Whilst the colour of 37 Tauri was fresh in my recollection," I placed the Equatorial upon 42 Leonis, when it had nearly the same altitude as that star; I instantly pronounced the star 42 Leonis to be blue—light blue. The attendant was now requested to look again at 37 Tauri, and to retain its colour in his mind as much as possible: 42 Leonis was next brought into the field; he said, "This star is certainly not red at all; I do not know what colour it is, unless a light blue." Alternating the examination of the one star with that of the other several times, and with various powers, he at last said, "It is certainly blue, and the first is certainly red."

The comparisons were repeated with the 12-feet Achromatic of $7\frac{3}{4}$ inches aperture, and the inferences drawn from them were the same; and, if just, the observations of 37 Tauri and 42 Leonis are perfectly reconcileable. Hypothesis, therefore, is not needed, to explain under similar circumstances with regard to Mars, the "red" colour of the one star, or the "blue" colour of the other.

XXIII. *On the Friction and Resistance of Fluids.* *By* George Rennie, *Esq. V.P.R.S.*

Read June 16, 1831.

When on a former occasion I communicated the results of a series of experiments on the Friction and Resistance of the Surfaces of Solids (Philosophical Transactions for 1828), I stated that they formed part only of a series of experiments on the nature of friction generally. My object at first was to trace the relation subsisting between the retardation produced by the surfaces of solids in motion when in contact with each other and with fluids; but finding that the subject connected with either of these branches was sufficiently extensive, I deemed it necessary to postpone the second part of the inquiry to a future occasion. Those experiments, however, established some important facts. They showed that (within the limits of abrasion) friction was the same for all solids, and that it was neither affected by surface nor velocity. Subsequent experiments upon rolling bodies of great weight and magnitude, when the resistance was reduced $\frac{1}{1000}$th part of the mass, and the surfaces in the ratio of 13 to 1, have corroborated the affinity of resistance between rolling and sliding bodies. Thus in connecting and continuing the isolated experiments of Coulomb and Vince, and assigning values to the abrasive resistances of most of the most useful solids, a considerable advance has been made in the science.

The subject of the present paper, however, involves difficulties of a more complicated kind. The theory of solids as deduced from the laws of mechanics, and independent of experiment, may be applied to any system of bodies; but the theory of fluids, in which the form and the disposition of the particles, or the laws of their action, are unknown, must necessarily be founded on experiment; and even with this aid, which can only be obtained through the intervention of a solid, our knowledge of the true properties of fluids must be vague

and uncertain. Accordingly we find that the subject of fluids attracted the attention of some of the most distinguished mathematicians and philosophers of Europe for the last two centuries; that is, from the year 1628, when CASTELLI first published his Treatise on the Measure of running Water, down to the hydraulic investigations of EYTELWEIN and YOUNG. Between these periods, Italy, France, Germany and England, added their contributions to the science. But it is to the Italians principally that we owe the foundation of it, in their numerous investigations and controversies on the rivers of Italy; hence the writings of CASTELLI, VIVIANI, ZENDRINI, MANFREDI, POLINI, FRISI, GULIELMINI, LECHI, MICHELLOTTI, and of many others *.

Each of them has endeavoured to establish a theory applicable to rivers and torrents, but in general with indifferent success. The science again received fresh accessions from the more valuable investigations of BOSSUT, DUBUAT, VENTURI, FÜNCK, BRUNNING, BIDONE, COULOMB, PRONY, EYTELWEIN and GIRARD; and among our own countrymen, of M'CLAURIN, VINCE, MATTHEW YOUNG, DR. JURIN, Professor ROBINSON, and the late DR. THOMAS YOUNG. Sir ISAAC NEWTON had already demonstrated, in his celebrated propositions 51, 52, and 53, of the Principia, (in the case of a cylinder in motion immersed in a fluid,) that the resistance arising from the want of a perfect lubricity in fluids is (cæteris paribus) proportional to the velocity with which the parts of a fluid separated from each other; and that, if a solid cylinder of infinite length revolves with a uniform motion round a fixed axis, in a uniform and infinite fluid, the periodical times of the parts of the fluid thus put in motion will be proportional to their distances from the axis. This theory (although conformable to experiment) was objected to by BERNOULLI and D'ALEMBERT, on the ground that Sir ISAAC NEWTON had not taken into consideration the centrifugal force or friction arising from the pressure of the concentric rings or filaments round the cylinder, the fluid being supposed in a state of permanence, and the friction of the rings equal throughout.

PITOT (1728), in his experiments on the water-works at Marly and Versailles, was the first to demonstrate that with equal velocities, and in the ratio of the volume of water, the friction of water in pipes was in the inverse ratio of their

* Raccotta d' Autori che trattano del Moto dell' Acque.

diameters; and Couplet (1733), Mariotte, and Deparcieux, estimated the difference between the real and calculated expenditures of glass tubes and pipes.

Chezy (in 1771 and 1786) was the first engineer who endeavoured to establish the relation subsisting between the inclination of an aqueduct and the transverse section of the volume of water it ought to carry,—on the supposition that the accelerating force, due to the inclination of the bed of the conduit, is counterbalanced by the resistances of the channel in the ratio of the surface, and increasing in proportion to the square of the velocity. What Chezy had remarked was concluded by Bossut, who cleared the investigation of most of its difficulties, and demonstrated it to be in accordance with theory. He found that small orifices discharged less water in proportion than great ones, on account of friction; that the vena contracta, and consequent expenditure, diminished with the height of the reservoir: he pointed out the law by which the discharge diminishes according to the inclination and number of bends in a pipe, and the influence of friction in retarding the velocity of waters moving in canals and pipes, in which he made the square of the velocity to be in the inverse ratio of the length of the pipe: he determined the co-efficients by experiment, and thus obtained a formula expressive of the conditions of the uniform motion of water in open canals. The greater part of these hypotheses may be said to have been removed by the more extensive researches of Dubuat. His great hydraulic work, published in 1779 and 1786, contains a series of the most valuable observations, whose results accord very nearly with the new formula of the motion of water in pipes and open conduits; and his experiments, with pipes inclined in various angles from the 40,000th part of a right angle to 90 degrees, and in channels which varied from a line and a half in diameter to areas of seven or eight square toises, seem to comprehend every case of inclination; so that by collecting a prodigious number of facts, both with compressible and incompressible fluids, he obtained a general expression for all cases relative to the friction and cohesion of fluids: but a logarithmic function which he introduces in it, by a sort of approximation, gives it a character of uncertainty, which restrains its use, and shows the necessity of fresh researches. Venturi, in 1798, "Sur la Communication latérale du Mouvements dans les Fluides," repeated and added many new facts to the experiments of Bossut, on

the expenditure of differently shaped orifices and tubes, but particularly on the lateral communication of motion by the cohesion of fluids. COULOMB first approximated to the solution of the question, by a very ingenious apparatus, consisting of discs of different sizes, fixed by their centres to the lower extremity of a brass wire, and made to oscillate in fluids by the force of torsion only; he concluded that the resistance was a function, composed of two terms, one proportional to the first, the other to the second powers of the resistance: again, that it was not sensibly increased by increasing the height of the fluid, but simply by the cohesion of the particles of the fluid which presented greater or less resistance, in proportion to the viscidity of the fluid, oil being to water in the ratio of 17.5 to 1. But whatever might be the conclusions of COULOMB, it is obvious that both the size and construction of his apparatus were ill calculated to produce results whereon to found a satisfactory theory; and accordingly both Messrs. PRONY and GIRARD, in expressing their formulæ of resistance, have not admitted that of COULOMB, but have adopted the mean of the best of experiments made by other authors: but as these formulæ give only the mean velocity, which is much greater than the velocity (of the fluid contiguous to the pipe) which ought alone to enter into the expression of the retarding force, it follows, that the coefficients deduced from the mean of all the experiments adopted by these gentlemen, have a value greatly inferior to the motion of the fluid contiguous to the side of the pipe or conduit. To ascertain correctly the value of this kind of resistance, M. GIRARD (vide les Mémoires des Sçavans etrangers for 1815), undertook a prodigious number of experiments on tubes of different diameters and length, from which he deduced that the retardation is as the velocity simply. The effects of temperature are very remarkable; if the velocity be expressed by 10, when the temperature is 0° centigrade thermometer, the velocity will be 42°, or increased four times when the temperature is 85°: these values must be deemed approximations only.

The contributions of British philosophers towards the improvement of this science have been, unfortunately, scanty; for, with the exception of Sir ISAAC NEWTON (who led the way), Dr. JURIN, Dr. MATTHEW YOUNG, Dr. DESAGULIERS, Dr. VINCE, Mr. SMEATON, Mr. BANKS, and the late Dr. THOMAS YOUNG, (see the paper of the latter gentleman in the Philosophical Transactions, and his commentaries on EYTELWEIN'S experiments,) we can scarcely find any


ELEVATION
of the apparatus for trying
THE RESISTANCE OF WATER.
Fig. 1.
Fig. 2.
Circular Discs.
a
b
b'
a'
e
e'
f
f
c
d
d'
c'
9
6
3
0
1
2
Feet
J. Basire, sc.

experiments on the subject*: whatever has been effected by our engineers or scientific men, has either been withheld from the public, or consigned to obscurity; and though we have tracts of marshes and fen land, consisting of many thousand acres, the dissertations on the mode of draining and carrying off their superfluous waters are confined to local pamphlets and reports, of comparatively minor interest to the science of hydraulics.

From the foregoing short but imperfect history, it is obvious that much has been done towards perfecting this science. It is however certain, that much yet remains to be accomplished; and although we are deeply indebted to both the French and English philosophers for their extensive investigations on the laws of capillary attraction, the descents of globes in fluids, and the adhesion of fluids to metal discs, the phenomena of fluidity, and the laws which govern the motion and equilibrium of their particles, must yet remain a problem purely geometrical; and as we possess no tangible means of approximating to the solution of the problem, but through the intervention of a solid, we must content ourselves, in like manner, with the imperfect formulæ deduced from experiments made on a small scale on the friction and adhesion of water in pipes and conduits, until we can ascertain more correctly the causes of the retardations of rivers as they occur in nature.

In the consideration of this question, therefore, I propose to examine, first, the retardations of the surfaces of solids moving in fluids at rest; secondly, the retardations of fluids over solids; and, thirdly, the direct resistance of solids revolving in fluids at rest.

To illustrate the first case, I caused an apparatus to be constructed, of which the annexed Plate XI. is a representation; it consists simply of a cylinder of wood ten inches and three quarters in diameter, and twenty-four inches long, and divided into eight sections of three inches in each, and fixed upon a spindle of iron about four feet in length, and one inch and a quarter thick. The apparatus was accurately turned and polished. Upon the upper part of the spindle, a small cylinder or pulley, six inches in diameter was fixed, and a fine flexible silken cord, communicating with the weight, was wound;

* The experiments of the Society for the Improvement of Naval Architecture, in the years 1793, 1794 1795, 1796, 1797, 1798, relate principally to the resistances of solids moving through fluids.

the apparatus was then fixed in an iron frame, and the frame let into a groove in two upright posts, driven into the bed of the river Thames.

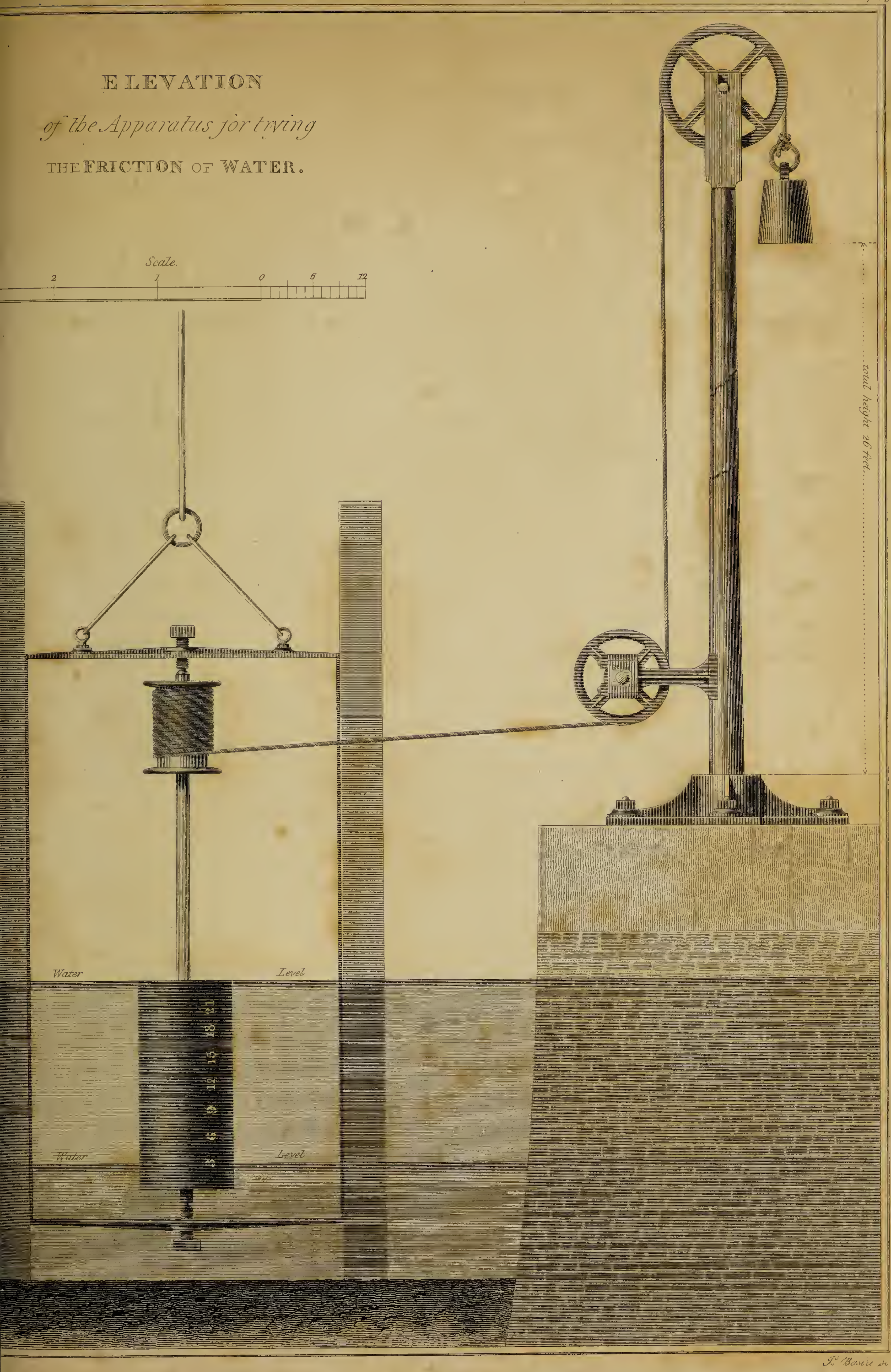
The object of the frame was to allow the cylinder to slide up and down with the level of the tide, and immerse it more or less according to the experiment required to be tried. The friction of the apparatus, or the time that the weight took to descend in the atmosphere, was first noticed; after which it was successively immersed in the water three, six, nine, twelve, fifteen, eighteen, twenty-one, and twenty-four inches, the difference of time showing the retardation according to the annexed Table.

Experiments on the Friction of the Surface of a Cylinder, twenty-four inches long and ten inches three quarters diameter, moving in air and in water.

Table I.

On Surfaces in Water.

Depth of immersion of cylinder.	Weight suspended.	Number of revolutions of cylinder falling the whole height of 26 feet.	Time in descending in water.	Velocity of periphery per second in water.	Time in descending in air.	Velocity of periphery per second in air.	Difference between air and water.	Remarks.
inches.	lbs.		seconds.	inches.	seconds.	inches.	seconds.	
3			15.00	36.021			5.00	
6		Sixteen turns in descending. Periphery moves through 540.32 inches.	18.00	30.017			8.00	Resistance increased by surface with slow velocities, but not in the ratio of the surfaces.
9			25.00	21.612			15.00	
12	1		28.00	19.297	10	54.032	18.00	
15			32.00	16.885			22.00	
18			37.00	14.603			27.00	
21			40.00	13.508			30.00	
24			55.00	9.824			45.00	
3			9.00	60.035			4.00	
6			10.00	54.032			5.00	
9			10.50	51.459			5.50	Resistance scarcely influenced by surface with increased velocities.
12			10.50	51.459			5.50	
15	2	Ditto.	10.50	51.459	5	108.064	5.50	
18			10.50	51.459			5.50	
21			11.00	49.120			6.00	
24			11.00	49.120			6.00	
On Velocities in Water.								
inches.	lbs.		seconds.	inches.	seconds.	inches.	seconds.	
24	4		8.0	67.54	2.45	196.48	5.15	
24	8	Ditto.	6.0	90.053	2.00	270.16	4.00	Could not be tried.
24	16		4.0	135.08	1.50	360.21	2.50	
24	32		2.5	216.128				


ELEVATION
of the Apparatus for trying
THE FRICTION OF WATER.
Scale.
2
1
0
6
12
total height 26 feet.
Water
Level
Water
Level
3 6 9 12 15 18 21
J. Basire sc.

Conclusions.

1. That the friction or adhesion of water against the surfaces of solids in motion, approximates the ratio of the surfaces with slow velocities; but that an increase of surface does not materially affect it with increased velocities.

2. That with equal surfaces the velocities do not seem to observe any fixed ratio, but approximate to the squares of the resistance.

With increased velocities the index of the power was found to be less than the duplicate ratio.

To exemplify the result of the foregoing conclusion in a different way,—the cylinder was removed, and circular discs of iron, ten inches and three quarters diameter and one eighth of an inch thick, accurately adjusted to the spindle and polished, were substituted. The friction of the apparatus was again tried, and immersed in the river Thames, as before.

Table II. (See Plate XII. Fig. 2.)

Experiments on the Friction in Water of Circular Discs ten inches and three quarters in diameter and one eighth of an inch thick, revolving with the planes parallel to the horizon, and six inches apart.

Number of discs.	Weight suspended.	Height fallen of weight.	Time of weight descending in water.	Velocity of periphery per second.	Time descending in air.	Velocity of periphery per second in air.	Difference.
lbs.	lbs.		seconds.	inches.	seconds.	inches.	seconds.
	1	Twenty-five feet, mean circle 16.88 would move through 422 inches.	10.00	42.200			8.00
	2		5.00	84.400			3.00
1	3		3.00	140.660	2	211	1.00
	4		3.00	140.660			1.00
	6		3.00	140.660			1.00
	1		15.00	28.133			13.00
	2		6.50	64.923			4.50
2	3	Ditto.	4.50	93.770	2	211	2.50
	4		4.00	105.500			2.00
	6		4.00	105.500			2.00
	1		17.00	24.823			15.00
	2		7.00	60.285			5.00
3	3	Ditto.	5.50	76.727	2	211	3.50
	4		4.00	105.500			2.00
	6		3.00	140.660			1.00
	1		33.00	12.787			31.00
	2		17.00	24.823			15.00
4	3	Ditto.	8.00	52.750	2	211	6.00
	4		6.00	64.923			4.00
	6		4.00	105.500			2.00

Conclusions.

That the friction or adhesion of water is not quite as the surfaces with slow velocities, being in the ratio of one to three instead of one to four, but diminishes rapidly, without observing any ratio in increased velocities*. Hence the resistance of a ship or vessel moving through the water, with an average or higher rate of velocity, forms an inconsiderable portion of the resistance resulting from the displacement of the fluid, and that the brightness observed on the copper of ships after a voyage, may be owing to other causes than the friction of the water simply.

An experiment was made to ascertain the comparative resistance of a pipe revolving in water, and with water running through a pipe; when the resistance was found to be as the surfaces in slow velocities, but to diminish greatly, as before, in high velocities, without observing any fixed ratio.

The above conclusions are in contradiction to those of Coulomb, who did not find that pressure augmented the resistance, but states that the resistance is greater when the immersion is partial.

This apparatus being applicable to fluids generally, advantage was taken of it to ascertain the direct resistance of solids to fluids (see Plate XII.)†, by causing plates and globes to revolve in them, with their planes perpendicular to the plane of the horizon.

As the resistance of solids in fluids does not form the object of this paper, it will be unnecessary to introduce many detailed observations on the subject of these experiments at present, connected as they are with another branch of hydrodynamics. But as it is important to show the relation subsisting between the resistances of cohesion and impulse, I have ventured to detail the following experiments.

* The experiments of the Society for the Improvement of Naval Architecture show a decreased resistance with increased velocities.

† In this case, the number of particles struck will be diminished in the ratio of the radius to the sine of inclination; wherefore the resistance will be diminished in a duplicate ratio of the radius to the sine of inclination. But as the sines of inclination of the two plates are equal, the resistances will be equivalent to the area of one plate (moving perpendicularly to its planes) into the duplicate ratio of the velocity of its motion, and the density of the fluid.

TABLE III.

Experiments on the Rotations of Iron Discs and Wooden Balls moving in Air, with their planes perpendicular to the plane of the horizon.

Weight suspended.	Height fallen.	Time in descending.					
		Two circular discs $10\frac{3}{4}$ inches diameter. Area 81 inches.	Velocity per second.	Two square fans. Area 81 inches.	Velocity per second.	Two wooden balls $10\frac{3}{4}$ inches diameter.	Velocity per second.
lbs.		seconds.	feet.	seconds.	feet.	seconds.	feet.
2	The spindle made 15.9 turns in falling 25 feet. Mean circle 51.83 would move through 68.67 feet.	10.00	6.867	10.00	6.867	23	2.984
4		6.00	11.445	7.00	9.810	13	5.282
9		4.50	15.261	4.50	15.261	8	8.584
16		3.00	22.891	3.25	21.130	7	9.810
20		2.50	27.469	3.00	22.891	6	11.445

Conclusions.

1. That the resistances are as the squares of the velocity.
2. That the comparative resistances between discs and globes are as two to one nearly.

TABLE IV.

Experiments on the Resistance of Iron Discs and Wooden Globes revolving in Water.

Weight.	Height fallen.	Time in descending.					
		Two circular discs, 81 inches area.	Velocity per second.	Two square fans, 9 inches square, 81 inches area each.	Velocity per second.	Two wooden balls. Area 81 inches.	Velocity per second.
lbs.		seconds.	feet.	seconds.	feet.	seconds.	feet.
16	The spindle made 15.9 turns in falling 25 feet. Mean circle 51.83 would move through 824.19 inches or 68.67 feet.	63	1.09	53	1.29	15.00	4.57
20		54	1.27	48	1.43	14.00	4.90
32		43	1.59	40	1.71	10.50	6.59
40		40	1.71	35	1.96	9.50	7.22
64		30	2.28	28	2.45	8.00	8.58
256		14	4.90	15	4.57	5.00	13.73

Conclusions.

1. That the resistances are as the square of the velocities.

2. That the mean resistances of circular discs, square plates, and globes in air, are as the numbers 25.180, 22.010, 10.627; and in water, 1.18, 1.36, 0.755; consequently the proportional resistance of air to water, with

Circular discs, is as	1	to	21.3
Plates and fans . .	1	to	16.2
Wooden balls . . .	1	to	2.2

Note.—A portion of the square fans, represented by the letters *a, b, c, d,* in Plate XII. fig. 1, and equal to one fourth of the area of each fan, was cut off, when the resistance was found to be the same as with the square fans.

Experiments on the Quantities of Water discharged by Orifices and Tubes of different diameters and lengths, and at different altitudes.

The phenomena incident to spouting fluids are,

First, The inequality observed in the velocity of the particles comprised in every horizontal section parallel to the orifice.

Secondly, The contraction of the fluid vein beyond the orifice, and consequent diminution of discharge as compared with theory.

Thirdly, The inversion and changes in the sections of the fluid vein at different distances from the orifices.

All these phenomena have been noticed and recorded by various writers, and formulæ adapted to the different circumstances of the expenditure have been given. But neither Bossut nor Du Buat (the most accurate of writers) have recorded a continuous and systematic series of experiments upon the comparative expenditure of orifices and tubes under the circumstances of area, altitude, and length.

The apparatus with which these experiments were performed, consisted of a wooden cistern very accurately made, two feet square inside, and four feet four inches in height. The water was kept at a constant altitude by a regulating cock; and a float having an index attached to it enabled the observer to ascertain the exact height at which the water stood in the cistern above the centre of the orifice.

The orifices were accurately made by Dollond in brass plates one sixtieth of an inch in thickness. The plates were accurately adjusted to a hole in the

side of the cistern, and closed by a valve of brass ground to each of the plates. The valve was opened by a lever, and the time noted by chronometers.

The diameters of the tubes, from having been drawn on mandrils, were as accurate as possible; their diameters at the extremities were carefully enlarged, to prevent any wire edges from diminishing their sections; and one extremity of the tube being inserted into a block of hard wood fastened to the cistern, and the other stopped by a valve, the experiments were recorded as before.

Table V.

Experiments on the Quantity of Water discharged by different-sized Orifices from a vessel kept constantly full and at different heights.

Constant height of the surface of the water above the centre of the orifice.	Real time in discharging one cubic foot.	Theoretical time in discharging one cubic foot, $t = \frac{Q}{2A\sqrt{gH}}$.	Ratio of the theoretical to the real discharges.	Vena contracta.
Circular Orifice made in a brass plate 1 inch diameter, $\frac{1}{60}$ inch thick.				
feet.	seconds.	seconds.		
4	19.50	11.4	1 : .584	Not accurately measured.
3	21	13.2	1 : .628	
2	26	16.1	1 : .619	
1	36	22.8	1 : .633	
Circular Orifice in a brass plate $\frac{3}{4}$ inch diameter, $\frac{1}{60}$th inch thick.				
4	33	20.3	1 : .614	At six tenths of an inch from the orifice, the diameter had contracted to 0.685 of an inch.
3	37	23.4	1 : .632	
2	44	28.7	1 : .652	
1	63	40.6	1 : .644	
Circular Orifice in a brass plate $\frac{1}{2}$ inch diameter, $\frac{1}{60}$th inch thick.				
4	73	45.7	1 : .626	At half an inch beyond the orifice, the diameter contracted to 0.37 of an inch.
3	83	52.8	1 : .636	
2	104	64.6	1 : .621	
1	144	91.4	1 : .634	
Circular Orifice in a brass plate $\frac{1}{4}$ inch diameter, $\frac{1}{60}$th inch thick.				
4	276	182.9	1 : .662	At a quarter of an inch beyond the orifice, the diameter contracted to one twentieth of an inch less than the orifice.
3	320	211.3	1 : .660	
2	396	258.6	1 : .653	
1	545	365.7	1 : .671	

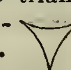
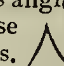
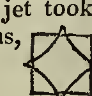
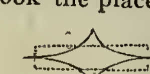
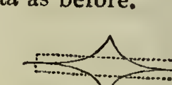
N.B. Each result shows the mean of four experiments.

Remarks.

The phenomena relative to the form and direction of veins of spouting fluids, and the remarkable inversion of the fluid veins at certain distances from their orifices, have been so fully noticed in "Experiences sur la Forme et sur la Direction des Veines et des Courans d'Eau; par George Bidone: Turin, 1829," that it is unnecessary to state further than that they have been completely corroborated in the foregoing experiments.

Table VI.

Experiments on the Quantities of Water discharged from Rectangular and Triangular Orifices in brass plates one sixtieth of an inch thick, and of equal areas, from a vessel kept constantly full and at different heights.

Constant height of the surface of the water above the centre of the orifice.	Time in discharging one cubic foot.	Theoretical time in discharging one cubic foot, $\frac{Q}{2A\sqrt{gH}}$.	Ratio of real to theoretical discharge.	Form of orifice.
Equilateral Triangle whose area is one inch, and angle uppermost.				
feet.	seconds.	seconds.		Vena contracta about half an inch beyond the orifice; but the jet with the angles reversed, and taking the sides of the triangle, the jet afterwards expanded and lost its form. 
4	15	8.9	1 : .593	
3	18	10.3	1 : .572	
2	22	12.7	1 : .577	
1	30	17.9	1 : .596	
Equilateral Triangle as before, with the angle downwards.				
4	15	8.9	1 : .593	Vena contracta the same as before, but the jet having its angle upwards, being the reverse of the former experiments. 
Rectangular Orifice of one square inch.				
4	15	8.9	1 : .593	Vena contracta about three quarters of an inch beyond the orifice, when each angle of the jet took the place of a side thus, and dissipated in spray. 
3	17	10.3	1 : .606	
2	20	12.7	1 : .635	
1	29	17.9	1 : .617	
Rectangular Orifice 2 inches long, ½ an inch wide, having the long side parallel to the surface of the water.				
4	15	8.9	1 : .593	Vena contracta as before. Each angle of the jet took the place of a side. 
3	17	10.3	1 : .606	
2	20	12.7	1 : .635	
1	29	17.9	1 : .617	
Rectangular Jet 1½ inch long, ⅛ wide, placed as before.				
4	15	8.9	1 : .593	Vena contracta as before. 
3	17	10.3	1 : .606	
2	19	12.7	1 : .668	
1	27	17.9	1 : .663	

Remarks.

That with equal areas, the expenditure by different orifices, whether circular, rectangular, or triangular, is nearly the same, the increase being in favour of rectangular orifices.

Table VII.

Experiments on the Quantity of Water discharged by Cylindrical Glass Orifices and Tubes, from one inch in length to one foot, and of different diameters, from a vessel kept constantly full, and at different heights.

Constant height of the surface of water above the centre of orifice.	Time in seconds in discharging one cubic foot.				Remarks.
	1 inch.	$\frac{3}{4}$ in.	$\frac{1}{2}$ in.	$\frac{1}{4}$ in.	
feet.					
4	11.5	24.5	55	145	In comparing these experiments with the time and quantity discharged by plate orifices, there is a diminution of time, and an increased discharge of from $\frac{1}{5}$ to $\frac{1}{4}$.
3	15.0	28.5	63	157	
2	17.5	35.0	77	205	
1	25.0	53.0	110	297	
From Glass Tubes one foot long.					
	1 inch.	$\frac{3}{4}$ in.	$\frac{1}{2}$ in.	$\frac{1}{4}$ in.	
4	14.0	30	63	200	Shows an increase of time and a diminution of discharge in the ratio of from $\frac{1}{9}$ to $\frac{1}{4}$.
3	17.0	33	73	227	
2	21.5	40	88	283	
1	30.0	58	130	410	

Conclusions.

1. That the quantities discharged in equal times by orifices and additional tubes, are as the areas of the orifices.

2. That the quantities discharged in equal times by the same additional tubes and orifices under different heads, are nearly as the square roots of the corresponding heights.

3. That the quantities discharged in equal times by the different additional tubes and orifices under different heights, are to one another in the compound ratio of the areas of the apertures, and of the square roots of the heights.

From the foregoing experiments the mean coefficient for altitudes of 4 feet with the circular orifices,
is 0.621
but with altitudes of 1 foot the coefficient is 0.645
with triangular orifices at 4 feet altitude 0.593
with triangular orifices at 1 foot altitude 0.596
with rectangular orifices at 4 feet altitude 0.593
with rectangular orifices at 1 foot altitude 0.616

Hence, allowing for the inaccuracies incident to experiments of this nature, we may safely adopt Messrs. Prony and Bossut's coefficients for altitudes of 4 feet 0.621
—— —— —— 1 foot 0.619

In the case of additional tubes of glass the coefficient is much higher than Bossut's, which was for 4 feet 0.806, and 1 foot 0.817.

Note.—Vide Venturi and Eytelwein's experiments.

Let A = area of orifice in square feet.
d = diameter of orifice if circular.
H = altitude of the fluid in feet.
T = time.
g = gravity in one second.
Q = quantity of water in cubic feet.

According to BOSSUT's experiments $Q = 0.61938\, A\, T \sqrt{2\, g\, H}$.

And as $2\,g$ is a constant quantity, and is equal to 7.77125, we have $Q = 4.818\, A\, T \sqrt{H}$ for orifices of any form, substituting d if circular, or $Q = 3.7842\, d^2\, T \sqrt{H}$.

From the second of these equations we obtain

$$A = \frac{Q}{4.818\, T \sqrt{H}}\; T = \frac{Q}{4.818\, A \sqrt{H}} \text{ and } H = \frac{Q}{(4.818\, A\, T)^2}.$$

For additional tubes the equation will stand thus: $Q = 0.81\, A\, T \sqrt{2\, g\, H}$; but since $2\,g$ is constant, and is 7.77125, we have $Q = 4.9438\, d^2\, T \sqrt{H}$, from which we deduce

$$d = \sqrt{\frac{Q}{4.9438\, T \sqrt{H}}} \quad T = \frac{Q}{4.9438\, d^2 \sqrt{H}} \quad H = \frac{Q}{(4.9438\, d^2\, T)}^2.$$

TABLE VIII.

Experiments on the Friction or Quantity of Water discharged by Leaden Pipes of different diameters and lengths, from a vessel kept constantly full, and at different heights.

Pipes 15 feet long each, straight.					
Constant height of the surface of the water above the centre of the pipe.	Time in discharging one cubic foot.				Remarks.
	1 inch.	$\frac{3}{4}$ in.	$\frac{1}{2}$ in.	No leaden pipes to be had $\frac{1}{4}$ bore.	
feet.	seconds.	seconds.	seconds.		
4	28	54	143		The time in discharging one cubic foot is nearly double the time occupied by glass tubes of equal lengths and areas.
3	33	63	164		
2	$41\frac{1}{4}$	79	208		
1	$61\frac{1}{4}$	117	312		

Table IX.

Experiments on the Quantities of Water discharged by Leaden Pipes $\frac{1}{2}$ inch bore, but of different lengths from one foot to thirty feet in length.

Glass tubes 1 inch long, $\frac{1}{2}$ inch diam.		Brass orifice $\frac{1}{2}$ diam.	1 foot long.	3 ft. 9 in.	7 ft. 6 in.	11 ft. 3 in.	15 ft.	30 ft.	Remarks.
feet.	seconds.	seconds.	seconds.	seconds.	seconds.	seconds.	seconds.	seconds.	
4	55	73	55	78	102	122	143	203	The ratio of discharge by glass tubes with pipes of 30 feet long, is as 1 : 4 } nearly.
3	63	83	63	92	120	145	164	240	Ditto with brass orifices, is as 1 : 3 } nearly.
2	77	104	93	113	151	184	208	303	
1	110	144	133	170	226	278	312	450	

Conclusions on Pipes of different lengths.

That the expenditures of water by pipes of equal diameters but of unequal lengths and under different altitudes, are nearly as follow.

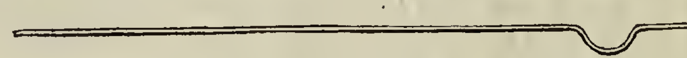
The length being as 30 to 1, the expenditures are as 3.7 to 1
Do. . . . 8 to 1 . . . do. . . . 2.6 to 1
Do. . . . 4 to 1 . . . do. . . . 2.0 to 1
Do. . . . 2 to 1 . . . do. . . . 1.4 to 1

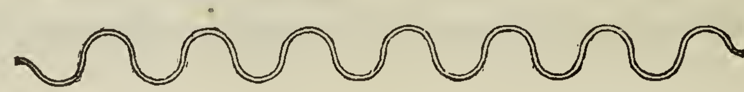
The discharges by glass and leaden tubes are nearly alike. The length of a pipe may be increased from 3 to 4 feet without diminishing the discharge as compared with the plate orifices.

Table X.

Experiments on Leaden Pipes with Flexures.

The straight pipe of $\frac{1}{2}$ an inch bore, on which the preceding experiments were made, was carefully bent into one, two, and fourteen semicircular bends respectively, each of $7\frac{1}{2}$ inches in the semidiameter, and two of $\frac{1}{4}$th part of a circle of $3\frac{1}{4}$ inches radius. One end of the pipe was fixed in the wooden orifice as before, and the following are the results.



Pipe 15 feet long, $\frac{1}{2}$ inch bore, with one semicircular and two $\frac{1}{4}$-circle bends.			
Constant height of the surface of the water above the centre of the orifice.	Time in discharging one cubic foot by a pipe with three bends.	Time discharging one cubic foot by a straight pipe.	Remarks.
feet.	seconds.	seconds.	
4	147	143	The position of the bends, whether vertical or horizontal, at either extremity of the pipe, does not affect the result.
3	175	164	
2	213	208	
1	316	312	
			
Pipe 15 feet long, $\frac{1}{2}$ inch bore, with 14 semicircular and two $\frac{1}{4}$-circle bends.			
feet.	seconds.	seconds.	
4	162	143	The expenditure is diminished by the bends from $\frac{1}{5}$ to $\frac{1}{3}$, which represents the friction of the pipe.
3	200	164	
2	247	208	
1	351	312	

Results.

1. That with one semicircular and two $\frac{1}{4}$ of a circle bends, as compared with a straight pipe of equal length and bore, the resistance varies from $\frac{1}{30}$th to $\frac{1}{70}$th part of the resistance of the straight pipe.

2. That with fourteen semicircular and two quarter of a circle bends, the resistance varies from $\frac{1}{19}$th to $\frac{1}{39}$th of the resistance of a straight pipe.

3. That the increased number of bends does not increase the resistance in the ratio of the number of bends, but merely shows an increased resistance, as compared with the four bends, of $\frac{1}{15}$th to $\frac{1}{35}$th.

Table XI.

Experiments on the Discharge of Water by Leaden Pipes of $\frac{1}{2}$ an inch bore, 15 feet long, but bent in the forms of from one to twenty-four right-angled elbows, each side being $6\frac{3}{4}$ inches long.

Height of the surface of the water above the centre of the orifice.	One right angle $8\frac{1}{2}$ inches from the end of the pipe.	Straight pipe 15 feet long.	Twenty-four right angles.	Remarks.
feet.	seconds.	seconds.	seconds.	
4	180	143	395	In the first three experiments we have a diminution of expenditure in the ratio of $2\frac{1}{2}$ to 1, and in the last experiment as 3 to 1 nearly.
3	214	164	465	
2	246	208	584	
1	371	312	872	

Conclusions.

From the foregoing experiments with one rectangular pipe, it would be reasonable to conclude that the diminution of discharge would be as the number of right angles ; but comparing the expenditure by one right-angled pipe with the expenditure of a pipe with twenty-four right angles, the difference is only in the ratio of about two to one.

General Remarks on the Expenditure of Horizontal and Bent Pipes.

Formulæ adapted to the different circumstances of the motion of water in pipes and conduits have been given by various authors.

By some, the retardations were supposed to be in the inverse ratios of the squares of the lengths of the pipes ; and by others, to be represented by a certain portion of the altitude of the reservoir above the centre of the pipe, the resistance being directly as the length and circumference of the pipe, and inversely as the area of the section.

M. Girard, in his beautiful experiments*, conceived the resistance to be compounded of the first and second powers of the velocity. So that, deducing the values from Dubuat's experiments, and expressing the resistance due to cohesion by $R x U$, R being the quantity to be obtained by experiment, and making

* Memoires des Scavans Etrangers.

the resistance due to the asperities equal to $R\,x\,U^2$, the sum of the resistance is $R\,(U + U)^2$.

M. Prony, applying his profound acquirements to the solution of all the cases of preceding authors, deduced from a selection of upwards of fifty experiments the following simple formula: $U = 26.79\sqrt{\frac{DZ}{\lambda}}$;

- U being the mean velocity of the section of the pipe;
- D the diameter of the pipe;
- Z the altitude of the water;
- λ the length of the pipe:

from which it appears that the velocity is directly in the compound ratio of the square roots of the diameter of the pipe and head of water, and inversely as the square roots of the length of the pipe; that is, for any given head of water and diameter of pipe, the velocity is inversely as the square root of the length of the pipe.

If we compare these results with those of Dubuat, Girard, and others, they approximate very nearly to each other.

In general, if we incline a pipe to an angle of about $6\frac{1}{2}$ degrees, or one ninth of its length, the discharge will be nearly equal to the discharge by additional tubes. The charge necessary to express the mean velocity of water issuing from straight pipes is by some authors equal to $\frac{V^2}{478}$ *; Dr. Young makes it $\frac{V^2}{550}$: the diminution of expenditure depending upon the contraction of the fluid vein and the friction of the pipe.

The change occasioned by bends and angles in the direction of the fluid vein tends to diminish the velocity in a very remarkable manner.

Dubuat undertook several experiments upon this subject, but the formula proposed by him does not solve the difficulty, where $\frac{V^2 S^2}{m}$ gives the resistance due to one bend, V being the velocity, S the sine of incidence or reflection, and m a constant quantity determined by Dubuat to be 2998.50.

Now although it is reasonable to suppose that the resistance should be proportionable to the squares of the sines of the angles of incidence, yet as all

* Dubuat and Langsdorf.

the particles of the fluid vein are not reflected in the same angle, and as a considerable portion of the velocity is destroyed by the first angle or bend the fluid meets with in the pipe, M. Dubuat's theory is fundamentally erroneous, the more especially as he has rejected more than one half of the twenty-five experiments mentioned by him. Dr. Young's suppositions, of the resistance being as the angular flexure and the power of the radius, of which the index is $\frac{7}{8}$, are equally erroneous, as is evinced by the foregoing experiments.

In conclusion, it is evident that the subject of friction admits of an immense variety of applications. To determine the measure of the resistances experienced by vessels and floating bodies in their motion through fluids ; the law of the retardations of rivers, and the cause of the obstructions presented to the waves of the ocean in the slopes assumed by its shores ; the equilibrium of earths, and their connection with solids and fluids,—all of them are questions of the utmost importance in the economy of nature, and their solution can only be attained by an accumulation of facts.

N.B. Since the foregoing paper was presented to the Royal Society, an abstract of an extensive series of experiments on the expenditure of water through rectangular orifices of large dimensions, has been submitted to the French Academy by Messrs. Poncelet and Lesbros, of the Corps de Genie at Metz ; and as these experiments were undertaken by order of the French government, no expense was spared to have them made as extensive as possible. Their objects were principally to ascertain the exact measure of the coefficient of contraction and the forms of the fluid veins under different altitudes and areas.

The results of which are :—

That with an orifice of 20 centimetres square, the coefficient is 0.600 under altitudes of 1 metre 68 centimetres. But when the altitude was reduced to four or five times the opening of the orifice, the coefficient increased to 0.605, but again diminished rapidly as the altitude diminished, to 0.593.

That with orifices of smaller dimensions, i. e. from 10 to 5 centimetres square, the same law was observed, the coefficients being respectively 0.611,

0.618, and 0.611, for opening of 10 and for 5 centimetres, 0.618, 0.631, 0.623 ; and for orifices of less dimension, the coefficient continually increased up to 0.698.

That for water running over weirs, the mean coefficient was 0.400, which differs very little from that of Bidone.

Hence we see little reason to deviate from the coefficients already given.

XXIV. *Further Experiments with a new Register-Pyrometer for Measuring the Expansion of Solids. By* J. Frederick Daniell, *Esq. F.R.S. Professor of Chemistry in King's College, London.*

Read June 16, 1831.

In my former communication on a new Register-Pyrometer, which has been honoured with a place in the Philosophical Transactions for 1830, I stated that I hoped, at some future period, to be able to lay before the Society the results of some experiments upon the dilatation of metals to their melting points; and I now purpose to redeem this pledge.

My previous observations upon the subject of expansion, were directed chiefly to the object of establishing what degree of confidence might be reposed in the instrument as a measure of temperature; and I was able, I trust, to exhibit such an accordance between the measures which it had afforded and those of the best experimenters, long previously obtained with various metals to the boiling point of water, as fully to establish its sufficient accuracy. The comparison however which I most relied upon, was with the experiments of MM. Dulong and Petit, upon the expansion of platinum and iron to the high temperature of 572° Fahr.; and as this is a point of fundamental importance, I shall still further strengthen it by a comparison with the results obtained by the same distinguished philosophers with copper, the only other solid metal to which they extended their inquiries.

Previously to this, I trust it may not be thought tedious, if I briefly relate the results of some trials for obtaining registers of uniform composition, which might preclude the necessity of determining the rate of expansion in each individual instance.

Exp. 23. For this I had recourse to Wedgwood's ware, of which I obtained some bars carefully constructed and highly baked for the purpose. The expansion of these I found precisely equal to that of platinum; so that when

the register was immersed in boiling mercury, the index was found not to have moved. When a bar of iron was substituted for that of platinum, the arc measured was 1° 7′.

With black-lead the same expansion gave a measure of 2° 49′, from which if we deduct the expansion of platinum in black-lead . 1 45

the remainder . . . . 1 4 is sufficiently near to confirm the result.

Exp. 24. My next trial was with registers of black-lead of various and known mixtures of plumbago and Stourbridge clay. Four fifths proportion of the former to one fifth of the latter produced a composition which was too tender for the purpose; but a mixture in the proportion of three fourths to one fourth formed a ware of a fine, even texture; whose expansion was very equal, and not exceeding the least of those which I had formerly tried.

Three different registers of this composition afforded me the following measures of the expansion of a platinum bar to the boiling point of mercury.

1° 45′
1 42
1 38

To which I may add a fourth, which gave for the expansion of an iron bar to the same point an arc of 2° 42′, which is equivalent to 1° 40′ for a platinum bar. For all common purposes, therefore, the mean expansion of 1° 42′ might have been adopted without any serious error in the final results. In investigations, however, which require the utmost precision, I still think it advisable to fix the expansion of each register by experiment.

Exp. 25. A bar of copper was adjusted in one of the registers and exposed, in the manner formerly described, to boiling mercury; the arc measured on the scale was 4° 10′, equivalent to an expansion of .03633.

Let us now compare this result with the determination of MM. Dulong and Petit, as we formerly did the expansions of platinum and iron.

The expansion of Copper.

		Length of Bar.		
From 32° to 212° = .0017182	×	6.5	=	.01116830
From 392° to 572° = .0018832	×	6.5	=	.01224080
				.02340910
From 212° to 392° = Mean of the above			=	.01170455
Total expansion from 32° to 572°			=	.03511365
Add for the expansion from 572° to 660°, the temperature of boiling mercury, calculated at the highest rate:—				
180° : .0018832 : : 88° : .00920675			=	.00920675
				.04432040
Deduct expansion for 32°, the experiment with the pyrometer having commenced at 64°			=	.00305457
Calculated at the lowest rate:— 180 : .0017182 : : 32° : .00305457				
Real expansion of the bar by Dulong and Petit .			=	.04126583

If from the real expansion thus obtained 04126
We deduct the apparent expansion obtained by the pyrometer .03633

The remainder .00493

will be the expansion of the black-lead.

We thus obtain the expansion of 6.5 inches of black-lead ware,

from 64° to 660°	by Platinum bar	00421
	by Iron bar	00457
	by Copper bar	00493
	Mean	.00457

in which the extreme results differ from the mean not .0004 inch, or one fourteenth of the whole.

When we take into consideration the great difference in the total expansion of these three metals, as well as the differences in their several rates of increase

with the increasing temperature, such an accordance appears to me to be perfectly decisive of the accuracy of the pyrometer.

It will be unnecessary for me to trouble the Society with the details of the experiments by which I determined the expansion of several other metals to the boiling point of mercury; it will be sufficient to state the results in a tabular form. I thought that it would add much to the interest of the determination of the total expansion to the fusing points, to determine previously the expansion of each to the points of boiling water and boiling mercury; that any alteration in the rates of expansion between these points might be detected.

I must, however, make a few observations upon the general method which I adopted to insure an accurate determination of the former.

Exp. 26. Judging from the action of the pyrometer at lower heats, I expected that the index would continue to be thrust forward by the progressive expansion of any bar of metal, till its cohesion gave way and it assumed the fluid form; and consequently that a register would be obtained of its maximum dilatation: but the difficulty consisted in applying the heat so equally that one part should not melt before another. The arrangement which I finally adopted to secure this purpose, and which was found to answer perfectly, was as follows. In the laboratory of the Royal Institution there is an excellent wind-furnace, from which proceeds a lateral horizontal flue, along which a flame may be drawn with any required degree of force. Into this flue open two muffle-holes, which give a complete view and command of the interior. From the equality of the draught, regulated by a register, the whole of this chamber may be kept at a low red, or an intense white, heat, by a proper management of the fuel in the body of the furnace.

The registers of the pyrometer were prepared for the experiment by drilling three holes on their under sides, communicating with the cavities in which the bars were placed; one at each extremity, and one in the centre. This was done for the purpose of allowing a vent for the melted metal, and to afford some criterion of the equality of the heat, by the time at which the metal ran from the different apertures. When the bar was properly adjusted in the register, it was carefully placed in the hot air-chamber, in a horizontal position, supported at each end by a small piece of brick, at a proper distance from the

body of the fuel, accordingly as a greater or less degree of heat was required. The muffle-holes were then closed with their stoppers; all but a narrow slit, through which the progress of the heating and the flow of the metal could be observed. The equality of the heat could be very accurately ascertained by the uniform colour of the register as it became red; and any irregularity could easily be corrected by advancing one or other end more towards the fuel. In this manner I succeeded in obtaining very satisfactory results; except in the case of gold; and this metal requiring for its fusion rather more heat than I could at the time command in the air-chamber, I laid the register upon the fuel in the body of the furnace, and it thus became only partially melted, and half the bar remained in the solid state. The amount of the expansion indicated is therefore evidently deficient, and must be discarded from the table. A similar accident happened once with brass; but this I have been able to rectify by subsequent trials.

I shall now arrange the results of my experiments in two tables:—the first showing, in arcs of the scale, the expansion of pure metals from 62° F. to 212°, 662° F., and their respective melting points; and the second exhibiting the expansion of certain alloys to the same points.

The bars were in all cases of the same length of 6.5 inches.

Table XIII.

Showing the progressive expansion of the following pure metals to their melting points.

From 62°	to 212°	to 662°	to Melting Point.
	° ′	° ′	° ′
Tin	0 55	......	2 30
Lead	1 33	......	6 17
Zinc	1 40	5 50?	8 44
Silver	0 59	4 9	13 45
Copper	0 45	4 10	16 0
Gold	0 35	3 11	(7 51 not correct)
Cast Iron	0 29	2 25	9 47

Table XIV.

Showing the progressive expansion of the following alloys to their melting points.

From 62°	to 212°	to 662°	to Melting Point.
	° ′	° ′	° ′
Brass. Common	0 54	4 42	(8 41 not correct)
Brass. Copper $\frac{3}{4}$, Zinc $\frac{1}{4}$	1 9	4 51	13 39
Brass. Copper $\frac{1}{2}$, Zinc $\frac{1}{2}$	1 27	5 3	15 34
Bronze. Copper $\frac{15}{16}$, Tin $\frac{1}{16}$	0 52	3 37	9 49
Bronze. Copper $\frac{7}{8}$, Tin $\frac{1}{8}$	0 54	4 11	10 16
Bronze. Copper $\frac{3}{4}$, Tin $\frac{1}{4}$	0 58	4 44	10 55
Bronze. Copper $\frac{1}{2}$, Tin $\frac{1}{2}$	1 0	4 7	4 7 ?
Pewter. Lead $\frac{4}{5}$, Tin $\frac{1}{5}$	1 5	......	2 28
Type Metal. Lead and Antimony	1 5	......	3 13

The first remark which I shall make upon these tables regards the fusing points of the pure metals. Having ascertained for each the expansion due to certain definite increments of temperature, and the utmost expansion which they undergo to their fusing points, it is clear that, had their expansion been equal for equal increments, we might have determined the true temperature of their melting points from these data. As it is, even, knowing something of the limits of error introduced into such a calculation by the increased rate of expansion at the upper part of the scale, and the direction in which it ought to affect the result, we may draw some important inferences with regard to the correctness of the determinations derived from other means. The following Table exhibits the results of such a calculation, compared with the melting points previously determined.

Table XV.

Fusing points of metals derived from their expansions to 212° and 662° supposed equable.

	From 212° rate.	From 662° rate.	Real Temperature.
	°	°	°
Tin	471	....	442 by Thermometer.
Lead	670	....	612 by Thermometer.
Zinc	848	960 ?	773 by Pyrometer.
Silver..............	2159	2049	1873 by Pyrometer.
Copper	3262	2366	1996 by Pyrometer.
Cast Iron	3096	2489	2786 by Pyrometer.

Now by these results, the accuracy of the pyrometer may, again, be placed beyond doubt, in a manner which was perfectly unforeseen at the time of instituting the experiments.

In the first place we have two metals, tin and lead, whose melting points being within the temperature of boiling mercury, have been accurately determined by the common thermometer. Upon calculating the same points from their several expansions to boiling water, measured by the pyrometer, upon the supposition that they maintain the same rate to their points of fusion, the temperature of the first comes out 29°, and of the second 58°, higher: that is to say, the rate of expansion of these two metals increases with the increase of temperature, as has been found to be the case with platinum, iron and copper, by the experiments of MM. Dulong and Petit. It is worthy of remark, that this increased rate in tin is equivalent to 29° in about 200°, and in lead to 58° in about 400°, above the boiling point of water. These results therefore indicate a very close agreement between the thermometer and pyrometer.

2ndly. The melting point of the next metal, zinc, is one of those which has been determined by immersion of the pyrometer into it, when it was in the act of fusion. Its temperature, so determined, falls short of the same point, calculated from the expansion supposed equal, by 75°. This again indicates an expansion increasing at nearly the same rate (75° in 560°), as in the preceding instances of tin and lead. I pass over at present the result obtained by calculating from the expansion to the boiling point of mercury, as it presents an anomaly upon which I shall presently make some observations.

3rdly. The melting point of silver, determined in the same way by immersion, differs from that calculated from expansion in the same direction; and the difference (286° in 1660°) is nearly in the same proportion. The calculation from the rate of expansion to the boiling point of mercury comes much nearer to the melting point directly determined, and only differs from it 176°: proving that the rate of expansion increases with the increasing temperature.

4thly. A similar comparison instituted with copper presents us with a rate of expansion increasing much more rapidly than in the preceding instances; so that the melting point, calculated from the expansion to boiling water, differs from the true melting point no less than 1266°. Taking the rate of expansion to boiling mercury, the difference is reduced to 370°. And here again I may

refer to the experiments of MM. Dulong and Petit in confirmation of the result; for they found that the temperature indicated by the expansion of a rod of copper was 50° Fahr. higher than the true temperature at 572° Fahr.

5thly. The interesting nature of the results which I obtained with iron, and the peculiar difficulties in arranging the experiments from which they were derived, will, I trust, excuse my entering more into their details than I have thought necessary in the preceding instances. I have already given the expansion of wrought iron to the temperatures of boiling water and boiling mercury, and shown that the measures obtained with the pyrometer agree essentially with those determined by very different means by MM. Dulong and Petit. I have also proved that the melting points of gold and silver, determined by the expansion of the same bar of iron, agreed very closely with the same points determined by the expansion of platinum. I was extremely anxious to complete this series of experiments by measuring the expansion of iron to its melting point. For this purpose I had a small bar of iron cast from the best gray iron, and afterwards cleaned of all oxide and reduced to the size of the other bars employed by filing. Upon measuring its expansion to the temperatures of boiling water and boiling mercury, I found the arcs upon the scale respectively 0° 29′ and 2° 25′; and this being considerably less than what I had obtained with the bar of wrought iron, I repeated the experiment with the latter in the same register that I had employed for the former, and obtained the measures of 0° 35′ and 2° 44′—nearly agreeing with the previous determination: so that there can be no doubt that cast iron expands less than wrought iron, though the rate of increase for the higher temperature appears to be the same in both.

I now arranged the two bars in two registers; and having strongly heated the furnace and filled the air-chamber itself with coke, I cleared out a space in which they could be placed, without coming in contact with the fuel on each side of them. Their two ends rested on pieces of fire-brick; the wrought iron was placed lowest, and, the thickness of the register, in advance of the cast iron; which was placed about two inches higher. The apertures were now all closed, and the draught increased to the utmost. At the expiration of a quarter of an hour the register with the cast iron was removed with a pair of tongs; and the metal upon lifting it, immediately flowed out at the two end

holes. The register, with the wrought iron was then taken out. The bar of the latter was found perfect, without any signs of oxidation or fusion.

The arc measured of the cast iron was . . . . 9° 47′

The arc of the wrought iron . . . . 7° 56′

I had some reason to think that the register, with the wrought iron bar, had not been exposed so fully to the heat as that with the cast iron: for, although placed slightly in advance of the latter towards the body of the furnace, it was not raised so high from the floor of the flue, which probably had a cooling influence; and as the flame was drawn upwards, it must have struck with greater force upon the higher register. I therefore replaced the wrought-iron bar in the register, and put it exactly in the position previously occupied by the cast iron; it was then covered with charcoal, and the fire urged to the utmost. At the expiration of twenty minutes it was removed: the bar was found uninjured, with a white metallic lustre, except over the apertures, where it was blue, and perfectly free from oxide. The arc now, however, measured 11° 16′.

Now from these experiments there are four ways of approximately determining the temperature of melting cast iron.

1st. By taking the expansion of cast iron to its melting point, and calculating from the expansion for 150° to the boiling point of water, upon the supposition that the same rate is maintained, and adding the initial temperature of 60°, we obtain 3096°.

2ndly. By calculating from the expansion of the same bar for 600° to the boiling point of mercury, supposed equal, we obtain 2489°.

3rdly. By assuming the expansion of a bar of wrought iron, at the point of melting cast iron, and calculating from the expansion of the same bar for 150° to the boiling point of water, we obtain 2957°.

4thly. By calculating from the expansion of the same bar for 600° to the boiling point of mercury, supposed equal, we obtain 2533°.

It is remarkable that the mean of these four determinations is 2768°; for it will be remembered that the corrected temperature, which I deduced from the expansion of a platinum bar plunged into melting cast iron, was 2786°.

It may be observed, that in both cast iron and wrought iron, the calculation from the rate of expansion to the boiling point of water gives a temperature higher than the true; and that, in both, the calculation from the point of boil-

ing mercury affords a result lower than the true. This might afford some grounds for conjecturing that, although the rate of expansion evidently increases beyond the temperature of boiling water, it does not continue to increase to the end; but there is another inference from the fact, which I am rather inclined to adopt.

In calculating the temperature of melting cast iron, from the expansion of the platinum bar, I applied a correction, upon the supposition that the same rate of increase of expansion which was exhibited by platinum between the boiling points of water and mercury continued to the higher degrees; whereas there is great reason to suppose that the rate must be an increasing one; and, although this might not sensibly affect the final result of the comparatively low temperature of melting silver, the calculation of the temperature of melting iron, which is more than one third higher, would be sensibly affected by it. I think it therefore extremely probable that the true temperature of melting cast iron is below 2786°.

The consistency of these results will, I trust, remove any doubts as to the competency of the pyrometer to determine fixed and comparable points of very high temperatures, and induce those connected with arts and manufactures to introduce its use, for the purpose of ascertaining many questions of the highest interest, both to practical and theoretical science. The experiments just detailed upon bars of wrought iron remove even the only trifling objection which could be brought against its general use; namely, the expense of a platinum bar: for it is quite proved that a bar of wrought iron is sufficient for every practical purpose, and it affords the important additional advantage of a much more open scale.

I proceed now to remark that zinc, as well as iron, appears by the Tables to present an exception to the law of an increasing rate of expansion with increasing temperature; the expansion for the 600° to boiling mercury not being so much as four times that for the 150° to boiling water. I cannot, however, from some peculiar circumstances attending the experiment, place entire confidence in the result. When, after boiling in mercury, the register was opened, the vapour was found to have gained admittance, and to have acted upon the zinc. It was firmly fixed in the cavity, and was not removed without considerable difficulty and piecemeal. At its upper end, the bar was

reduced almost to a point, and was very considerably thickened at its lower end, and moulded to the bottom of the register, as if it had been partially fused. It was hard and brittle. The vapour of the mercury had probably combined with it at some temperature below the boiling point; the amalgam so formed had flowed down to the bottom of the bar, and the mercury was afterwards expelled by the boiling temperature.

I may here observe, as not unworthy of attention, that in no instance have I seen a metal acted upon by the vapour of mercury at its full boiling temperature;—even gold, which has so strong an affinity for it, comes out of it with its yellow colour perfectly unstained; but when the mercury is in the fluid form at the same temperature, the gold is immediately dissolved by it.

Under these circumstances there certainly may exist some doubt whether the full amount of expansion in zinc to the boiling point of mercury was properly registered.

On the other hand, in confirmation of the result so recorded, it may be seen, in Table XIV. of the expansion of the alloys, that a composition of half copper and half zinc presents the same anomaly; the expansion for the 600° to boiling mercury is not quite four times that of the 150° to boiling water. In the alloy of three fourths copper to one fourth zinc, the rate of expansion increases in a small degree; and in common brass, where the proportion of zinc is still less, it increases still more rapidly.

My purpose in instituting these experiments upon the alloys, was to observe the relation which might exist between the expansions of the pure metals and those of their mixtures: and the better to illustrate any such, I made alloys of copper with known multiple proportions of zinc and tin. I shall here present, in a tabular form, the temperatures of their melting points, as derived from their expansions to the boiling points of water and mercury; as, although I am not able to compare them with results directly obtained by immersion, we can judge, by comparison with the similar calculation of the pure metals, within what limits any error is probably confined.

Table XVI.

Fusing points of alloys, derived from their expansions to 212° and 662° supposed equable.

	From 212° rate.	From 662° rate.
Brass. Copper $\frac{3}{4}$, Zinc $\frac{1}{4}$	1842°	1750°
Brass. Copper $\frac{1}{2}$, Zinc $\frac{1}{2}$	1672	1910
Bronze. Copper $\frac{15}{16}$, Tin $\frac{1}{16}$	1761	1690
Bronze. Copper $\frac{7}{8}$, Tin $\frac{1}{8}$	1773	1534
Bronze. Copper $\frac{3}{4}$, Tin $\frac{1}{4}$	1755	1446
Pewter. Lead $\frac{4}{5}$, Tin $\frac{1}{5}$	403	
Type Metal. Lead and Antimony	507	

I have not included in the foregoing Table the alloy of half copper and half tin, but have exhibited its expansion to the boiling point of mercury in Table XIV. This mixture was very hard and brittle, and resembled the speculum metal of reflecting telescopes. After it had been exposed to boiling mercury, it appeared as if it had undergone partial fusion; it was set fast in the cavity of the register, and had thickened towards the lower extremity. I am inclined to think that it had nearly attained its melting point, but it was broken in removing it; and I had not an opportunity of trying any further experiment with it.

With regard to these alloys, the experiments are not numerous enough to enable us to deduce with precision the general laws by which their expansions and points of fusion are governed; but enough is discernible to show that the subject is well worthy of further investigation. It appears

1st. That the expansion of the compounds is not the mean of the expansions of the simple metals of which they are composed, but bears some proportion to their relative quantities. Thus we may observe that the expansion of brass increases with the quantity of zinc which it contains, as does bronze or bell-metal with the quantity of tin.

2ndly. That the expansion of brass is in an increasing ratio to the increase of temperature till the quantity of zinc amounts to one half, when it seems to assume a decreasing rate, as we have reason to suppose is the case with pure zinc. On this account the melting points both of this mixture and zinc appear

to be higher when derived from their expansions to the boiling point of mercury, than when calculated from their expansions to the boiling points of water. With this exception, there is great reason to suppose that the melting points of the alloys, from the higher rate of expansion, cannot be very far removed from the true temperatures.

3rdly. That the melting point of copper is reduced by an admixture of one fourth of zinc to nearly the average which results from the proportions of the two ingredients; but by an admixture of an equal quantity of tin it is reduced in a much greater proportion. The temperature derived from the average with zinc would be 1690°, and the corresponding temperature in the Table is 1750°. The temperature derived from the average with tin would be 1607°, but the corresponding temperature is only 1446°.

4thly. That a similar power in tin to depress the melting point of another metal is exhibited in pewter; in which we may observe that a mixture of one fifth of tin with lead reduces the melting point actually below that of either of the pure metals; and we may recall to recollection the fact, that an alloy of eight parts of bismuth, whose fusing point is 476°; five of lead, whose fusing point is 612°; and three of tin, whose fusing point is 442°,—liquefies at 212°.

I shall here subjoin a Table, in the usual form, of the progressive linear dilatation by heat of such solids as I have measured with the pyrometer to the boiling point of water, the boiling point of mercury, and their respective melting points, where they have been ascertained. I have added to their apparent expansions by the register the corresponding expansion of the black-lead; upon the assumption that the latter continues at an equal rate to temperatures above 662°; in which it is not probable, from the preceding observations, that there is any error of material importance.

Table XVII.

Linear Dilatations of Solids by Heat.

Dimensions which a bar takes whose length at 62° is 1.000000.

	At 212° (150°).	At 662° (600°).	At Point of Fusion.
Black-lead ware	1.000244	1.000703	
Wedgwood ware	1.000735	1.002995	
Platinum	1.000735	1.002995	(1.009926 maximum, but not fused.)
Iron (wrought)	1.000984	1.004483	(1.018378 to the fusing point of cast iron.)
Iron (cast)..........	1.000893	1.003943	1.016389
Gold	1.001025	1.004238	
Copper	1.001430	1.006347	1.024376
Silver..............	1.001626	1.006886	1.020640
Zinc	1.002480	1.008527	1.012621
Lead	1.002323	....	1.009072
Tin................	1.001472	....	1.003798
Brass. Zinc $\frac{1}{4}$	1.001787	1.007207	1.021841
Bronze. Tin $\frac{1}{4}$......	1.001541	1.007053	1.016336
Pewter. Tin $\frac{1}{5}$......	1.001696	....	1.003776
Type Metal	1.001696	....	1.004830

The regularity of these several expansions is very striking. As long as the metal retains the solid form, the dilatation proceeds according to a fixed law, without any sudden starts or changes; till assuming the form of a liquid it doubtless is subject to a different mode of action.

I shall conclude these observations with the results of some experiments which I made to determine, if possible, the cause of the singular change of texture in platinum, when intensely heated in the black-lead registers, which I described in my former paper. Upon showing the bar so changed to those who were best acquainted with the working of this metal, they universally ascribed it to the action of sulphur: but nobody could explain to me why this action should require such a very intense heat; as up to the temperature of melting cast iron, to which it had several times been exposed, no change took place; but the bar remained perfectly soft and malleable.

In De Ferussac's Bulletin for November 1830, there is an abstract of my paper on the Pyrometer, which the Editor concludes with the observation, that "unfortunately I inclosed in the crucible which contained the register

and the bar of platinum some pieces of iron, without being aware of the fact, which is known to all the workmen who manufacture platinum, that the mere presence of iron is enough to communicate brittleness to that metal."

Upon inquiry amongst workmen in this country I cannot find that such a property has ever been observed in the course of their experience; and when I consider that the bar in the cavity of the register was perfectly preserved from contact with the iron nails; and moreover, that it had actually been plunged into melted iron without any change of properties; I cannot suppose that the alteration depended in any way upon this circumstance.

To resolve these doubts I took 116 grains of the brittle platinum, which had been ground without difficulty to a fine powder in a steel mortar, and boiled them in nitro-muriatic acid till I had effected a complete solution;—a little of this solution produced a scarcely perceptible cloudiness in a solution of muriate of baryta. This I have reason to think was owing to a slight impurity in the acids employed; I infer therefore that there was no sulphur in the metal. I proceeded to evaporate the solution; which towards the end of the process assumed a gelatinous appearance. When in this state, I poured alcohol upon it; and as the acid still remained in excess, a violent reaction took place with extrication of nitrous gas. I then evaporated to dryness and continued the heat; till the salt of platinum kindled spontaneously, and finally was left in a spongy state. This was again digested in nitro-muriatic acid, and the solution carefully evaporated to dryness. The muriate of platinum was then dissolved in water, and a sandy residue remained; which, when well washed and heated to redness, was of a grayish-white colour, and had all the properties of silica: it weighed 3.5 grains. There can therefore, I think, be little doubt that at the high temperature to which it was exposed, platinum took up as much as 3 per cent of silica; or, more probably, a quantity of its base equivalent to that quantity of the earth, to which it owed all its change of character and properties. A temperature considerably above that of melting cast iron appears to be necessary to this combination; which is analogous in many respects to the absorption of carbon by iron in the process of making steel by cementation.

Errata in the former Paper, Phil. Trans. 1830.

The following errata in my former paper have arisen from my having omitted to add the initial 32° in the reduction of the Centigrade scale to that of Fahrenheit.

Page 266,	line 24,	*for* 360°,	*read* 392°.
—— 266,	— 26,	— 360°,	— 392°.
—— 266,	— 27,	— 360°,	— 392°.
—— 269,	— 3,	— 360°,	— 392°.
—— 269,	— 5,	— 360°,	— 392°.
—— 269,	— 23,	— 360°,	— 392°.
—— 269,	— 25,	— 360°,	— 392°.

XXV. *Experiments on the Length of the Seconds Pendulum at the Royal Observatory of Greenwich. By Captain* Edward Sabine, *of the Royal Regiment of Artillery, F.R.S.*

Read June 16, 1831.

These experiments were made with the original convertible pendulum constructed by Captain Kater, and employed by him in 1817 in Portland Place, London.

Prior to its employment in the present experiments I made the following alterations in the pendulum.

1. The tail pieces were removed altogether; coincidences being observed by the bar itself (the ends being blackened for the purpose), and by a disc of corresponding diameter on the pendulum of the clock.

2. The "moveable weight" employed by Captain Kater was dispensed with altogether; and the pendulum rendered convertible, within the limits of more exact adjustment by the slider, by filing away a small quantity of metal from one extremity of the bar. By this alteration there remained nothing moveable about the pendulum except the slider; and that was so placed in these experiments, that a change in its position, of so great magnitude as the tenth of an inch, did not occasion an alteration of so much as the tenth of a second in the daily rate of the pendulum, when suspended with the weight below, which is the position in which the rate is determined. The slider was clamped to the bar, and was moved by a screw for slow motion, by which it could be adjusted with tolerable precision to the hundredth of an inch. The graduation on the bar, by which the place of the slider was regulated, was on the side of the bar next the observer, who could thus at all times assure himself that no change occurred in its position by the inversion of the pendulum.

Neither of these alterations interfered with the distance between the knife edges, which was referred in 1817 to Sir George Shuckburgh's scale by Cap-

tain Kater, and found to measure 39.44085 inches of that scale at the temperature of 62°.

The experiments were made, unless where it is otherwise noticed, in the vacuum apparatus, established in the S.W. angle of the pendulum room in the Royal Observatory, being the place assigned for that apparatus by the Astronomer Royal.

The thermometer, by which the temperature of the pendulum was observed, is the same which I have used and described on former occasions; particularly in the Phil. Trans. for 1830, where its comparison is given with a standard thermometer of M. Bessel's, at the same temperatures at which it was used in these experiments. In a paper on the construction and use of the vacuum apparatus, in the Phil. Trans. for 1829, I have shown that when the pressure of the air is withdrawn from the exterior of the bulb of this thermometer, an index correction of $+0^{\circ}.75$ is required to make its indications in a highly rarified medium correspond with its measure of the same temperatures when under the pressure of the atmosphere. This index correction is consequently applied whenever the air is withdrawn from the apparatus. The thermometer was inclosed with the pendulum within the glasses, with its bulb suspended midway between the knife edges. When the air is withdrawn from the apparatus, and the shutters of the apartment are kept closed (except when light is required for the observation of coincidences), a great uniformity as well as steadiness of temperature is maintained within the glasses.

Whenever the indications of the barometer were required, a reference was made to the standard barometer of the observatory, which is stationed in an adjoining room on the same level.

The scale by which the arcs of vibration were observed was graduated in degrees, each degree being the 0.833th of an inch. In making the observation, the division of the scale coinciding with one side of the bar was noticed at each extremity of the vibration, and the same repeated with the other side of the bar; a mean was then taken of the two included spaces, and half the mean registered as the arc on either side the vertical. The scale being placed in both positions of the pendulum 45.5 inches below the point of suspension, the registered arcs multiplied by 1.05 produce the true arcs of vibration; by employing these in the usual formula for that purpose, what is usually considered

as the compensation for the arcs is obtained. During the progress of the experiments, however, I had occasion to suspect that, particularly when the pendulum was vibrated with the weight above, the retardation of the vibrations, in consequence of their being performed in circular arcs, was greater than the compensations computed by the customary formula $N \cdot \frac{\sin(A + a) \cdot \sin(A - a)}{32\, M \cdot (\log. \sin A - \log. \sin a)}$. To examine this more closely, and to obtain practically a just compensation for the arcs in which the pendulum had been vibrated in the course of its employment, I made several series of experiments distinct from those made to determine the rate, and which I shall proceed to describe in the first instance, though they were not the first in the order of time.

Correction for the Arcs of Vibration.

§ 1. Weight above.

On the 27th of January 1830, I made the following observations, purposing to compare, under circumstances otherwise similar, the rate of the pendulum in different arcs; 1st, commencing with 1°.32 and ending with 0°.73; 2nd, commencing with 0°.70 and ending with 0°.42; and 3rd, commencing with 0°.42 and ending with 0°.19. Having withdrawn the air from the apparatus, the resistance to the vibration was so far diminished, that the time which the pendulum took to reduce its arc from 1°.32 to 0°.73, from 0°.70 to 0°.42, and from 0°.42 to 0°.19, was in each case sufficient to give the rate of the pendulum within the respective arcs with tolerable exactness.

The rate of the clock on this day was taken from its average rate in several days, viz. from the 24th to the 30th of January; exactness in the daily gain or loss of the clock was not required, as the observations were only to be used in their relation to each other.

January 27, 1830. Clock by DENT making 86311.0 Vibrations in a mean solar day.

No. of Coincid.	Therm.	Gauge.	Times of Disapp.	Times of Reapp.	Times of Coincidence.	Arcs of Vibration.	Mean Temp.	Mean Gauge.	Mean Interval.	Reductions to mean tem. 36°.	Reductions to a Vacuum.	Vibrations per diem.
	°	in.	m s	m s	h m s	° °	°	in.				
1	....	....	11 16	11 27	10 11 21.5							
2	34.9	0.86	23 31	23 43	10 23 37	1.26 × 1.05 = 1.32	35.1					
3	....	....	35 45	35 59	10 35 52		+ 0.75	1.275	737.90	−0.07	+0.60	86077.59
15	35.3	1.69	3 15	3 44	1 3 29.5	0.70 × 1.05 = 0.73						
							35.85					
	Air pumped out to reduce the gauge.											
							35.35					
16	35.3	0.92	15 35	16 02	1 15 48.5	0.67 × 1.05 = 0.70	+ 0.75	1.225	742.53	+0.04	+0.58	86079.14
31	35.4	1.53	21 06	21 47	4 21 26.5	0.40 × 1.05 = 0.42						
							36.10					
31	35.4	1.53	21 06	21 47	4 21 26.5	0.40 × 1.05 = 0.42	35.45					
46	....	....	26 37	28 03	7 27 20		+ 0.75	1.725	743.41	+0.08	+0.82	86079.66
47	35.5	1.92	38 55	40 26	7 39 40.5	0.18 × 1.05 = 0.19						
48	....	....	51 17	52 48	7 52 02.5		36.20					

Examining these observations we have

	I.	II.	III.
Arcs	1°.32 to 0°.73	0°.70 to 0°.42	0°.42 to 0°.19
Vibrations per diem, uncorrected for the arcs	86077.59	86079.14	86079.66
Compensation for the arcs by the formula	+1.67	+0.50	+0.14
These numbers should agree if the corrections for the arcs computed by the formula were a just compensation .	86079.26	86079.64	86079.80

From the differences that are here seen to exist, it appears that when the pendulum is vibrated with the weight above, the compensation for the arcs as computed by the customary formula is in defect; and more in defect as the arcs are larger. To make the results agree, it is necessary to multiply the computed correction by 1.36. By the employment of this multiplier, the result obtained in the arcs of least magnitude (which as being the nearest to infinitely small arcs are presumed to be the most correct,) is not altered more than five-hundredths of a vibration; whilst the results obtained in larger arcs

are brought in accordance with it. The corrections and the corrected vibrations then become

	I.	II.	III.
Corrections for the arcs	+2.27	+0.68	+0.19
	86077.59	86079.14	86079.66
Corrected vibrations	86079.86	86079.82	86079.85

On the 29th January the same process was repeated in arcs differing somewhat from the preceding, and designed therefore to render the differences in the results, should they be found to exist, still more conspicuous.

No. of Coincid.	Therm.	Gauge.	Times of Disapp.	Times of Reapp.	Times of Coincidence.	Arcs.	Mean Temp.	Mean Gauge.	Mean Interval.	Reductions to 36°.	Reductions to a Vacuum.	Vibrations per diem.
	°	in.	m s	m s	h m s	° °	°	in.				
1 17	34.6 34.5	0.80 1.05	17 53 34 26	18 01 34 54	1 17 57 4 34 40	1.39×1.05=1.46 0.76×1.05=0.80	34.55 + 0.75 35.30	0.925	737.69	−0.31	+0.42	86077.11
35	34.4	1.23	17 20	18 04	8 17 42	0.39×1.05=0.41	34.45 + 0.75 35.20	1.14	743.44	−0.35	+0.54	86078.98
53	34.1	1.36	0 39	02 04	12 01 21.5	0.17×1.05=0.18	34.25 + 0.75 35.00	1.295	745.53	−0.44	+0.61	86079.60

Examining these observations, we have

	I.	II.	III.
Arcs	1°.46 to 0°.80	0°.80 to 0°.41	0°.41 to 0°.18
Vibrations, uncorrected for the arcs	76077.11	76078.98	76079.60
Corrections for the arcs by the formula	+2.03	+0.58	+0.13
Numbers which should agree if the corrections computed by the formula were a just compensation	76079.14	76079.56	76079.73

The inference from this day's experiment is to the same effect as before. The multiplier required in this instance is 1.32, altering the result in the smallest arcs four hundredths of a vibration, and bringing the others in accordance with it. The corrections and the corrected vibrations become

	I.	II.	III.
Corrections	+2.68	+0.77	+0.17
	86077.11	86078.98	86079.60
Corrected vibrations . . .	86079.79	86079.75	86079.77

On February 3rd I made a third series with the pendulum in the same position; the pendulum of the clock had been shortened since January 29th, and was now making about 86371.0 vibrations; but the temperature being at this time very low, its going was less regular than usual. To obviate that inconvenience I commenced with a series in large arcs; made then two series in small arcs, and concluded with a second series in large arcs; and finally took a mean between the two series in large arcs and the two series in small arcs, as the results to be compared with each other.

No. of Coincid.	Therm.	Gauge.	Times of Disapp.	Times of Reapp.	Times of Coincidence.	Arcs.	Mean Temp.	Mean Gauge.	Mean Interval.	Reductions to 25°.	Reductions to a Vacuum.	Vibrations per diem.
	°	in.	m s	m s	h m s	° °	°	in.				
1 13	25.0 25.0	1.30 1.30	20 07 18 40	20 15 18 55	2 20 11 4 18 47.5	1.26×1.05=1.32 0.85×1.05=0.89	25.0 + 0.75 25.75	1.30	593.04	+0.33	+0.64	86080.67
Fresh impulse.												
1 13	24.9 24.7	1.30 1.30	26 23 25 51	26 55 26 46	4 26 39 6 26 18.5	0.35×1.05=0.37 0.25×1.05=0.26	24.8 + 0.75 25.55	1.30	598.29	+0.24	+0.64	86083.16
26	24.7	1.34	35 25	36 35	8 36 00	0.15×1.05=0.16	24.70 + 0.75 25.45	1.32	598.58	+0.20	+0.65	86083.27
Fresh impulse.												
1 16	24.8 25.0	1.34 1.40	04 03 31 41	04 09 31 57	9 04 06 11 31 49	1.37×1.05=1.44 0.80×1.05=0.84	24.90 + 0.75 25.65	1.37	590.87	+0.29	+0.67	86079.58

Examining these observations, we have

	I.	IV.	II.	III.
Arcs	$1^{\circ}.32$ to $0^{\circ}.89$	$1^{\circ}.44$ to $0^{\circ}.84$	$0^{\circ}.37$ to $0^{\circ}.26$	$0^{\circ}.26$ to $0^{\circ}.16$
Vibrations	86080.67	86079.58	86083.16	86083.27
Correction for the arcs by the formula . .	+1.98	+2.08	+0.16	+0.06
Numbers which should agree if the corrections computed by the formula were a just compensation	86082.15		86083.32	

The inference here is again to the same effect. The multiplier required to produce accordance is 1.61. The corrections and the corrected vibrations then become

	I. & IV.	II. & III.
Corrections	+3.27	+0.18
Corrected vibrations	86083.39	86083.39

For a purpose foreign to the present object, the slider was on this day at 1.75 inch from the middle of the pendulum towards the weight.

The irregular going of the clock (which stopped altogether soon after the fourth series was completed) may have rendered the result of this day's experiments of less value than that of either of the preceding days, in determining the amount of the multiplier. It was not until the 5th of May following, that, the clock being repaired and replaced, I was able to resume the inquiry by the following observations conducted in the same manner as before. The clock was making 86665.0 vibrations, on an average of several days at this period.

No. of Coincid.	Therm.	Gauge.	Times of Disapp.	Times of Recapp.	Times of Coincidence.	Arcs.	Mean Temp.	Mean Gauge.	Mean Interval.	Reductions to 58°.	Reductions to a Vacuum.	Vibrations per diem.
	°	in.	m s	m s	h m s	° °	°	in.				
1	....	....	53 31	53 33	2 53 32							
2	56.5	2.1	58 21	58 25	2 58 23	1.36 × 1.05 = 1.43	56.60					
3	....	....	03 13	03 15	3 03 14		+ 0.75					
20	....	....	25 39	25 43	4 25 41			2.10	291.035	−0.29	+0.96	86065.61
21	56.7	2.1	30 31	30 35	4 30 33	0.93 × 1.05 = 0.98	57.35					
22	....	...	35 23	35 25	4 35 24							
	Fresh impulse.											
1	57.0	....	47 36	47 43	4 47 39.5							
2	....	2.1	52 29	52 35	4 52 32	0.24 × 1.05 = 0.26	57.32					
3	57.1	....	57 21	57 27	4 57 24		+ 0.75					
21	57.6	....	25 06	25 13	6 25 09.5			2.10	292.46	+0.03	+0.96	86068.85
22	....	2.1	30 56	31 05	6 30 00.5	0.18 × 1.05 = 0.19	58.07					
23	57.6	....	34 49	34 57	6 34 53							

Examining these observations, we have	I.	II.
Arcs	1°.43 to 0°.98	0°.26 to 0°.19
Vibrations	86065.61	86068.85
Correction for the arcs by the formula	+2.35	+0.08
Numbers which should agree if the corrections computed by the formula were a just compensation .	86067.96	86068.93

The inference is the same as on the former occasions. The multiplier is in this instance 1.42, and the corrections and corrected vibrations become

Corrections	+3.34	+0.11
	86065.61	86068.85
Corrected vibrations	86068.95	86068.96

If now we collect in one view the different multipliers which have been found from these four series of experiments, we have January 27, 1.36; January 29, 1.32; February 3, 1.61; May 5, 1.42: of which four results, that of February 3rd (1.61), from the unfavourable circumstances of the experiments in the irregular going of the clock already noticed, is least entitled to confidence. The arithmetical mean of the four is 1.43; and that of the three, omitting the result of February 3rd, is 1.37. If then we take 1.4 as

the multiplier, to be employed in computing the compensations for the arcs in the experiments with the weight above, to be hereafter related, we may infer with probability that we employ a more correct multiplier than either 1.3 or 1.5. Now, if either of those numbers were substituted for 1.4, the effect would be to alter the rate of the pendulum with the weight above, as deduced from the experiments in the succeeding pages, one tenth of a vibration per diem. To correct the influence of this tenth of a vibration on the convertibility (the object for which the pendulum is vibrated with the weight above), the slider would require to be moved a certain quantity, which would alter the final deduction of the rate, due to the distance between the axes of suspension, less than one hundredth of a second per diem. In employing 1.4 therefore as a multiplier, any uncertainty in the final result, arising from this cause, is limited in all probability to less than .01 of a second per diem.

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§ 2. Weight below.

On the 31st of January and 1st of February 1830, I made the following observations, for the purpose of ascertaining the compensation for the arcs when the weight was below. The clock, by Dent, was making 86313.3 vibrations. The clock had been stopped on the 30th of January, and was stopped again on the 1st of February, to alter its pendulum. The rate was assigned by comparison with the transit clock during the two days it was going.

No. of Coincid.	Therm.	Gauge.	Times of Disapp.	Times of Reapp.	Times of Coincidence.	Arcs.	Mean Temp.	Mean Gauge.	Mean Interval.	Reductions to 30°.	Reductions to a Vacuum.	Vibrations per diem.
	°	in.	m s	m s	h m s	° °	°	in.				
1	....	....	16 00	16 35	9 16 17.5	0.40 × 1.05 = 0.42	29.9					
2	30.3	1.20	28 19	28 55	9 28 37		+ 0.75	1.25	739.8	+0.29	+0.52	86080.77
21	29.5	1.30	22 25	23 18	1 22 51.5	0.23 × 1.05 = 0.24						
22	....	....	34 49	35 41	1 35 15		30.65					
	Fresh impulse.											
							28.75					
1	29.5	1.30	49 28	49 36	1 49 32	1.25 × 1.05 = 1.31	+ 0.75	1.35	737.4	−0.22	+0.56	86079.54
39	28.0	1.40	36 18	36 48	9 36 33	0.50 × 1.05 = 0.52						
							29.50					
							28.20					
88	28.4	1.52	41 48	43 26	19 42 37	0.14 × 1.05 = 0.15	+ 0.75	1.46	742.12	−0.46	+0.61	86080.81
							28.95					

Examining these observations, we have

	I.	II.	III.
Arcs	0°.42 to 0°.24	1°.31 to 0°.52	0°.52 to 0°.15
Vibrations	86080.77	86079.54	86080.81
Correction for the arcs by the formula	+0.18	+1.29	+0.17
Numbers which should agree if the corrections computed by the formula were just compensations	86080.95	86079.83	86080.98

On the 28th of April following, I repeated the experiment in arcs designed to make the differences between the experimental and computed results still more apparent. The clock, by Dent, was making 86660.5 vibrations on an average of several days about that period.

No. of Coincid.	Therm.	Gauge.	Times of Disapp.	Times of Reapp.	Times of Coincidence.	Arcs.	Mean Temp.	Mean Gauge.	Mean Interval.	Reductions to 58°.	Reductions to a Vacuum.	Vibrations per diem.
	°	in.	m s	m s	h m s	° °	°	in.				
1	53.4	1.60	27 46	27 48	12 27 47	$1.42 \times 1.05 = 1.49$	53.70 + 0.75 = 54.45	1.605	291.58	−1.56	+0.65	86065.17
13	54.0	1.61	26 05	26 07	1 26 06	$1.25 \times 1.05 = 1.31$						
26	55.4	1.62	29 18	29 21	2 29 19.5	$1.07 \times 1.05 = 1.12$	54.7 + 0.75 = 55.45	1.615	291.81	−1.12	+0.67	86066.09
	Fresh impulse.											
1	56.0	1.62	51 49	51 52	2 51 51.5	$0.38 \times 1.05 = 0.40$	56.2 + 0.75 = 56.95	1.62	292.68	−0.46	+0.67	86068.53
15	56.4	1.62	00 07	00 11	4 00 09	$0.31 \times 1.05 = 0.33$						
25	56.2	1.63	48 55	49 00	4 48 57.5	$0.28 \times 1.05 = 0.30$	56.3 + 0.75 = 57.05	1.625	292.85	−0.42	+0.68	86068.92
	Fresh impulse.											
1	56.2	1.63	56 49	56 51	4 56 50	$1.30 \times 1.05 = 1.36$	56.3 + 0.75 = 57.05	1.635	291.46	−0.42	+0.68	86066.10
15	56.4	1.64	04 49	04 52	6 04 50.5	$1.12 \times 1.05 = 1.17$						

Examining these observations, we have

	I.	II.	III.	IV.	V.
Arcs	1°.49 to 1°.31	1°.31 to 1°.12	0°.40 to 0°.33	0°.33 to 0°.30	1°.36 to 1°.17
Vibrations	86065.17	86066.09	86068.53	86068.92	86066.10
Correction for the arcs by the formula . . .	+3.13	+2·42	+0.22	+0.16	+2.73
Numbers which should agree if the corrections computed by the formula were just compensations	86068.30	86068.51	86068.75	86069.08	86068.83

In both these series of experiments the same indication is afforded, viz. that the retardation of the vibration is greater in large arcs than is covered by the correction computed by the formula. The difference, however, between the results in large arcs and in small arcs, with the computed corrections applied, is much less than takes place when the pendulum is vibrated with the weight above. With the weight below, the computed corrections being multiplied by 1.13, the results are rendered accordant with each other on both days of experiment, and the vibrations become as follows:—

January 31st, large arcs, Exp. II. 86080.99; small arcs, Exp. I. & III. 86081.00.
April 28th, large arcs, Exp. I. II. & V. 86068.92; small arcs, Exp. III. & IV. 86068.94.

The alteration produced by the employment of this multiplier, in the final deduction of the rate of the pendulum from the experiments to be subsequently narrated, is an addition of 0.12 vibration per diem to the number of vibrations which would otherwise have been derived. I should have been glad to have employed a greater number of observations in the more assured determination of this multiplier, but circumstances did not permit me to pursue the inquiry further; and I have only now to refer to the consistency and concurrent indication of those results that were obtained, as an evidence that the multiplier derived from them is in all probability very near the truth. I may also notice, that a change of one unit in the second figure of decimals of this multiplier, would produce an alteration of something less than one hundredth of a second per diem in the rate of the pendulum, derived from the experiments which form the subject of this paper.

It was my intention to have investigated experimentally the cause of the retardation being greater in large arcs than accords with the formula of reduction

to infinitely small arcs; but circumstances not permitting me to do so at present, I have only to state my conjecture, that it is caused by the gliding of the knife edges on the planes, a consequence of the elasticity of the planes. It has been found by M. Bessel, that small movements of this kind always took place in a pendulum vibrating on a knife edge, whatever might be the nature of the supporting planes; that its direction was the same as the motion of the pendulum,—to the right when the pendulum moved to the right, and vice versâ; and that its amount was proportioned to the arcs of vibration. Supposing that the cause is as I have conjectured, it would have been satisfactory to have measured the amount of the gliding corresponding to particular arcs, *directly*, in the manner that M. Bessel has done, and to have compared the correction of the *length*, which is the mode of compensation adopted by M. Bessel, with the correction of the *rate*, which is the method that has been adopted here; by either mode the experiments ought to give the same length for the seconds pendulum.

In discussing the correction for the arcs of vibration, it has appeared the most satisfactory course, to introduce the detail of the experiments relating to that branch of the subject into the body of the discussion itself. In the remainder of the paper I shall pursue the more usual course, of placing together, at the close, the detail of all the experiments, in the order and succession in which they were made; introducing into the discussion, abstracts and results, with proper references to the part of the paper containing the details.

Before we enter on the examination of the rate of the pendulum, it is necessary to ascertain the reduction to a vacuum for the small residue of air which cannot be pumped out of the apparatus, as well as for the small additional quantity which occasionally leaks in. I proceed to collect in one view the results of the experiments which were made at suitable opportunities to determine the amount of this reduction.

Reduction to a Vacuum.

§ 1. Weight above.

The plan of experiment was to compare the vibrations made in full atmospheric pressure, with those in the exhausted apparatus, in circumstances in

other respects as nearly similar as possible. By this comparison, the correction for nearly the full atmospheric pressure is ascertained; the proportional part of which is taken as the equivalent for the small residue of air which cannot be wholly withdrawn, and for the occasional small leakage of the apparatus during the experiments; the pressure withinside the glasses being duly observed and registered by means of a syphon gauge placed by the pendulum. In regarding the correction for the last remaining inch of pressure as a proportional part, namely the thirtieth, of the correction for the whole atmosphere, I refer to the experiments related in a former paper in the Philosophical Transactions, in which it was shown that the corrections for a half atmosphere, and for a quarter atmosphere, bore corresponding proportions to the correction for the full atmosphere.

The following is an abstract of two series of such experiments with the weight above.

	Reference to the details at the close of the paper.	Therm.	Vibrations.	Reduction to mean temp. 57°.	Vibrations at 57°.		Barom. or Gauge.	
In the free air .	E	57°.5	86056.88	+0°.22	86057.10	86057.03	inch. 29.650	29.433
	G	59.6	86055.82	+1.14	86056.96		29.217	
Air withdrawn .	F	59.8	86068.56	+1.23		86069.79		1.045
(Slider at 1.5)			Differences			12.76		28.388

In the free air .	O	56°.05	86055.95	−0°.41	86055.54	inch. 29.376
Air withdrawn .	P to S (mean)	56.87	86068.48	−0.06	86068.42	1.572
(Slider at 1.633)			Differences		12.80	27.804

We have then, from these two series, the following differences:

12.76 vibr. corresponding to 28.388 in. of air at 58°.5; or 13.48 vibr. to 30 in.
12.88 —— —— 27.804 —— at 56.1; or 13.89. ——

Whence we obtain the reduction, when the weight is above, in the proportion of 13.68 vibrations per diem for 30 inches of air at 57°.3.

§ 2. Weight below.

Abstract of three series of experiments from which the reduction to a vacuum is derived for the vibrations of the pendulum with the weight below.

	Reference to the details.	Temp.	Vibrations.	Reduction to mean temp. 57°.	Vibrations at 57°.	Barom. or Gauge.
						inch.
In free air	A	56°.95	86057.30	−0.02	86057.28	29.727
	B	57.60	86056.88	+0.26	86057.14 } 86057.25	29.756 } 29.740
	D	56.70	86057.48	−0.13	86057.35	29.737
Air withdrawn .	C	56.95	86068.28	−0.02	86068.24	1.950
(Slider at 1.5)			Differences		10.99	27.790
In free air	N	57°.50	86056.84	+0.22	86057.06	29.457
Air withdrawn .	H to M	56.87	86068.67	−0.06	86068.61	1.211
(Slider at 1.633)			Differences		11.55	28.246
In free air	FF	57°.50	86056.88	+0.22	86057.10 } 86057.04	29.361 } 29.349
	GG	58.00	86056.53	+0.44	86056.97	29.337
Air withdrawn .	W to EE	56.75	86068.71	−0.11	86068.60	0.976
(Slider at 1.566)			Differences		11.56	28.373

From these three series we have the following differences; viz.

10.99 vibr. corresponding to 27.790 in. of air at 57°.1 ; or 11.86 vibr. to 30 in.
11.55 —— —— 28.246 —— at 57.5 ; or 12.26 ——
11.56 —— —— 28.373 —— at 57.7 ; or 12.22 ——

Whence we obtain the reduction when the weight is below, in the proportion of 12.10 vibrations per diem for 30 inches of air at 57°.4.

The mean pressure within the glasses during the fourteen experiments with the weight below, (from which, as will be seen hereafter, the rate of the pendulum was finally derived,) was 1.07 inch: the reduction for which, according to the proportion found above, is 0.43 of a vibration per diem. This amount would be altered only one hundredth of a vibration, were the partial result most distant from the mean substituted for the mean of the three results by which the reduction was determined.

Abstract of the Experiments for the Rate of the Pendulum.

Reference.	Gauge.	Temp.	Duration of experiments.	Vibrations.	Correction to M. T. 57°.	Reduction to a Vacuum.	Corrected Vibrations at 57° in a vacuum.
I. Slider at 1.633. *a.* Weight below.							
	inches.	°	h m s				
H	1.207	57.37	13 58 29	86068.49	+0.16	+0.48	86069.13
I	0.930	56.61	9 17 09.5	86069.04	−0.17	+0.37	86069.24
K	1.365	56.25	14 35 26.5	86068.80	−0.33	+0.55	86069.02
L	0.938	56.67	8 48 33	86069.01	−0.14	+0.37	86069.24
M	1.615	57.47	14 05 25	86067.99	+0.21	+0.65	86068.85
	1.211	56.87	60 15 03	86068.67	−0.06	+0.48	86069.10
I. Slider at 1.633. *b.* Weight above.							
P	1.442	56.70	4 55 48	86068.67	−0.13	+0.66	86069.20
Q	1.650	56.70	4 41 01	86068.45	−0.13	+0.75	86069.07
R	1.685	56.55	8 32 43.5	86068.45	−0.20	+0.76	86069.01
S	1.512	57.55	6 15 10	86068.34	+0.24	+0.69	86069.27
	1.572	56.87	24 24 42.5	86068.48	−0.06	+0.72	86069.14
II. Slider at 1.566. *b.* Weight above.							
T	1.222	58.00	5 46 36.5	86068.76	+0.44	+0.56	86069.76
U	1.460	57.00	9 45 39	86068.83	0.00	+0.67	86069.50
V	1.115	56.60	6 44 40	86069.24	−0.17	+0.51	86069.58
	1.266	57.20	22 16 55.5	86068.94	+0.09	+0.58	86069.61
II. Slider at 1.566. *a.* Weight below.							
W	0.950	56.85	15 03 24	86068.72	−0.07	+0.38	86069.03
X	1.055	56.40	13 01 03	86068.78	−0.26	+0.42	86068.94
Y	1.165	56.50	13 01 05.8	86068.82	−0.22	+0.46	86069.06
Z	0.985	56.95	10 21 25.7	86068.70	−0.02	+0.39	86069.07
A A	0.900	57.30	15 46 32	86068.56	+0.13	+0.36	86069.05
B B	0.895	57.60	9 01 09.5	86068.41	+0.26	+0.36	86069.03
C C	0.920	56.80	9 30 48.5	86068.67	−0.09	+0.37	86068.95
D D	1.070	56.50	9 23 49	86068.97	−0.22	+0.42	86069.17
E E	1.030	56.50	17 36 24	86068.89	−0.22	+0.42	86069.09
	1.000	56.82	112 45 41.5	86068.72	−0.08	+0.40	86069.04

The letters of reference in the first column are to the detail of the experiments at the close of the paper.

If, now, we take a mean of the results in both positions of the pendulum *a* and *b*, and in both positions of the slider I. & II., we have 86069.38 vibrations with the weight above, and 86069.07 vibrations with the weight below, corresponding to the slider at 1.6 inch (being the mean between 1.633 and 1.566) from the middle of the pendulum towards the weight.

The vibrations with the weight above are in excess 0.31 of a vibration. To ascertain the fraction of the tenth of an inch which it would be necessary to move the slider in order to produce perfect convertibility, and the effect of such change in the position of the slider, upon the vibrations with the weight below, we have from the experiments the following data (the letters referring as before to the detail of the experiments at the close of the paper).

Weight above.

		Vibrations.	
Slider at 1.5	E to G	86070.26	making a difference of 1.12 vibration per diem corresponding to 0.133 inch of the slider.
——— 1.566	T to V	86069.61	
——— 1.633	P to S	86069.14	

Weight below.

		Vibrations.	
Slider at 1.5	A to D	86069.00	making a difference of 0.10 vibration per diem corresponding to 0.133 inch of the slider.
——— 1.566	W to E E	86069.04	
——— 1.633	H to M	86069.10	

Whence we obtain by proportion 1.637 as the position of the slider which renders the pendulum perfectly convertible: and the vibrations, in the position of perfect convertibility, 86069.09 with the weight above, and 86069.10 with the weight below;—a result which, it will be seen, is in the closest accordance with the experiments H to S, during which the slider was at 1.633.

Expansion of the Pendulum.

The rate of the pendulum having thus been found for the temperature of 57°, we have to seek its rate at 62°, being the temperature at which Captain Kater referred the distance between the knife edges to Sir George Shuckburgh's scale. To obtain the alteration of rate corresponding to each degree of Fahrenheit from the data furnished by the experiments, we have the rate with the weight below, observed on the 31st of January and 1st of February

at 30°, pp. 467 and 469, to compare with the rate at 57° which has been deduced as above from the experiments in the preceding August and September. The position of the slider in the winter experiments was at 1.6; consequently the result is strictly comparable with the mean of the two results H to M, and W to EE, made in summer, in one of which series the slider was at 1.633, and in the other at 1.566, the mean of which is 1.6. But in fact a small change of position of the slider when the weight is below, has so little influence on the rate, that had there been even a slight difference it might have been safely disregarded for the purpose now in view.

Winter experiments . . .	86081.00	vibrations at	30°
Summer experiments . . .	86069.07	———	57
Differences . . .	11.93	———	27

Equivalent to 0.441 for each degree of Fahrenheit.

In the Phil. Trans. for 1830, Art. XIX. I have reported the results of a similar comparison of the rates of an invariable pendulum, made also of plate brass, in winter and in summer; by which it was shown that a degree of Fahrenheit corresponded to a change of 0.44 in the rate of that pendulum also. These two results are strongly confirmatory of each other.

It results, then, that the vibrations of Captain Kater's pendulum, which at 57° have been found 86069.10, are 86066.90 at 62°. The distance between the knife edges as measured by Captain Kater is 39.44085 inches of Sir George Shuckburgh's scale, the pendulum and scale being at 62°. The vibrations in a vacuum corresponding to this distance are 86066.90, the temperature of the pendulum being 62°. We have then for the seconds pendulum in the Royal Observatory, at 62° and in a vacuum,

$$\text{In } 86066.90^2 : \overset{\text{In.}}{39.44085} : : 86400^2 : \overset{\text{In.}}{39.13734}.$$

Stability of the support of the Pendulum in the preceding experiments.

Being desirous of assuring myself of the stability of the support of the pendulum in the vacuum apparatus, I undertook a distinct series of experiments to obtain the rate of the pendulum on the iron frame, which is permanently affixed to the south wall of the pendulum room in the Royal Observatory, and designed for the use of observers with invariable pendulums, who wish to obtain a basis for their experiments on the variation of gravity at other stations. With this intention, I transferred the agate planes, which had been employed in the experiments in the vacuum apparatus, to the iron frame; and placed the pendulum on them, with the weight below, and the slider at 1.6 inch, making the observations H H to R R as detailed at the close of this paper. The experiments were necessarily made in the free air of the apartment, and are reduced to a vacuum by the reduction already found for the pendulum with the weight below. The rate of the clock, by Graham, which is attached to the wall of the room beneath the iron frame, was furnished me by Mr. Thomas Glanville Taylor, in a memorandum which is subjoined to the detail of the observations. By a mean of the ten experiments H H to R R on the iron frame, the pendulum was found to make 86070.98 vibrations at 53° reduced to a vacuum. The equivalent at 57° is 86069.20, which differs by only 0.10 of a vibration per diem from 86069.10, the rate ascertained in the vacuum apparatus. The rate on the iron frame was obtained by thirty-three hours' vibration of the pendulum; that in the vacuum apparatus by 173 hours. So near an approximation, obtained in less than one fifth of the time that the experiments in the vacuum apparatus were continued, satisfied me that no permanent cause of difference existed, and that it only required that the experiments on the iron frame should be persevered in for the same length of time as those were in the vacuum apparatus, to produce the closest accordance. We may regard therefore the result of the experiments H H to R R, as establishing the equal stability of both supports; and as affording a fair inference that both supports are perfectly stable.

Detail of the Experiments referred to in this paper.

Preliminary Experiments to adjust the Slider, and to obtain the reduction to a vacuum for portions of air remaining in the Apparatus.—The registered arcs must be multiplied by 1.05 throughout, to give the true arcs.

Slider at 1.5 in. from the middle of the pendulum towards the weight. Weight below.

Exp. A. In full atmospheric pressure. Aug. 21, 1829 A.M. to P.M. Dent making 86465.92 vibr.

Barom. {beginning 29.766; ending .. 29.796} 29.781 at 57°; +.019 Capill.; −.073 to 32° = 29.727.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arcs.	Mean Temp.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°		m s	m s	h m s	°	°			°	
1	56.7	....	50 42	50 58	9 50 50	0.65	56.95	422.95	....	+0.26	86057.30
33	57.2	....	35 57	36 52	1 36 24.5	0.13					

Exp. B. In full atmospheric pressure. Aug. 21 P.M. Dent making 86465.92 vibrations.

Barom. {beginning 29.796; ending .. 29.830} 29.813 at 58°; +.019 Capill.; —.076 to 32° = 29.756.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arcs.	Mean Temp.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°		m s	m s	h m s	°	°			°	
1	....	....	49 59	50 18							
2	57.6	....	57 02	57 19	1 57 10.33	0.62					
3	....	....	04 02	04 22			57.6	422.52	....	+0.28	86056.88
25	....	....	38 49	39 28							
26	57.6	....	45 51	46 30	4 46 10.83	0.18					
27	....	....	52 53	53 54							

Exp. C. In rarified air. Aug. 21 P.M. to Aug. 22 A.M. Dent making 86465.92 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arcs.	Mean Temp.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°	in.	m s	m s	h m s	°	°			°	
1	....	....	25 53	26 03							
2	57.4	1.00	33 05	33 16	5 36 47.25	1.15					
3	....	....	40 18	40 30			56.2				
4	....	....	47 30	47 43			+ 0.75	434.17	1.95	+0.68	86068.26
122	....	....	01 11	01 46							
123	....	....	08 25	09 05	20 12 22	0.16	56.95				
124	55.0	2.90	15 40	16 20							
125	....	....	22 53	23 36							

Exp. D. In full atmospheric pressure. Aug. 22 A.M. Dent making 86465.92 vibrations.

Barom. {beginning 29.816, ending .. 29.760} 29.788 at 55°5; +.019 Capill.; −.070 to 32° =29.737.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidences.	Arcs.	Mean Temp.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°		m s	m s	h m s	°	°			°	
1	56.5	....	03 41	03 53	9 03 47	1.00	56.7	422.71	....	+0.67	86057.48
26	....	....	59 43	00 07	12 06 57.33	0.23					
27	56.9	....	06 44	07 10							
28	....	....	13 46	14 14							

Pendulum removed. Planes examined. Pendulum replaced, Weight above.

Exp. E. In full atmospheric pressure. Aug. 22 P.M. Dent making 86465.92 vibrations.

Barom. {beginning 29.760, ending .. 29.650} 29.705 at 57°.5; +.019 Capill.; .074 to 32° = 29.650.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidences.	Arcs.	Mean Temp.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°		m s	m s	h m s	°	°			°	
1	....	....	33 50	34 00	12 40 54.5	1.03	57.5	421.78	....	+0.99	86056.88
2	57.5	....	40 49	41 00							
3	....	....	47 48	48 00							
26	....	....	29 00	30 13	3 43 40.7	0.095					
27	....	....	35 56	37 12							
28	57.5	....	43 04	44 20							
29	....	....	50 02	51 26							
30	....	....	57 04	58 30							

Exp. F. In rarified air. Aug. 23 A.M. to P.M. Dent making 86465.92 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidences.	Arcs.	Mean Temp.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°	in.	m s	m s	h m s	°	°			°	
1	58.3	0.94	18 40	18 50	9 18 45	1.24	59.05 + 0.75 = 59.80	433.85	1.045	+1.27	86068.56
66	59.8	1.15	08 30	09 01	5 08 45.5	0.30					

Exp. G. In full atmospheric pressure. Aug. 24 A.M. Dent making 86466.00 vibrations.

Barom. {beginning 29.200, ending .. 29.360} 29.280 at 60°; +.019 Capill.; −.082 to 32° = 29.217.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidences.	Arcs.	Mean Temp.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°		m s	m s	h m s	°	°			°	
1	59.8	....	4 32	4 42	8 4 37	1.19	59.6	420.52	....	+1.06	86055.82
26	59.4	....	59 24	00 16	10 59 50	0.12					

The ends of the bar of the pendulum were then blackened with lamp-black and varnish, to make a stronger contrast with the white disc on the clock pen-

dulum. The slider was moved to 1.633 inch from the middle of the pendulum towards the weight. The pendulum was then replaced, and the experiments were commenced for determining the rate due to the distance between the axes of suspension.

Slider at 1.633 inch from the middle towards the weight. Weight below.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidences.	Arcs.	Mean Temp.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
Exp. H. August 24 P.M. to August 25 A.M. Dent making 86466.29 vibrations.											
	°	in.	m s	m s	h m s	°	°			°	
							56.62				
1	58.45	0.965	42 57	43 07	5 43 02	1.24	+ 0.75	433.70	1.207	+0.95	86068.49
117	54.8	1.450	41 16	41 46	19 41 31	0.26	57.37				
Exp. I. August 25 A.M. to August 25 P.M. Dent making 86466.31 vibrations.											
	°	in.	m s	m s	h m s	°	°			°	
1	55.0	0.76	09 53	10 04	20 09 58.5	1.08	55.86				
	55.4	....	....	....	22 0 0	....	+ 0.75				
	56.0	....	....	....	24 0 0	....		434.15	0.930	+1.05	86069.04
	56.4	....	....	....	2 0 0	....	56.61				
78	56.5	1.10	26 56	27 20	5 27 08	0.42					
Exp. K. August 25th P.M. to August 26th A.M. Dent making 86466.33 vibrations.											
	°	in.	m s	m s	h m s	°	°			°	
							55.50				
1	56.6	1.10	32 23	32 34	5 32 28.5	1.22	+ 0.75	434.10	1.365	+0.85	86068.80
122	54.4	1.63	07 39	08 11	20 07 55	0.21	56.25				
Exp. L. August 26th A.M. to August 26th P.M. Dent making 86466.15 vibrations.											
	°	in.	m s	m s	h m s	°	°			°	
1	....	....	32 02	32 14							
2	55.0	0.685	....	....	20 39 22	0.99					
3	....	....	46 30	46 42			55.92				
	55.8	....	....	....	23 45 0		0.75	434.42	0.938	+0.93	86069.01
	56.4	....	....	....	2 45 0						
74	....	....	20 29	20 52			56.67				
75	56.5	1.190	27 43	27 06	5 27 55	0.40					
76	....	....	34 58	35 22							

Exp. M. August 26th P.M. to August 27th A.M. Dent making 86465.98 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidences.	Arcs.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°	in.	m s	m s	h m s	°	°			°	
							56.73				
1	56.7	1.24	51 10	51 19	5 51 14.5	1.27	+ 0.75	433.55	1.615	+0.91	86067.99
118	56.75	1.99	56 22	56 57	19 56 39.5	0.21					
							57.48				

Exp. N. August 27th A.M. Dent making 86466.06 vibrations.

Barom. {beginning 29.520 57.5; ending .. 29.508 58.0} 29.514 ; +.019 Capill.; −0.76 to 32° = 29.457.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidences.	Arcs.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°		m s	m s	h m s	°	°			°	
1	57.5	....	47 23	47 31	20 47 27	1.17					
24	....	....	28 51	29 15			57.50	421.61	....	+0.95	86056.84
25	57.5	....	....	....	23 36 05.75	0.30					
26	....	....	42 58	43 19							

The Pendulum was then removed; the planes examined, and found truly horizontal; and the Pendulum replaced with the Weight above.

Exp. O. In Atmospheric Pressure. August 28th A.M. Dent making 86466.25 vibrations.

Barom. {beginning 29.382; ending . 29.470} 29.426 55°.5 ; +.019 Capill.; −.069 to 32° =29.376.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°		m s	m s	h m s	°	°			°	
1	56.0	....	18 52	19 02	20 18 57	1.20	56.05	420.33	....	+1.11	86055.95
25	56.1	....	06 35	07 35	23 07 05	0.13					

Exp. P. August 28th P.M. Dent making 86466.35 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°	in.	m s	m s	h m s	°	°			°	
1	55.5	0.895	46 52	47 01	00 50 32.25	1.29	55.95				
2	55.7	....	54 03	54 13			+ 0.75	432.88	1.442	+1.82	86068.67
42	56.3	....	42 32	42 56	5 46 20.25	0.48					
43	56.3	1.990	49 45	50 08			56.70				

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
Exp. Q. August 28th P.M. Dent making 86466.40 vibrations.											
	°	in.	m s	m s	h m s	°	°				
1	56.3	1.13	14 33	14 43	6 14 38	1.38	55.95 + 0.75 = 56.70	432.33	1.650	+2°.07	86068.45
40	55.6	2.17	55 24	55 54	10 55 39	0.51					
Exp. R. August 28th P.M. to August 29th A.M. Dent making 86466.44 vibrations.											
	°	in.	m s	m s	h m s	°	°				
1	55.6	0.85	9 24	9 37	11 09 30.5	1.32	55.80 + 0.75 = 56.55	433.29	1.685	+1°.15	86068.45
72	56.0	2.52	41 51	42 37	19 42 14	0.20					
Exp. S. August 29th A.M. to August 29th P.M. Dent making 86466.43 vibrations.											
	°	in.	m s	m s	h m s	°	°				
1	56.0	0.935	3 13	3 35	20 3 24	1.25	56.80 + 0.75 = 57.55	432.89	1.512	+1°.40	86068.34
53	57.6	2.09	18 19	18 49	2 18 34	0.34					

Slider removed to 1.566 &c. from the middle towards the weight. Planes examined, and found horizontal. Pendulum replaced, Weight above.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
Exp. T. August 29th P.M. Dent making 86466.40 vibrations.											
	°		m s	m s	h m s	°	°				
1	58.0	0.985	42 13	42 23	4 42 18	1.20	57.25 + 0.75 = 58.00	433.26	1.222	+1°.50	86068.76
49	56.5	1.460	28 37	29 12	10 28 54.5	0.41					
Exp. U. August 29th P.M. to August 30th A.M. Dent making 86466.39 vibrations.											
	°	in.	m s	m s	h m s	°	°				
1	56.6	0.960	41 17	41 27	10 41 22	1.32	56.25 + 0.75 = 57.00	433.81	1.460	+1°.09	86068.83
82	55.9	1.960	26 35	27 27	20 27 01	0.17					

Exp. V. August 30th A.M. to August 30th P.M. Dent making 86466.32 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
		in.	m s	m s	h m s						
1	55°.9	1.38	34 39	34 48	20 34 43.5	1°.38	55°.85 + 0.75 56.60	433.57	1.115	+1°.78	86069.24
57	55.8	0.40	19 10	19 37	3 19 23.5	0.40					

Pendulum removed and replaced. Weight below.

Exp. W. August 30th P.M. to August 31st A.M. Dent making 86466.26 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
		in.	m s	m s	h m s						
1	56°.7	0.76	52 59	53 06	4 53 02.5	1°.43	56°.10 + 0.75 56.85	433.63	0.95	+1°.26	86068.72
126	55.5	1.14	56 12	56 41	19 56 26.5	0.30					

Exp. X. August 31st A.M. to August 31st P.M. Dent making 86466.26 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
		in.	m s	m s	h m s						
1	55°.5	0.78	05 00	05 10	20 05 05	1°.22	55°.65 + 0.75 56.40	433.92	1.055	+1°.02	86068.78
109	55.8	1.33	05 49	06 27	9 06 08	0.30					

Exp. Y. August 31st P.M. to September 1st A.M. Dent making 86466.31 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°	in.	m s	m s	h m s						
1	....	....	23 33	23 45							
2	56.0	0.80	30 44	30 56	9 30 50.33	1°.25	55°.75 + 0.75 56.50	433.94	1.165	+1°.03	86068.82
3	....	....	37 56	38 08							
109	....	....	24 27	24 58							
110	55.5	1.53	31 42	32 10	22 31 56.17	0.29					
111	....	....	38 55	39 25							

Exp. Z. September 1st A.M. to September 1st P.M. Dent making 86466.29 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
		in.	m s	m s	h m s						
1	55°.6	0.82	48 11	48 20	22 51 51	1°.26	56°.2 + 0.75 56.95	433.55	0.985	+1°.28	86068.70
2	....	....	55 22	55 32							
87	....	....	9 26	9 52	9 13 16.75	0.41					
88	56.8	1.15	16 41	17 08							

Exp. A A. September 1st P.M. to September 2nd P.M. Dent making 86466.24 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°	in.	m s	m s	h m s	°	°				
1	57.0	0.665	23 35	23 44	9 23 39.5	1.40	56.55				
132	56.1	1.140	9 58	10 25	25 10 11.5	0.30	+ 0.75	433.53	0.90	+1°.23	86068.56
							57.30				

Exp. B B. September 2nd P.M. Dent making 86466.24 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°	in.	m s	m s	h m s	°	°				
1	57.0	0.72	15 52	16 01	1 15 56.5	1.30	56.85				
76	56.7	1.07	16 54	17 18	10 17 06	0.55	+ 0.75	432.93	0.895	+1°.63	86068.41
							57.60				

Exp. C C. September 2nd P.M. to September 3rd A.M. Dent making 86466.24 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°	in.	m s	m s	h m s	°	°				
1	56.7	0.70	25 49	26 02	10 25 55.5	1.20	56.05				
80	55.4	1.14	56 33	56 55	19 56 44	0.48	+ 0.75	433.53	0.92	+1°.33	86068.67
							56.80				

Exp. D D. September 3rd A.M. to September 3rd P.M. Dent making 86466.23 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°	in.	m s	m s	h m s	°	°				
1	55.5	0.80	06 37	06 46	20 06 41.5	1.30	55.75				
79	56.0	1.34	30 17	30 44	5 30 30.5	0.48	+ 0.75	433.70	1.07	+1°.47	86068.97
							56.50				

Exp. E E. September 3rd P.M. to September 4th A.M. Dent making 86466.23 vibrations.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°	in.	m s	m s	h m s	°	°				
1	56.0	0.89	40 31	40 40	5 40 35.5	1.30	55.75				
147	55.5	1.17	16 42	17 17	23 16 59.5	0.19	+ 0.75	434.14	1.03	+0°.99	86068.89
							56.50				

The air was then admitted, and the following observations made to obtain the reduction to a vacuum, for the small portion of air remaining in the apparatus, indicated by the gauge, in those of the preceding experiments in which the weight was below.

Slider at 1.566. Weight below.

Exp. F F. September 7th A.M. to September 7th P.M. Dent making 86465.83 vibrations.

Barom. {beginning 29.422 57.5; ending . 29.420 60} 29.421 59°; + .019 Capill.; —.079 reduction to 32° = 29.361.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°		m s	m s	h m s	°	°			°	
1	57.0	....	09 05	09 13	10 09 09	1.22	57.5	422.00	....	+0.85	86056.88
31	58.0	....	39 54	40 24	1 40 09	0.21					

Exp. G G. September 7th P.M. Dent making 86465.80 vibrations.

Barom. {beginning 29.420 60; ending . 29.382 60} 29.401 60°; + .019 Capill.; — .083 reduction to 32° = 29.337.

No. of Coincid.	Therm.	Gauge.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arc of Vibration.	Mean Therm.	Mean Interval.	Mean Gauge.	Correction for Arc.	Vibrations in 24 hours.
	°		m s	m s	h m s	°	°			°	
1	58.0	....	50 37	50 47	1 50 42	1.18	58.0	421.73	....	+0.79	86056.53
31	58.0	....	21 18	21 50	5 21 34	0.20					

Mr. Taylor's *memorandum of the computation of the rate of the Clock by* Dent.

The clock's rate in these experiments has been deduced from daily comparisons between it and the Greenwich transit clock; the daily rate of the latter being determined from the observations of several stars and of the sun. With reference to these observations, it may be necessary to remark that they are made by three several observers, as is sufficiently shown in the Greenwich observations by the initials of each observer's name; and in making use of these for determining right ascensions, or the rate of the clock, it becomes necessary to apply to each observed transit the constant difference which is found to exist between the observer, and the observer of the same star on the following day, with which it is compared: for this purpose I have employed the difference as pointed out by a great many observations made for the express purpose; whence it appears that if the observer T. noted the passage of a star at the time τ, the observers T. T. and N. would observe it to pass at $\tau + 0^s.16$ and $\tau + 0^s.35$ respectively. Making use of these, the rate of the Greenwich transit clock comes out:

1829.	s		s
August 21 to 24	−0.36	September 1	−0.24
——— 25	−0.22	——— 3	−0.32
——— 26	−0.18	——— 4	−0.20
——— 27	−0.42	——— 6	−0.38
——— 28	−0.25	——— 7	−0.31
——— 29	−0.34	——— 8	−0.11
——— 30	−0.15	Mean .	−0.266
——— 31	−0.24		

From the near accordance of these results with one another, and from a consideration of the inability of two or three observations (which could only be taken in some cases, on account of the cloudy weather) to determine very correctly the rate, I have preferred taking the general mean $0^s.27$ rather than each individual result. This was accordingly employed with the following comparisons.

Comparison of the Clock Dent *with the Greenwich Transit Clock.*

1829.	Dent. d h m	Transit Clock. h m s	Daily rate of Dent. m s	Rates employed. m s	
August	20 21 52	7 51 50.44	+1 5.92	+1 5.92	Aug. 23
———	24 5 24	15 33 13.45	1 6.66	1 6.29	— 24
———	24 22 9	8 20 11.95	1 6.01	1 6.33	— 25
———	25 22 37	8 51 5.40	1 5.96	1 5.98	— 26
———	26 23 16	9 33 0.20	1 6.32	1 6.14	— 27
———	27 23 40	9 59 52.88	1 6.60	1 6.46	— 28
———	28 23 11	9 33 39.00	1 6.18	1 6.39	— 29
———	29 22 28	8 53 23.90	1 6.34	1 6.26	— 30
———	30 23 6	9 34 18.20	1 6.28	1 6.31	— 31
Sept.	1 0 3	10 34 14.80	1 6.19	1 6.24	Sept. 1
———	2 0 8	10 42 5.30	1 6.28	1 6.23	— 2 & 3
———	3 23 28	10 7 40.33	1 6.19	1 6.23	— 4 & 5
———	6 0 0	10 45 24.00			
Clock was stopped for the re-adjustment of the disc.					
———	7 2 41	13 30 44.20	1 6.09	1 6.14	— 7
———	8 2 59	13 51 36.40		1 6.14	— 8

The column entitled 'Rates employed' will be found to be the mean between the two daily rates in the preceding column; these are those which the clock may be supposed to have attained at midnight, the middle time between the two comparisons; but since the comparisons were taken in the middle of

the day, with observations of coincidences before and after, it will be advisable to take the mean of these daily rates as above for the rate to be employed throughout on any one day, rather than employ the former for coincidences observed in the morning, and the latter for afternoon coincidences.

(Signed) THOMAS GLANVILLE TAYLOR.

Experiments with Captain KATER's Pendulum on the fixed iron frame on the south wall of the Pendulum Room at the Royal Observatory. The agate planes the same which had been used in the vacuum apparatus. The same thermometer also, and arc. Barometer, the standard of the Observatory. Slider at 1.6. Weight below. Clock by GRAHAM; its rate furnished by Mr. THOMAS GLANVILLE TAYLOR in the subjoined memorandum. The correction for the arcs computed by the usual formula, and multiplied by 1.13. (p. 469.)

EXP. H H. September 30. GRAHAM making 86520.71 vibrations.

Barom. $\left\{\begin{matrix} 30.110 & 51 \\ 30.116 & 51 \end{matrix}\right\}$ 30.113; +.019 Capill.; —.058 to 32° = 30.074.

No. of Coincid.	Therm.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arcs of Vibration.	Mean Temp.	Mean Interval.	Correction for Arc.	Reducⁿ to mean temp. 53°	Reduction to a vacuum.	Vibrations in vacuo at 53°.
	°	m s	m s	h m s							
1	....	3 33	3 43								
2	49.65	9 47	9 58	11 09 52.3	0°.80 × 1.05 = 0°.84						
3	....	16 00	16 13			50°.5	375.38	+0°.43	−1.10	+12.30	86071.35
30	....	4 49	5 16								
31	51.35	....	....	2 11 18.2	0.19 × 1.05 = 0.20						
32	....	17 20	17 48								

EXP. I I. Fresh impulse. Barom. $\left\{\begin{matrix} 30.118 & 51.25 \\ 30.144 & 51.25 \end{matrix}\right\}$ 30.131; +.019 Capill.; —.058 to 32° = 30.092.

No. of Coincid.	Therm.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arcs of Vibration.	Mean Temp.	Mean Interval.	Correction for Arc.	Reducⁿ to mean temp. 53°	Reduction to a vacuum.	Vibrations in vacuo at 53°.
		m s	m s	h m s							
1	51°.3	35 47	35 56	2 35 51.5	0°.80 × 1.05 = 0°.84	51°.1	375.02	+0°.54	−0.84	+12.30	86071.28
24	50.9	59 24	59 50	4 59 37	0.28 × 1.05 = 0.29						

EXP. K K. October 1. GRAHAM making 86520.71 vibrations.

Barom. $\left\{\begin{matrix} 30.150 & 50 \\ 30.152 & 54 \end{matrix}\right\}$ 30.151; +.019 Capill.; — .060 to 32° = 30.110.

No. of Coincid.	Therm.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arcs of Vibration.	Mean Temp.	Mean Interval.	Correction for Arc.	Reducⁿ to mean temp. 53°	Reduction to a vacuum.	Vibrations in vacuo at 53°.
		m s	m s	h m s							
1	50°.8	41 33	41 40	9 41 36.5	0°.81 × 1.05 = 0°.85	51°.65	375.10	+0°.47	−0.59	+12.28	86071.52
31	52.5	48 57	49 22	0 49 09.5	0.21 × 1.05 = 0.22						

Exp. L L. Fresh impulse. Barom. $\begin{Bmatrix}30.152\\30.115\end{Bmatrix}$ 30.133 at 55°; +.019 Capill.; — .069 to 32° = 30.083.

No. of Coincid.	Therm.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arcs of Vibration.	Mean Temp.	Mean Interval.	Correction for Arc.	Reducn to mean temp. 53°	Reduction to a Vacuum.	Vibrations in Vacuums at 53°.
		m s	m s	h m s							
1 36	54°.8 52.8	17 11 55 15	17 22 55 32	1 17 16.5 4 55 23.5	0°.80×1.05=0°.84 0.15×1.05=0.16	53°.8	373.91	+0°.39	+0.35	+12.21	86070.85
Exp. M M. October 3. Graham making 86520.80 vibrations. Barom. $\begin{Bmatrix}29.697\\29.650\end{Bmatrix}$ 29.673 at 57°; +.019 Capill.; — .072 to 32° = 29.620.											
1 36	55°.0 54.6	25 36 03 23	25 48 03 38	10 25 42 2 03 30.5	0°.78×1.05=0°.82 0.15×1.05=0.16	54°.8	373.39	+0°.36	+0.79	+12.02	86070.53
Exp. N N. Fresh impulse. Barom. $\begin{Bmatrix}29.650\\29.648\end{Bmatrix}$ 29.649 at 56°; +.019 Capill.; —.070 to 32° = 29.598.											
1 30	54°.7 54.4	8 51 9 15	9 01 9 40	2 8 56 5 9 27.5	0°.84×1.05=0°.88 0.22×1.05=0.23	54°.55	373.5	+0°.50	+0.68	+12.01	86070.71
Exp. O O. October 4. Graham making 86520.89 vibrations. Barom. $\begin{Bmatrix}29.860\\29.856\end{Bmatrix}$ 29.858 at 52°; +.019 Capill.; — .061 to 32° = 29.816.											
1 40	52°.5 53.7	01 25 04 31	01 34 04 52	10 01 29.5 2 04 41.5	0°.82×1.05=0°.86 0.14×1.05=0.15	53°.1	374.16	+0°.39	+0.04	+12.12	86070.97
Exp. P P. Fresh impulse. Barom. $\begin{Bmatrix}29.856\\29.850\end{Bmatrix}$ 29.853 at 54°; +.019 Capill.; — .066 to 32° = 29.806.											
1 28	53°.7 54.3	10 37 58 41	10 47 59 08	2 10 42 4 58 54.5	0°.77×1.05=0°.81 0.23×1.05=0.24	54°.0	373.80	+0°.49	+0.44	+12.11	86071.03
Exp. Q Q. October 5. Graham making 86520.93 vibrations. Barom. $\begin{Bmatrix}29.470\\29.472\end{Bmatrix}$ 29.471 at 55°; +.019 Capill.; — .068 to 32° = 29.422.											
1 40	54°.2 54.3	40 43 43 35	40 55 43 54	10 40 49 2 43 44.5	0°.80×1.05=0°.84 0.13×1.05=0.14	54°.25	373.73	+0°.36	+0.55	+11.93	86070.74

Exp. R R. Fresh impulse. Barom. $\left\{\begin{matrix}29.472\\29.500\end{matrix}\right\}$ 29.486 at 53°.5 ; +.019 Capill. ; −.065 to 32° = 29.440.

No. of Coincid.	Therm.	Times of Disap.	Times of Reap.	Times of Coincidence.	Arcs of Vibration.	Mean Temp.	Mean Interval.	Correction for Arc.	Reduc^{n} to mean temp. 53°	Reduction to a Vacuum.	Vibrations in Vacuums at 53°.
	°	m s	m s	h m s							
1	54.4	50 07	50 17	2 50 12	0°.80×1.05=0°.84	53°.7	373.80	+0°.35	+0.30	+11.97	86070.64
43	53.0	11 39	12 04	7 11 51.5	0.12×1.05=0.13						

Mr. Taylor's *memorandum of the computation of the rate of the Clock by* Graham.

The rate of the clock by **Graham** has been determined by comparisons between it and the Greenwich transit clock, made at intervals of twelve hours nearly; the rate of the latter being determined by transits of the sun and of several stars, from which it appears that the transit clock's rate was,

				s
1829.	September 29 evening	to September 30 evening		+0.06
	——— 30 ———	to October	1 ———	−0.14
	October 1 ———	to ———	2 ———	−0.04

These being employed with the following comparisons give the rate of the clock Graham:

	Graham. h m	Transit Clock. h m s	Rate of Graham. m s
September 30 morning	18 33	11 35 27.05	
			+2 0.76
——— 30 evening	6 23	23 26 14.10	
			2 0.63
October 1 morning	19 20	12 24 26.48	
			2 0.79
——— 1 evening	6 33	23 38 20.45	
			2 0.67
——— 2 morning	18 58	12 4 20.30	

The very near accordance of these results with their mean $+2^{m}\ 0^{s}.71$, will seem to justify the use of this quantity for the experiments of September 30 and October 1.

Daily rate of clock Graham, October 3 to 6.

		m s
1829.	October 3 noon to 4 noon	+2 0.80
	——— 4 ——— to 5 ———	2 0.89
	——— 5 ——— to 6 ———	0.93

(Signed) Thomas Glanville Taylor.

XXVI. *On the Sources and Nature of the Powers on which the Circulation of the Blood depends.* By A. P. W. Philip, *M.D. F.R.S. L. & E.*

Read June 16, 1831.

It is remarkable that, notwithstanding the great importance of the circulation in the animal economy, the length of time which has elapsed since its discovery, and the constant attention it has obtained, there is hardly any department of physiology respecting which there appears to be greater uncertainty and contrariety of opinion than the sources and the nature of the powers on which this function depends. I propose in the following paper, by comparing the principal facts on the subject, and by such additional experiments as seem still to be required, to endeavour to determine these points. Much has lately been written and many experiments have been made with this view, and it has become customary to look for the causes which support the circulation to other sources beside the powers of the heart and blood-vessels.

It has been supposed that what has been called the resilience of the lungs, that is, their tendency to collapse, by relieving the external surface of the heart from some part of the pressure of the atmosphere, is a principal means of causing it to be distended with blood, the whole weight of the atmosphere acting on its internal surface through the medium of the blood which is thus propelled from the veins into its cavities; and in this way it has been supposed that the motion of the blood through the whole of the venous part of the circulation is maintained. A similar effect has been ascribed to the act of inspiration, which it is evident must operate on the same principle; and this opinion has even been sanctioned by the Report of a Committee of the Royal Academy of Sciences of Paris *, and in this country by men whose authority is deservedly high; and the effect of these causes, it is asserted, is increased by the elastic power of the heart itself.

* Report on Dr. Barry's paper, by Baron Cuvier and Professor Dumeril.

However successfully such opinions might be combated by reasoning on the data we already possess, as direct experiment is the most simple as well as decisive way of determining the question, as reasoning on physiological subjects has so often deceived, and the experiments may here be made on the newly dead animal, and consequently without suffering of any kind, I have thought it better that the point should be determined in this way, especially as it is by experiments, which at first view seem to countenance the foregoing opinions, that their supporters attempt to establish them, with the effect, as it appears to me, of withdrawing the attention from the powers on which the circulation actually depends, and introducing considerable confusion respecting a question so immediately connected with the phenomena and treatment of disease.

With a view, therefore, to submit the foregoing opinions to this test, the following experiments were made, in which Mr. Cutler was so good as to assist me.

Exp.—A rabbit was killed in the usual way by a blow on the occiput, and the chest opened on both sides so as freely to admit the air. The lungs were then inflated eight or ten times in the minute by means of a pipe introduced into the trachea; the circulation was found to be vigorous. On laying bare one of the femoral arteries, it was observed to pulsate strongly; and on wounding it, the blood, of a florid colour, indicating that it had undergone the proper change in its circulation through the lungs, gushed out with great force; and on introducing the hand into the thorax, the heart was found to be alternately distended and contracted as in the healthy circulation.

Exp.—All the vessels attached to the heart in the newly dead rabbit being divided, and the heart removed, it was allowed to empty itself. Its contractions continued to recur, and in their intervals it assumed a perfectly flat shape, proving that the elasticity of the heart in this animal is so small that it cannot even maintain the least cavity after the blood is discharged.

It appears from these experiments that the circulation was vigorous when none of the causes to which the motion of the blood in the veins have been ascribed existed. In the first experiment the chest being freely opened on both sides, so that the play of the lungs on inflating them could be seen, all effect on the heart either of the resilience of the lungs or the act of inspiration, was evidently prevented; and in the second, it was proved that no sensible

elasticity of the heart existed; yet while artificial respiration was performed we could perceive no abatement in the vigour of the circulation.

It is to be observed, that all these means can act only in one way in promoting the circulation, namely, by giving to the heart the power of suction; that is, by producing a tendency to vacuum in its cavities, in consequence of which the pressure of the atmosphere propels the blood from the veins into them, that of the arteries being prevented from returning to the heart by the valves at their origins. But all, as far as I know, who have either made experiments with a view to prove the supposed effect of these means on the circulation, or who have sanctioned the inferences from such experiments, have overlooked the circumstance that the veins being tubes of so pliable a nature that when empty they collapse by their own weight, whatever may be said of the effect of such causes in favouring a horizontal or descending motion of the blood, it is impossible that an ascending motion could be produced in them on the principle of suction. As far as the heart may possess any such power, its tendency must be to cause the vessel to collapse, not to raise the fluid it contains.

That the resilience of the lungs as far as they possess this property, and the act of inspiration, tend to dilate the heart and large vessels within the chest, is evident; but the former is very trifling, if it exist at all, except as far as it depends on the mere weight of the lungs; and the latter in common breathing is little more efficient, although the effect of respiration on the brain, when any part of the cranium is removed, sufficiently attests that it has a certain effect. When the breathing is so laborious as essentially to influence the circulation, it evidently tends to derange the regular flow of the blood towards the heart, inspiration of course acting interruptedly; whereas it is only necessary to inspect the chest of any of the more perfect animals immediately after death, and while artificial respiration is being performed, provided death has not been caused by great loss of blood, or an extreme and instantaneous impression on the nervous system, to see that the blood flows uniformly towards the heart with no interruption but that which the contraction of the heart itself occasions.

The elasticity of the heart is greater in some animals than in the rabbit; but it is in all cases very inconsiderable. The heart of the tortoise is the most elastic I have examined; yet even it may be compressed during its diastole by a force not sensibly greater than is sufficient to compress other muscles in a

state of relaxation. Besides, the auricles possess little or no elasticity; and whatever the elasticity of the ventricles may be, it can have no effect on the blood in the veins, because they receive their blood from the auricles which are contracting during the diastole of the ventricles. To these statements it may be added, that in many of the inferior animals the foregoing supposed causes of the venous part of the circulation evidently have no existence, and that, with the exception of the elasticity of the heart, they have no existence in the fetal state in any.

We have just seen from direct experiment, that the circulation of the blood goes on as usual when all these causes have wholly ceased to operate.

I shall now take a rapid view of the facts which, as far as I am capable of judging, leave no room of doubt respecting the sources of the power on which this function depends.

It is so evident to those in the least acquainted with the animal economy that the contractile power of the heart is one of the chief of these sources, that it would be superfluous to enumerate the proofs of it; yet even this position has been denied, and that by a writer of no mean abilities. The opposite error, however, is the more common; and not a few have ascribed, and even still do ascribe, the motion of the blood throughout the whole course of circulation to the contractile power of the heart alone, although it would not be difficult to prove that to drive the blood through one set of capillary vessels, and still more through two or three sets of such vessels,—for in man himself, in one important part of the circulation, it is carried through two, and in some animals through three, sets of capillaries before it returns to the heart,—I say it would not be difficult to prove that to drive it through one set of capillaries, at the rate at which the blood is known to move, would require a force capable of bursting any of the vessels. But here, as in the former instance, it is better to appeal to the evidence of direct facts than to any train of reasoning; and there is no want of such facts to determine the point before us, some of which I formerly had the honour to lay before the Society, and others are stated in my Treatise on the Vital Functions. The most decisive is, that the motion of the blood in the capillaries continues long after the heart has ceased to beat and the animal in the common acceptation of the term is dead, even in the warm-blooded animal, for an hour and a half or two hours, and it is not for some time sen-

sibly affected by the heart's ceasing to beat; nor does this arise from some imperceptible impulse still given by the heart, because when all the vessels attached to this organ are secured by a ligature and the heart cut out, the result is the same.

That the circulation in the capillary vessels is independent of the heart, may be shown by various other means. On viewing the motion of the blood in them, with the assistance of the microscope, it may generally be observed that it is moving with different degrees of velocity in the different vessels of the part we are viewing, frequently more than twice as rapidly in some than in others. Were the motion derived from a common source, this could not be the case. It is impossible, in the motion of the blood in the capillaries, in the least degree to perceive the impulse given by the beating of the heart, which causes the blood in the arteries to move more or less per saltum, the motion of the blood in the former being uniform as long as they retain their vigour, and the necessary supply of blood is afforded from the larger vessels. I have found by experiments very frequently repeated*, that the motion of the blood may be accelerated or retarded in the capillaries by stimulants or sedatives, acting not through the medium of the heart, but on these vessels themselves. Nay, so little effect has the action of the heart on the motion of the blood in the capillaries, that I have found that when the power of the capillaries of a part is suddenly destroyed by the direct application of opium to them, the motion of the blood in them instantly ceases, although the vigour of the heart and that of every other part of the sanguiferous system is entire †.

If the circulation in the capillaries be thus independent of the heart, it is evident that the influence of that organ cannot extend to the veins. On comparing the whole of the foregoing circumstances, is it not a necessary inference that the motion of the blood in the veins, like that in the capillaries, depends on the power of these vessels themselves? But that we may not trust to any train of reasoning, where it is possible to have recourse to direct proof, I made the following experiment, with the assistance of Mr. Cutler.

Exp.—In the newly dead rabbit, in which the circulation was maintained by artificial respiration, the jugular vein was laid bare for about an inch and a half; a ligature was then passed behind the part of the vessel nearest to the

* My Treatise on the Vital Functions. † Ibid.

head, and the animal was so placed that the vein was brought into the perpendicular position, the head of the animal being undermost, so that it was necessary for the vein, in conveying the blood to the heart, to convey it perpendicularly against its gravity. The ligature, which was placed at what was now the lowest part of the exposed portion of the vein, was suddenly tightened, while Mr. Cutler and myself observed the vessel. The blood in the part of the vein between the ligature and the heart was instantly and completely expelled, as the transparency of the vessel enabled us to perceive. The vessel itself wholly collapsed, proving that all its blood had entered the heart, so that to a superficial view there seemed to be no vessel in the part where a large dark-coloured vein had just before appeared. In the mean time, on the other side of the ligature, the vein had become gorged with blood.

In the foregoing experiment we see the blood rising rapidly against its gravity, where all causes external to the vessel on which the venous part of the circulation has been supposed to depend, had ceased to exist, and the vis à tergo was wholly destroyed by the ligature.

By a similar experiment, the power of the arteries in propelling the blood may also be demonstrated.

Exp.—In a newly dead rabbit, the circulation being supported by artificial breathing, the carotid artery was laid bare for about an inch and a half. The animal was so placed as to keep the vessel in the perpendicular position, the head being now uppermost. A ligature was passed behind that part of the vessel which was next the heart, and Mr. Cutler and myself observed the vessel at the moment the ligature was tightened. The artery of course did not collapse as the vein had done in the preceding experiment; but the blood was propelled along the vessel, so that it no longer appeared distended with it. It was at once evident, from the change of appearance in the vessel, that the greater part of the blood had passed on in a direction perpendicularly opposed to its gravity. It is worthy of remark, that the blood of the artery was propelled neither so rapidly nor so completely as that of the vein, the cause of which will be evident in the observations I am about to make on the nature of the function and powers of these vessels.

When the whole of the preceding facts are considered, it will, I think, be admitted that the circulation is performed by the combined power of the heart

and blood-vessels themselves, and that no auxiliary power is necessary for its perfect performance. Here, as in other cases, the more we study the operations of nature, the more direct and simple we find them. The resilient power of the lungs and elasticity of the ventricles of the heart, as far as they exist, favour the free entrance of the blood into these cavities, an office adapted to the feebleness of such powers, which, in many animals we have seen, have no existence. Their operation is similar, but probably much inferior, to the elastic power of the arteries, by which the ingress of the blood suddenly impelled into them by the systole of the heart, is rendered more free than it would have been had these vessels tended to collapse in the intervals of its contractions. Had the blood flowed into them in a continued stream, and been carried through them by their own powers alone, their elasticity would evidently have impeded, not promoted, the circulation through them. Thus the veins, where these conditions obtain, are so pliable that they collapse by their own weight, and hence it was that in the preceding experiments the vein carried on its blood so much more rapidly and completely than the artery, which felt the want of the impulse it receives from the heart, that at once assists in propelling its blood, and through the blood stimulates the vessel itself. The action of the vein was perfect; it possessed all its usual powers, which reside in itself alone.

It only remains for us to inquire into the nature of the power by which the heart and blood-vessels maintain the circulation. Respecting the nature of the power of the heart there cannot be two opinions. It is evidently a muscular power. The structure of its parietes is similar to that of other muscles, and they obey all the usual laws of the muscular fibre.

Is the power of the vessels of the same nature? This is a question which has frequently been discussed. The chief arguments which have been adduced in favour of the affirmative are, the nature of their function; the fibrous appearance observed in some of the vessels, which is more evident in some other animals than in man; and the minuteness of most of the vessels, which, if they are muscular, accounts for the difficulty with which the muscular structure is detected in them. The chief arguments against the muscularity of the vessels have been, that they could not be made to obey an artificial stimulus in the way that the heart and other muscles are found to do, and that their chemical analysis gives no evidence of fibrin. Of the latter of these ob-

jections Dr. Young observes, that a part may be muscular although it does not contain fibrin, and refers in support of this opinion to the crystalline lens. The former of these objections no longer exists, the vessels having been found to obey both stimulants and sedatives as readily as parts more evidently muscular. It appears from many experiments related in my Treatise on the Vital Functions, that the action of the capillary vessels is as easily influenced both by stimulants and sedatives as the heart itself; and although the larger vessels are not so easily excited artificially as the heart and muscles of voluntary motion, yet several physiologists have succeeded in exciting them both by mechanical and chemical agents. But there is another argument in favour of the muscularity of the vessels, which, I think, may be regarded as no less powerful. I endeavoured, in papers which I had the honour to present to the Society, and which appeared in the Philosophical Transactions for 1815, to ascertain the relation which the heart bears to the nervous system, which is different from that of the muscles of voluntary motion. It appears from the facts there adduced, that this organ is not only independent of that system, although capable of being influenced through it either by means of stimulants or sedatives, and that even to the instantaneous destruction of its power; but that it equally obeys either set of agents, whether applied to the brain or spinal marrow; while the muscles of voluntary motion obey no stimulus acting through the nervous system, unless it be applied to their nerves themselves or to the particular parts of that system from which their nerves arise. I found from repeated experiments that the vessels bear the same relation to the nervous system as the heart does, their power being independent of this system, but equally with the heart capable of being influenced by either stimulants or sedatives applied either to the brain or spinal marrow, and that even to the instantaneous destruction of their power. They in all respects bear the same relation to the nervous system with the heart, which affords the strongest argument for believing that their power is of the same nature *.

From the various facts stated or referred to in the foregoing paper, the following inferences appear to be unavoidable ;—That the circulation is maintained by the combined power of the heart and blood-vessels; and that the power of both is a muscular power.

* My Treatise on the Vital Functions.

Fig. 1.

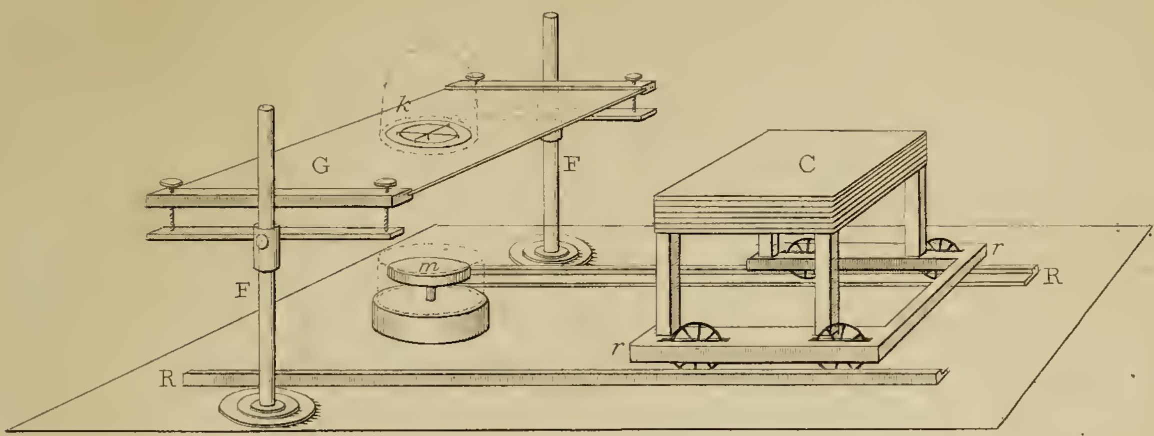
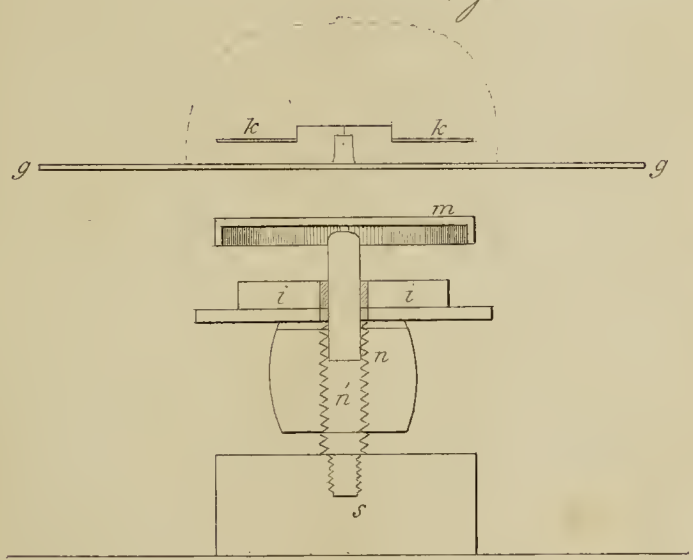


Fig. 2.



XXVII. *On the Influence of Screens in arresting the Progress of Magnetic Action.* *By* William Snow Harris, *Esq.* *F.R.S.*

Read, June 16, 1831.

1. WHILST engaged in some experiments with a thin plate of iron, employed as a screen to intercept the action of a magnet, the following curious fact presented itself to my notice. Although the single plate of iron, which was about the $\frac{1}{16}$th of an inch in thickness, very effectually screened the action of a revolving magnet on a disc of copper, the magnet and disc being placed at a certain distance from each other; yet the same effect did not follow when the disc acted on by the magnet was also of iron: in the latter case it was found requisite to increase the quantity of intervening iron very considerably. This was done by piling several plates similar to the first, one on the other, by which the quantity of iron requisite to intercept completely the action of the magnet could be in some measure estimated.

From this I have been led to some new inquiries concerning the screening influence of substances generally, which, I trust, may possess some claim to the consideration of persons engaged in scientific pursuits. I have therefore to beg the favour of being allowed to lay them before the Royal Society.

2. Having carefully repeated the experiments from which the result just mentioned was obtained, I subsequently endeavoured to determine if the screening influence of the iron plates depended actually on the mass, or was otherwise confined to the plates immediately on the surface. With this view the interior ones were removed, and a ring of wood equal in thickness substituted in their place; the position of the upper and under plate therefore remained unchanged.

In this instance, however, the magnetic action was no longer effectually intercepted, nor could it be completely shut out, except by restoring all the interior plates as before; that is to say, two plates above and two below were insufficient, and so on.

3. From this circumstance, it seemed not unreasonable to infer, as was subsequently proved by the fact, that a screening power might possibly be obtained in a similar way, by means of substances not containing iron, provided such substances were employed in large masses, and were in any degree susceptible of a transient magnetic state; notwithstanding that from the few experiments hitherto tried, it was rather to be inferred that such substances were deficient in this peculiar property*.

4. After a few unsuccessful attempts, I succeeded in making the screening power of several substances, not supposed to contain iron, very evident; the mechanical arrangements resorted to for the purpose were similar to those already described in my paper on the transient magnetic state, of which substances are susceptible, and which has been honoured by a place in the Royal Society's Transactions; it will be unnecessary therefore to describe them again; they will however be easily understood in the detail of the following experiments.

(*a*). A circular magnetic disc *m*, Plate XIII., fig. 1, being delicately balanced on a fine central point, by means of a rim of lead, was put into a state of rotation, on a small agate cup, at the rate of 600 revolutions in a minute; and a light ring of tinned iron *k* also finely balanced on a central pivot, placed immediately over it, at about four inches distance, by means of a thin plate of glass G.

The glass plate was supported on two sliding bars, and stands F F, by which it could be set to any required height; it was also furnished with four levelling screws, passing through two clamps of wood in which the extremities of the plate were fixed, so as to be further adjusted with the required precision: when the iron began to move slowly on its pivot, by the influence of the magnet revolving below, a large mass of copper C, about three inches thick, and consisting of plates a foot square, was carefully interposed; it being sustained on a convenient carriage *r r*, moveable on a rail-way R R, in order to be easily transferred without deranging the subject of experiment.

5. The copper thus interposed soon diminished sensibly the motion of the disc *k*, and at length arrested it altogether: on again withdrawing the copper,

* See Transactions of the Royal Society for the Year 1825. p. 469.

the motion of the disc was restored, and this effect could be obtained as often as required.

In this experiment both the magnet and disc were very completely inclosed by glass shades adapted to the nature of the experiment, and were also supported on a firmly fixed base.

(*b*). When a mass of silver or zinc of about the same dimensions was substituted for the copper, a similar result ensued; the motion of the disc was completely arrested by the screening power of the intervening mass.

(*c*). The screening property evinced by these substances depended, as in the foregoing experiment with the iron plates (2), on the whole mass interposed, as subsequently appeared by removing the interior laminæ, in which case the motion of the disc was no longer impeded.

6. It may therefore be reasonably inferred that this power of intercepting magnetic action is more or less common to every class of substance, and that to render it sensible it is only requisite to employ different bodies in masses bearing some direct ratio to their respective magnetic energies.

(*d*). Thus in substituting a similar mass of lead in the above-mentioned experiments *a b*, the motion of the disc *k* could not be completely checked, and it was subsequently found requisite to increase the quantity of the intervening mass very considerably before the screening effect became sensible to any great extent; the magnetic energy of lead being so much less than that of copper.

7. The screening influence of substances is best shown by employing a powerful magnet, and by placing the disc *k* just within the limit of the action; thus a sufficient mass may be interposed, and the screening effect made very evident. To exemplify the influence of distilled water in this way at about 32° of Fahrenheit's scale, I am led to believe it would be requisite to obtain a slight action on the disc, at a distance of rather more than thirty feet, so as to interpose, about the same thickness of ice.

8. This curious property seems to be intimately connected with a principle, which may be termed a neutralization of force; by which the magnetic action is, as it were controulled, as in the following experiment.

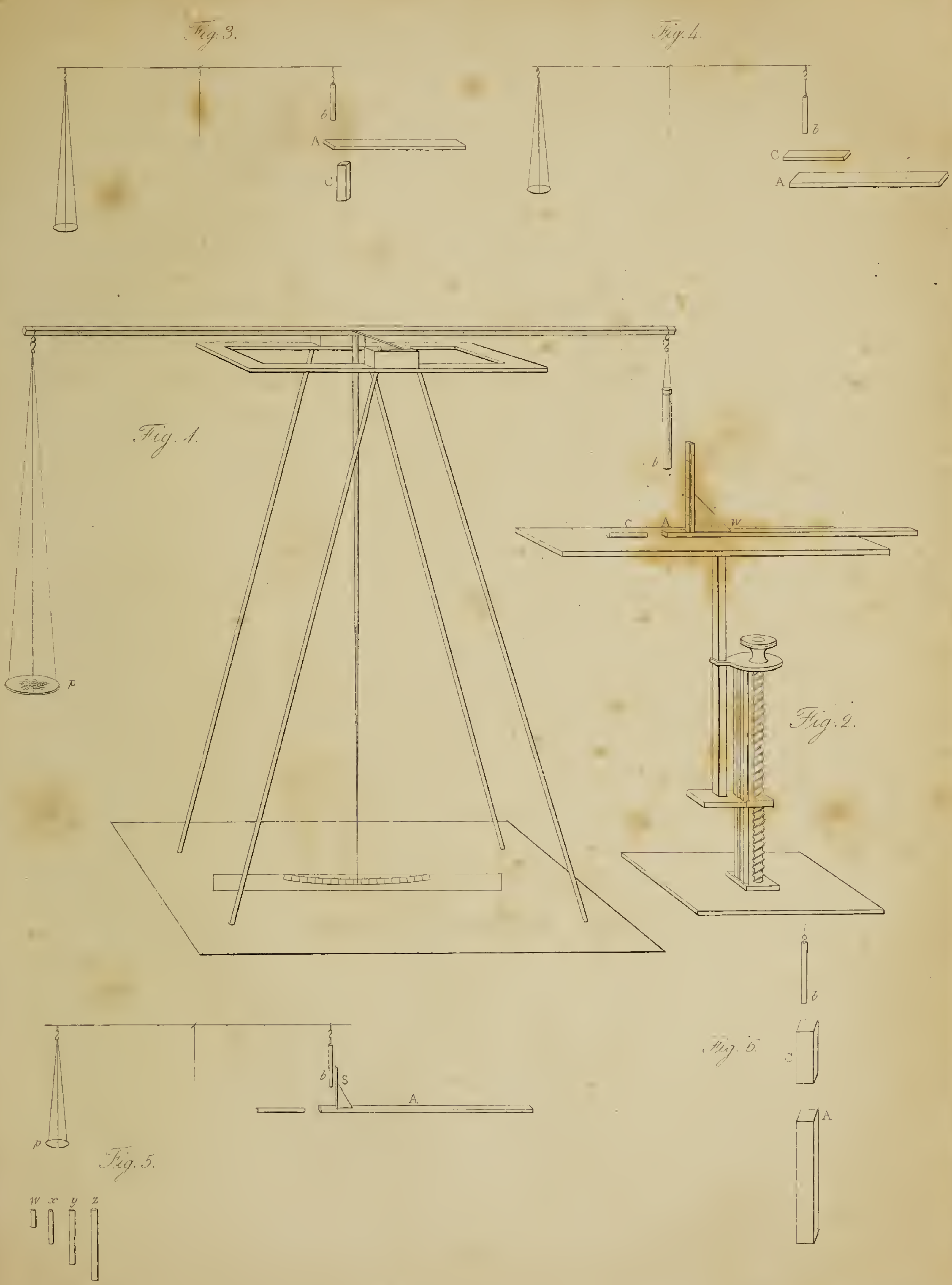
(*e*). A circular magnetic disc *m*, Fig. 2, being put into a state of rapid rotation, a light ring of copper *k* movable on a fine centre was placed imme-

diately over it as in Fig. 1, on a thin plate of glass *g g*, and at such a distance as to be just within the limit of the action; when the copper disc began to move slowly on its pivot, a mass of iron *i* about half an inch thick was carefully raised immediately under the revolving magnet, by means of a nut *n* and screw *n'* affixed to the block *s* on which the whole was sustained; when the iron *i* was near the under surface of the magnet, the action on the disc above began to diminish, so that the motion of the disc was finally checked altogether: by depressing the mass of iron *i* the motion of the disc was again restored, and this result could be obtained as often as required.

9. It does not therefore appear essential to place the iron *i* immediately between the magnet and disc in order to screen the action of the former; since the same effect is produced when the iron is placed immediately beneath the magnet.

10. This subject of screens seems to possess great scientific interest, and if fully investigated is not unlikely to bring us further acquainted with one of those wonderful agencies, on which the phænomenon of attraction may be supposed to depend, the more complete elucidation of which is of the utmost consequence in the present state of physical science.

Plymouth, April 3, 1831.



J. Basire sc.

XXVIII. *On the power of masses of Iron to controul the Attractive Force of a Magnet.* By WILLIAM SNOW HARRIS, *Esq. F.R.S.*

Read June 16, 1831.

1. IT may not be unfavourable to the further elucidation of the interesting subject of screens, treated of in my last paper, and which I had the honour of laying before the Royal Society, to give a short account of some subsequent experiments concerning the effects of masses of iron on a magnet, as they have more particularly arisen out of the investigation above alluded to.

2. The principal part of the apparatus which I employed in these experiments is represented in Plate XIV., fig. 1 & 2; it is extremely simple, and will be readily understood by the following description of it.

A light beam of dry clean-grained deal, of about fourteen inches in length, and somewhat less than a quarter of an inch square, is allowed to rest freely, by means of a delicate axis, on two horizontal bars of glass; the glass bars are secured on a convenient frame and stand, and the axis of the beam, as in many similar cases, is formed of a fine sewing-needle; it was passed in this case through the opposite angles, directly at the centre. From the extremity of one of the arms is suspended a small cylindrical piece of iron, about an inch and a half long and one fifth of an inch diameter, which is counterpoised by an equivalent weight, placed in a small pan suspended from the other; the method of suspension is by light hooks and rings, so as to obtain every possible degree of motion. The under part of the centre of the beam carries an index of about a foot in length, constructed of short pieces of straw, which, being tubular, are easily secured at their extremities one within the other; an index thus formed is very straight and true. By means of this index and a graduated arc the slightest motion of the beam is apparent.

3. The beam and index may be so managed, that for a short distance the deviations from the horizontal position, with equal and very small weights placed in the pan, will correspond to equal divisions on the arc, or very nearly so.

4. To regulate the distance between a magnet or other bodies, and the suspended iron, when placed immediately under the latter, either vertically or horizontally, I employed a simple screw and nut, attached to a brass frame similar to that shown in Fig. 2: thus any required altitude could be obtained, the distance between the magnet and the suspended iron being estimated by a graduated scale resting on the magnet.

5. Wherever magnetic attraction is exerted between two bodies, it seems to be accompanied by a sort of neutralization of the same force in respect of a third substance.

(*a*). Thus if the magnet A, Fig. 1, be attracting the suspended iron *b*, the vicinity of a mass of iron C will diminish the apparent force of A upon *b*: if, therefore, when the force of A upon *b* is exerted so as just to depress the beam, the force being measured by the inclination of the index (3), we place a mass of iron C close to A, the beam will immediately tend to recover its previous position.

(*b*). This power of the iron C to controul the attractive force of the magnet upon *b*, seems only to extend to a given point within the magnet, the distance between the magnet and iron remaining the same; for if the small iron *b* be suspended over a point at some distance from the extremity, as at *w*, then the action of C will not be felt at that point *w*, except by decreasing the distance between the iron and magnet, or otherwise by increasing the neutralizing power of the iron C.

6. That this depends on a sort of action which is not inappropriately termed a neutralization of force, in regard to the suspended iron *b*, is evident from the following experiments.

(*c*). The mass of iron C may be placed immediately below A, as in Fig. 3, the effect will be the most apparent when the thickness of A is not considerable; but if the magnet A be very thick, then the neutralizing effect of C is not so evident, on account of the intervening mass of which the magnet is composed, as also on account of the iron C being kept as it were at a greater distance from the immediate surface of attraction.

(*d*). Conversely, the mass of iron C may be placed above, immediately between *b* and A as in Fig. 4; but in this case we have to take into account the inductive effect on C, by which it becomes itself a temporary magnet, and con-

sequently takes on an attractive power; the experiment should therefore be so managed as to have the distance between C and *b* such, that whilst by the intervention of C the action of A is neutralized, its induced magnetic state does not become sensible upon *b* at that distance; this will be always very evident when C is of some considerable thickness, and the previous distance of A and *b* taken just within the limit of the attraction.

7. In this case C is said to screen or stop out the attraction of A upon *b*, and this probably explains the way in which screens operate in impeding the magnetic influence. It seems therefore not unreasonable to infer, that substances, possessing the greatest inductive energy, are at the same time the most powerful neutralizers. Hence in employing various bodies as screens, those are the most efficient which are susceptible of the greatest transient magnetic state: thus zinc is more efficient than lead; copper more efficient than zinc; silver than copper, and iron the most efficient of any.

8. As the distance within the magnet, to which the neutralizing force can extend, must necessarily depend on the magnetic energy of the substance employed, it would be difficult with non-ferruginous bodies to controul any very sensible portion of the action of a magnet by placing them at its extremity, Fig. 1, or beneath it, Fig. 3, except in the latter case we suppose the magnet to be extremely thin; but by intervening a considerable mass, Fig. 4, immediately between the magnet and the substance acted on, we operate directly on the contiguous attracting surface of the bar, and thus the neutralizing effect at length becomes sensible.

9. The attractive force exerted between a magnet and a mass of iron is in the direct ratio of this neutralizing power of the iron; the distance between the magnet and the iron being the same.

(*e*). Let a magnet A, Fig. 5, of about ten inches in length, and three eighths of an inch square, be placed at some convenient distance, immediately under the suspended iron *b*, and the observed force carefully counterpoised by small weights placed in the opposite pan at *p*, so as to bring the index of the beam, Fig. 1, to zero of the graduated arc; then the neutralizing power of a few small pieces of very soft iron, *w*, *x*, *y*, *z*, about the same diameter as the magnet, and varying from a quarter of an inch to two inches in length, may be easily estimated on the graduated arc, Fig. 1, by bringing each piece successively in

contact, or very nearly so, with the extremity of the magnet *a* (3). Let these small iron cylinders be now substituted in succession for the suspended iron *b*; and being first nicely counterpoised, let the attractive forces be determined at a constant distance from the magnet A by means of the graduated scale *s*; then these respective forces will be found to be very nearly in the same ratio as the previous powers of neutralization: in a great variety of cases they were found to be exactly in the same ratio.

(*f*). Where the neutralizing power is equal, there the attractive force is also equal; thus the neutralizing power, with a given magnet, not being greater in a cylindrical mass of iron of two inches in length than in one of an inch and half in length, no difference was subsequently found in the respective forces of attraction.

10. The foregoing illustrations seem to throw some light on the manner in which magnetic action may be supposed to pervade bodies.

(*g*). Having assigned any given distance, A *b*, Fig. 6, through which we know the influence of a magnet A can extend as estimated by some sensible measure *b*, then in interposing a third substance C in the space A *b*, the latter may receive a temporary magnetic state in two ways, either by the immediate action of the magnet A upon every particle of C, or otherwise by the propagation of magnetism from particle to particle, or by both: now these operations seem to be in some inverse ratio of each other; thus when the induced magnetic energy in C is considerable, the influence of the magnet A is more or less arrested by the laminæ first acted on, which operate as screens on the succeeding ones; so that the magnetic development after a certain distance proceeds entirely by propagation from one particle to another, until it is finally as it were expended; and a body *b* which was before attracted at the distance A *b* will at the same distance now remain at rest *c*. Such is the case in interposing a screen of iron between a revolving magnet and a metallic disc; but if the body C be low in the scale of magnetic energy, then the induced magnetic state is so weak that little or no screening influence is exerted between its particles, and the body *b* may be attracted as before: hence each particle of C will owe its magnetic development to the direct operation of the exciting magnet; and it is only by the successive action of a great number of particles that we at length neutralize or cut off the magnetic force by means of such a substance employed as a

screen: it therefore follows in this instance, that whilst an inconsiderable portion of the magnetic action is neutralized, a considerable mass of the screen is pervaded; at the same time a very thin stratum only of the magnet is penetrated *b* (8).

11. The diminished influence of a magnet on a metallic disc, observed to ensue on intersecting the surface of the disc by radiating grooves varying in depth *, may possibly depend on the above-mentioned circumstance (10); for in this case we actually take away a portion of the substance in which the magnetic development takes place, and thus diminish the force. I wish, however, to be understood as speaking with some degree of reserve on this point, although the conclusion is by no means unwarranted, as in the following experiment.

(*h*). The number of vibrations of a delicately suspended bar in a given arc taken in vacuo, in two similar rings of copper of equal weights and quality as nearly as may be, did not materially differ, although one of the rings was made up of separate concentric laminæ, the other being a perfectly solid mass; whereas the removal of a very thin lamina externally from the former caused a very decided change in the number of the vibrations of the bar †.

12. The preceding inquiries appear calculated to modify in some measure our views concerning the operation of a magnet, which by experiment (*e*) is rather the patient than the agent in the production of the observed effects: it cannot therefore be considered as a purely active force, much less can it be viewed as a substance, from which emanations of an unknown subtile fluid are constantly proceeding; for it may be shown, Exp. (*c*), that a magnetic lamina of steel, supposed without sensible thickness, cannot act at the same time on two masses of iron, in every respect alike, when placed between them, and at an equal distance from each; as in this case we should have an annihilation of power as regards the magnet: hence each mass of iron, if at the same time drawn by some other force, and free to move, would drop away from the magnetized steel in opposite directions. If therefore the attractive energy of a magnet be supposed to arise out of any subtle principle emanating from it,

* Philosophical Transactions for 1825, p. 481.

† These vibrations were determined in the way described in the Royal Society's Transactions for 1831, Part I., p. 76.

such emanations cannot pass off in opposite directions at the same instant: now there is no sufficient reason why they should pass rather in one direction than another, and it therefore remains that an hypothesis which supposes them to pass in either, is quite unwarranted. The arrangement assumed by fine particles of iron, sifted on paper immediately over a magnet, arises out of the circumstance that the bar has generally a very sensible thickness; whilst the small particles of iron cannot operate beyond a certain distance *a, b, c,* and this equally applies to other cases in which the opposite sides of a magnet appear to attract at the same time. Moreover, the superficial boundaries of a magnet may be considered as so many distinct magnetic laminæ of uncertain thickness, as is evident from the circumstance, that the magnetic centre and poles of one surface of a bar very frequently fall in a different way from those of another surface according to the trifling variations in the progress of magnetizing; and sometimes all the surfaces differ in this respect in the same bar, that is to say, the centre and poles do not correspond to the same relative points on any two sides.

13. The wonderful phænomenon of magnetic attraction then is evidently the result of an impression first made on the magnet *e*, since with different masses of iron the attractive force at the same distance is unequal *e*: hence a magnet must be considered as a body in a peculiar state or condition, by which it may be caused to exhibit given powers or capabilities in consequence of external excitation.

15. It is always difficult in inquiries of this nature to employ terms which shall seem altogether without objection. I trust therefore that those resorted to in the course of this paper, will be taken only in the arbitrary sense in which they have been used, and not as having any necessary connection with a particular set of opinions: thus the expressions neutralizing force, magnetic development, magnetic excitation, and so on, must be taken merely as arbitrary terms, employed of necessity to facilitate the progress of inquiry, and to render its description as intelligible as possible, according to the general and unembarrassed acceptation of such terms.

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METEOROLOGICAL JOURNAL,

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MDCCCXXXI.

METEOROLOGICAL JOURNAL FOR JANUARY, 1831.

1831. Jan.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	External Thermometer. Fahrenheit. 3 P.M.	External Thermometer. Self-registering. Lowest.	External Thermometer. Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
♄ 1	29.766	38.7	29.782	41.2	34	34.6	42.3	31.5	42.3		WSW	A.M. Hoar frost—hazy. P.M. Fine and clear.
☉ 2	29.856	40.3	29.843	41.6	36	38.2	39.3	33.1	39.7		SSE	Fair—lightly cloudy.
☽ 3	29.904	39.7	29.896	40.9	38	38.4	39.8	32.8	39.8		S	Overcast.
♂ 4	29.908	40.3	29.867	40.2	37	37.3	37.5	36.4	37.5		ESE	Foggy—light wind.
☿ 5	29.830	40.3	29.875	40.8	37	37.7	39.2	36.3	39.2		E	Foggy.
♃ 6	30.197	38.8	30.301	40.0	35	35.3	38.5	33.3	38.5		N	Fine and cloudless—hazy.
♀ 7	30.592	36.5	30.580	37.0	29	31.0	32.4	29.3	33.7		N	Foggy—light wind.
♄ 8	30.584	33.6	30.499	35.2	27	27.5	33.8	24.3	33.8		N	P.M. Hoar frost—hazy. P.M. Fine —light clouds.
☉ 9	30.254	34.3	30.114	36.3	31	31.7	38.9	26.0	41.2		WNW	Foggy.
☽ 10	29.897	38.9	29.892	39.8	40	40.9	40.8	30.4	40.8		NNE	Overcast—light wind and fog.
♂ 11	30.082	36.7	30.051	37.8	33	33.7	35.2	32.6	36.3		ENE	Lightly cloudy and foggy.
☿ 12	30.080	38.2	30.041	39.0	37	37.7	38.7	32.7	38.7		NNE	Foggy—light wind.
♃ 13	30.120	38.3	30.128	39.3	33	34.7	38.2	33.2	38.3		NE	Foggy—light wind.
● ♀ 14	30.172	41.3	30.130	41.9	37	39.4	38.3	33.7	38.5		NE	Foggy—light wind.
♄ 15	29.989	39.3	29.893	38.2	32	32.6	32.4	31.4	32.6		E	Foggy—light wind.
☉ 16	29.817	37.7	29.771	37.7	32	32.5	33.7	31.3	39.6		E	Overcast—light fog.
☽ 17	29.605	39.4	29.565	40.6	40	40.7	43.2	31.6	43.2		E	Foggy—light rain and wind.
♂ 18	29.552	40.3	29.554	41.9	38	38.7	43.2	36.6	44.3		E	Foggy—light rain.
☿ 19	29.652	43.6	29.625	45.0	45	45.3	46.6	37.7	46.6		S	Foggy—light wind.
♃ 20	29.411	44.6	29.273	45.5	42	42.3	43.6	39.7	43.6		ESE	Foggy—light rain and wind.
♀ 21	29.131	45.5	29.091	47.0	42	42.6	47.8	40.3	47.8		E	Foggy—light rain and wind.
♄ 22	29.218	48.3	29.204	49.2	47	47.7	48.5	41.7	48.5		E	Foggy—light rain.
☉ 23	29.325	49.0	29.385	49.2	47	47.3	44.5	45.7	46.7	0.250	E	Light rain and fog.
☽ 24	29.488	42.7	29.502	41.2	32	35.7	35.5	33.1	35.5	0.083	NNE	A.M. Snow. P.M. Light clouds and wind.
♂ 25	29.802	38.8	29.881	39.3	30	33.5	34.2	30.4	34.2		NNW	Light clouds and wind.
☿ 26	30.085	34.2	30.132	35.8	22	28.7	32.6	24.7	32.6		N	Fine and cloudless.
♃ 27	29.939	33.6	29.522	36.0	21	29.7	38.8	25.3	38.8		SW	Overcast—light rain and wind.
○ ♀ 28	29.650	37.6	29.656	38.4	27	33.0	35.0	28.7	35.0	0.194	NNW	Fine and cloudless—light haze and wind.
♄ 29	29.874	36.2	29.857	37.0	26	32.5	33.5	30.7	33.5	0.014	N	Overcast—light wind.
☉ 30	29.833	34.6	29.834	36.1	20	30.0	32.3	26.3	32.3		E	Lightly cloudy.
☽ 31	29.727	33.7	29.623	36.2	27	30.5	33.8	25.2	33.8		SSE	Lightly overcast. Fall of snow early A.M. and at 1 P.M.
	Mean 29.850	Mean 39.2	Mean 29.818	Mean 40.3	Mean 34.0	Mean 36.2	Mean 38.5	Mean 32.5	Mean 38.9	Sum 0.541		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. .... 9 A.M. 29.832 — 3 P.M. 29.797

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge........................ =83 feet 2½ in.

.... above the mean level of the Sea (presumed about) .... =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR FEBRUARY, 1831.

1831. Feb.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	External Thermometer. Fahrenheit. 3 P.M.	External Thermometer. Self-registering. Lowest.	External Thermometer. Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
♂ 1	29.099	33.4	29.047	36.5	31	31.9	34.9	26.7	35.7		ESE	A.M. Snow. P.M. Fine—light clouds.
☿ 2	29.173	35.3	29.126	37.8	32	34.4	36.0	29.7	38.7	0.175	SSE	Overcast. Light rain and snow P.M.
♃ 3	29.418	34.9	29.235	37.0	33	34.3	35.9	28.3	41.4		S	Overcast. Light rain P.M.
♀ 4	28.954	41.3	29.204	41.2	42	42.3	38.3	33.3	42.3	0.250	E	Overcast. Foggy A.M.
♄ 5	29.642	40.4	29.782	41.6	33	37.5	38.7	33.7	38.7	0.072	W	A.M. Overcast. P.M. Fine—light clouds.
☉ 6	29.888	37.6	29.775	40.3	30	31.8	37.7	29.4	44.4		SSE	Lightly overcast.
☽ 7	29.512	40.7	29.612	44.4	46	46.4	51.2	30.5	51.7	0.325	SSW	Cloudy—light wind.
♂ 8	29.893	46.5	29.918	49.5	51	51.3	52.8	45.7	54.3		S	Overcast. A.M. Light wind and fog. P.M. Rain.
☿ 9	30.004	51.3	30.054	54.4	53	54.5	58.7	50.7	59.5		SSW	Lightly cloudy—light wind. P.M. Fine and clear.
♃ 10	30.234	53.4	30.231	55.2	47	52.6	57.8	49.3	57.7		S	Light clouds and wind.
♀ 11	30.254	53.7	30.274	57.0	49	49.5	55.5	46.7	55.7		SSW	Fine—light clouds.
● ♄ 12	30.292	53.7	30.289	55.2	50	50.3	52.8	48.4	52.8	0.025	W	Fine—lightly cloudy and foggy.
☉ 13	30.236	52.8	30.193	53.8	48	48.7	49.4	47.3	49.4	0.017	SSE	Overcast—light fog. Rain P.M.
☽ 14	30.237	51.5	30.229	53.0	45	46.5	48.6	45.3	48.6	0.011	WSW	Lightly cloudy.
♂ 15	30.131	48.7	29.984	50.6	41	41.0	48.9	40.3	48.9		ESE	A.M. Overcast—light fog. P.M. Fine and cloudless.
☿ 16	29.893	49.7	29.956	52.5	46	46.7	52.0	40.6	52.3	0.028	WSW	Fine—light clouds. At 3½ P.M. Heavy rain with hail.
♃ 17	29.949	48.7	30.085	49.8	43	43.6	47.0	41.3	47.1	0.097	NW	Lightly cloudy—light wind.
♀ 18	30.201	46.7	30.176	49.9	40	40.6	46.4	37.1	46.6	0.014	W	Lightly cloudy—light wind and fog.
♄ 19	30.045	45.0	30.030	48.5	43	43.8	45.7	37.9	47.7		W	A.M. Foggy. P.M. Fine—light clouds and wind. Rain.
☉ 20	29.955	45.3	29.958	46.3	36	41.0	41.8	38.7	41.8	0.069	NNW	Lightly cloudy and foggy—light wind.
☽ 21	30.053	41.4	30.045	44.5	32	36.8	42.0	32.7	42.0		NNW	Lightly overcast—light wind.
♂ 22	30.027	42.3	30.145	43.2	38	38.7	39.9	36.3	39.9	0.017	SSW	Overcast.—P.M. Light rain and wind.
☿ 23	30.403	41.3	30.354	43.8	37	38.5	41.6	35.3	43.8		NNW	Overcast.
♃ 24	30.139	43.7	30.079	44.8	44	44.6	46.6	37.2	46.6		W	Foggy.
♀ 25	29.824	46.7	29.780	48.3	44	44.6	45.8	42.0	45.8		W	A.M. Foggy. P.M. Fine—light clouds and wind.
○ ♄ 26	29.190	46.6	29.227	49.3	45	45.3	46.8	39.7	51.3	0.039	WSW	Fine—light clouds. Showery.
☉ 27	29.272	45.2	29.220	48.6	41	41.7	50.3	36.7	51.3	0.125	S	A.M. Rain. P.M. Fine and clear—brisk wind.
☽ 28	29.537	44.7	29.576	47.4	39	39.8	45.7	36.3	46.5	0.181	WSW	Fine and clear—light haze.
	Mean 29.837	Mean 45.1	Mean 29.842	Mean 47.3	Mean 41.4	Mean 42.8	Mean 46.0	Mean 38.5	Mean 47.2	Sum 1.445		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. .... 9 A.M. 29.802 — 3 P.M. 29.800

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge .... =83 feet 2½ in.

.... above the mean level of the Sea (presumed about) .... =95 feet.

The external Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR MARCH, 1831.

1831. March.	9 o'clock, A.M.		3 o'clock, P.M.		Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer.				Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
	Barom.	Attach. Therm.	Barom.	Attach. Therm.		Fahrenheit.		Self-registering.				
						9 A.M.	3 P.M.	Lowest.	Highest.			
♂ 1	29.844	40.1	29.897	45.4	38	38.8	46.5	34.3	46.5		W	Fine and cloudless—light haze.
☿ 2	29.751	44.7	29.691	48.6	47	47.3	53.5	37.8	53.7	0.036	WSW	Overcast. Light fog A.M.
♃ 3	29.622	49.7	29.641	52.5	51	51.4	54.0	46.6	54.3	0.075	SW	A.M. Foggy—light rain. P.M. Overcast.
♀ 4	29.764	52.7	29.897	55.0	49	49.8	52.8	49.8	53.7	0.222	NW	A.M. Foggy. P.M. Fine—light clouds and wind.
♄ 5	29.830	52.8	29.641	54.2	49	49.4	51.3	47.3	51.3		SSW	Overcast—light wind.
☉ 6	29.154	53.2	29.161	54.3	51	51.3	51.3	48.6	51.8		S var.	Lightly cloudy—light unsteady wind.
☽ 7	29.684	51.3	29.776	53.4	47	47.7	50.5	43.8	50.7		NW	Fine and clear—light clouds.
♂ 8	29.753	48.6	29.594	52.5	45	45.3	50.8	38.3	51.4	0.022	ESE	Cloudy—light wind and fog.
☿ 9	29.595	49.4	29.612	52.6	44	44.4	49.8	40.5	50.3		SW	A.M. Fine and cloudless. P.M. Hail and rain with light wind.
♃ 10	29.926	46.8	29.938	50.7	39	39.7	50.6	35.3	50.6	0.061	W	Fine and cloudless.
♀ 11	29.714	51.7	29.597	53.4	49	49.6	51.6	38.7	51.6	0.028	SSE	Overcast—light rain and wind.
♄ 12	29.937	49.0	29.756	53.0	43	43.4	49.8	37.6	51.3		WSW	A.M. Fine. P.M. Light rain and wind.
☉ 13	29.660	49.7	29.607	51.0	45	46.8	46.7	40.8	48.7		WSW	Overcast. Heavy showers, with light brisk wind.
● ☽ 14	29.690	47.9	29.681	50.3	41	44.3	45.5	39.3	49.7	0.086	WSW	Fine and clear—light clouds. Hail and rain P.M.
♂ 15	29.778	47.3	29.659	50.0	43	43.7	49.7	40.3	52.3	0.042	SW	Light rain.
☿ 16	29.648	50.8	29.741	53.3	51	52.7	55.7	43.2	56.3	0.014	W	Cloudy—light wind.
♃ 17	29.851	53.3	29.893	55.8	53	54.3	57.6	52.3	57.7		SSW var.	Fine—cloudy.
♀ 18	30.184	52.3	30.196	55.4	52	46.3	50.8	40.3	41.0		WNW	Fine and clear—light clouds.
♄ 19	30.217	49.7	30.172	53.5	42	43.8	51.0	37.7	51.5		WSW	Fine—lightly cloudy.
☉ 20	30.112	49.6	30.072	52.7	43	44.7	52.7	41.8	52.7		W	Lightly cloudy—light wind and fog.
☽ 21	30.092	51.4	30.097	54.2	49	49.5	54.4	44.3	54.4		N	Overcast—hazy.
♂ 22	30.262	50.7	30.283	53.0	45	46.7	48.3	43.7	48.3	0.083	ESE	Fine—light clouds.
☿ 23	30.369	46.9	30.297	47.8	34	42.4	42.3	37.7	44.6		E	Fine—lightly cloudy—light brisk wind.
♃ 24	29.958	41.3	29.811	43.2	35	36.1	38.4	31.7	40.3		N	Overcast—light brisk wind.—Snow A.M.
♀ 25	29.653	44.3	29.570	47.7	39	41.7	44.6	34.3	44.7		E	Lightly cloudy—light wind.
♄ 26	29.345	45.3	29.429	49.8	43	43.7	54.3	40.3	54.7	0.061	E	A.M. Showers, with fog and light wind. P.M. Fine.
☉ 27	29.906	51.4	29.941	53.7	51	51.7	53.3	42.6	59.4	0.153	S	Fine and clear—light clouds.
○ ☽ 28	30.056	51.3	30.053	55.3	47	47.7	56.7	42.3	59.4		N	Fine and clear—light clouds.—Hazy A.M.
♂ 29	30.165	47.7	30.168	48.0	42	42.7	44.7	39.7	45.0		N	Overcast—light wind.
☿ 30	30.281	45.3	30.277	49.6	35	41.6	48.2	38.5	49.7		N	Overcast. Hazy A.M.
♃ 31	30.401	44.6	30.398	48.6	38	42.4	48.0	37.3	48.3		N	Fine and clear. Clouds and haze A.M.
	Mean 29.877	Mean 48.7	Mean 29.856	Mean 51.6	Mean 44.5	Mean 45.8	Mean 50.2	Mean 40.9	Mean 50.8	Sum 0.883		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. .... 9 A.M. 29.834 — 3 P.M. 29.801

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge .... =83 feet $2\frac{1}{2}$ in.

.... above the mean level of the Sea (presumed about)............... =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR APRIL, 1831.

1831. April.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	External Thermometer. Fahrenheit. 3 P.M.	External Thermometer. Self-registering. Lowest.	External Thermometer. Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
♀ 1	30.448	45.9	30.335	48.3	37	44.0	45.7	37.5	46.3		N	Cloudy—light wind. Rain at night.
♄ 2	30.071	44.7	30.025	47.2	37	41.3	44.7	36.4	45.7	0.083	SE	Lightly cloudy.
☉ 3	29.963	46.3	29.853	50.2	43	44.3	51.3	40.3	51.3		N	Lightly cloudy—light wind.
☽ 4	29.702	45.3	29.590	48.5	40	41.3	48.4	36.7	48.5		N	Overcast—hazy.
♂ 5	29.503	45.3	29.468	49.2	40	42.7	49.5	35.3	51.3		SSW	Cloudy—hazy.
☿ 6	29.504	49.7	29.468	52.1	43	47.3	54.5	39.7	55.3		SW	Fine—lightly cloudy.
♃ 7	29.375	49.7	29.261	54.5	48	49.7	59.0	43.3	59.7		ESE	Overcast—light wind.
♀ 8	29.284	52.3	29.306	52.8	49	50.7	48.6	48.3	52.7		S	Overcast—light rain.
♄ 9	29.503	52.9	29.534	56.6	51	53.5	56.6	45.7	58.3	0.058	SSE	Lightly cloudy. Light rain at night.
☉ 10	29.610	56.7	29.698	58.6	52	54.7	57.4	49.4	58.6	0.325	S	Clear—lightly cloudy—light wind.
☽ 11	29.981	54.7	29.976	57.2	50	52.3	58.7	45.7	58.7		N	Lightly overcast.
● ♂ 12	29.925	54.6	29.830	58.3	53	53.6	60.8	48.3	61.8		NE	Fine—lightly cloudy. At 9½ P.M. Thunder and lightning, with rain.
☿ 13	29.820	57.7	29.794	61.6	57	57.6	63.8	48.4	64.8	0.153	N	Lightly cloudy.
♃ 14	29.969	58.3	29.948	59.7	43	50.7	56.8	48.6	56.8		N	Fine and clear—light clouds and wind.
♀ 15	30.015	57.3	29.985	58.3	46	52.7	55.8	43.7	57.1		W	Lightly cloudy.
♄ 16	30.064	58.2	30.006	59.9	50	56.7	57.4	57.2	59.3		N	Lightly cloudy.
☉ 17	29.989	55.4	29.994	57.7	49	49.7	52.3	49.2	55.3		N	Cloudy—clear. Light rain A.M.
☽ 18	30.044	52.7	29.977	55.9	39	46.6	56.2	38.3	56.3		N	Fine and clear—light clouds.
♂ 19	29.953	49.7	29.877	54.7	45	45.7	57.8	39.9	57.8		N	Light wind.—A.M. overcast. P.M. Fine and clear.
☿ 20	29.815	53.3	29.720	55.9	41	48.6	56.2	40.4	56.7		N	Fine—lightly cloudy.
♃ 21	29.533	51.4	29.453	54.3	47	47.7	53.4	43.7	53.8		NNE	Overcast.
♀ 22	29.429	54.3	29.429	58.9	53	54.4	60.2	45.8	61.7		E	Fine—lightly cloudy.
♄ 23	29.508	53.8	29.513	60.3	52	52.3	60.8	47.8	65.2		NNW	A.M. Overcast. Noon, hail and rain. P.M. Fine.
☉ 24	29.825	56.7	29.873	57.3	50	52.6	54.7	49.7	54.7	0.019	N	Overcast.
☽ 25	30.005	56.7	29.960	60.3	51	54.7	62.8	50.3	63.7		NNW	Cloudy—light wind.
○ ♂ 26	29.817	59.3	29.753	62.3	49	57.6	61.8	48.7	63.3		E	Lightly cloudy.
☿ 27	29.646	56.6	29.546	61.7	49	52.5	58.7	49.7	60.7		SE	Lowering—light wind and fog. Rain at 10 A.M.
♃ 28	29.347	56.3	29.316	58.4	51	51.7	54.0	50.3	55.6	0.036	S	Overcast and foggy—deposition of moisture.
♀ 29	29.203	56.3	29.216	61.3	50	50.5	61.0	45.5	61.3	0.367	E	A.M. Foggy. P.M. Fine—cloudy—brisk wind—showery.
♄ 30	29.243	60.4	29.195	62.4	51	58.0	61.0	49.7	63.7	0.033	S	Fair—cloudy.
	Mean 29.736	Mean 53.4	Mean 29.697	Mean 56.5	Mean 47.2	Mean 50.5	Mean 56.0	Mean 45.1	Mean 57.2	Sum 1.074		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. .... { 9 A.M. 29.676   3 P.M. 29.631 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge........................ =83 feet 2¼ in.

.... above the mean level of the Sea (presumed about) .... =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR MAY, 1831.

1831. May.	9 o'clock, A.M.		3 o'clock, P.M.		Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer.				Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
						Fahrenheit.		Self-registering.				
	Barom.	Attach. Therm.	Barom.	Attach. Therm.		9 A.M.	3 P.M.	Lowest.	Highest.			
☉ 1	29.462	58.7	29.418	60.3	53	57.3	59.0	46.7	63.3	0.019	SSW	Clear—cloudy—showery.
☽ 2	29.557	59.7	29.583	60.6	51	57.8	56.5	46.7	60.2	0.269	E	Cloudy and foggy—showery.
♂ 3	29.638	56.5	29.652	61.7	54	56.3	62.2	49.0	63.4	0.178	SE	A.M. Cloudy—showers. P.M. Fine and clear—light brisk wind.
☿ 4	29.599	61.3	29.527	61.9	53	60.0	54.7	49.4	62.3	0.017	SSW	A.M. Fine and clear. P.M. Overcast.
♃ 5	29.530	55.6	29.502	56.7	42	51.6	50.3	43.7	55.7		WSW	Fine—lightly cloudy.
♀ 6	29.783	58.3	29.813	55.7		48.5	48.7	40.3	55.7	0.153	NW var.	Cloudy.—Fine and clear A.M.
♄ 7	30.026	50.4	30.063	54.3		44.7	50.3	33.8	54.5		N	Fine—light clouds.
☉ 8	30.264	55.3	30.276	55.2	38	50.4	51.3	36.7	52.3		ENE	Cloudy.—Fine A.M.
☽ 9	30.337	55.3	30.258	55.3	37	51.2	55.5	38.3	55.7		NNE	Fine and clear.—Cloudless P.M.
♂ 10	30.215	51.3	30.162	55.7	47	47.5	54.4	43.5	54.6		ESE	Cloudy.—Light rain A.M.
● ☿ 11	30.201	56.7	30.197	56.5	45	53.8	57.5	38.8	58.5		NE	Fine and clear—brisk wind. Cloudless P.M.
♃ 12	30.125	55.3	30.038	57.2	47	53.7	60.0	39.5	61.7		NNE	Cloudless—hazy.
♀ 13	29.965	61.4	29.942	61.9	51	57.7	61.7	45.5	63.8		NNE	A.M. Hazy. P.M. Fine and clear—light clouds.
♄ 14	30.088	56.8	30.077	58.7	32	50.0	52.3	41.7	55.4		NNE	Lightly cloudy.
☉ 15	30.058	58.5	30.010	59.9	35	53.5	62.6	39.4	63.7		ENE	Fine.—A.M. Light haze. P.M. Clear—light clouds.
☽ 16	30.132	59.7	30.141	61.6	43	56.7	65.7	44.4	67.3		WSW	Fine—lightly overcast.
♂ 17	30.215	63.4	30.151	63.7	48	63.3	64.3	48.3	67.2		ESE	Clear and cloudless.
☿ 18	30.036	66.5	29.964	65.6	49	63.7	66.4	50.5	66.5		ENE	Clear and cloudless—light wind.
♃ 19	29.811	67.7	29.704	65.3	45	61.7	63.8	52.3	66.2		E	Light wind. A.M. Cloudless. P.M. Cloudy.
♀ 20	29.832	68.2	29.830	66.9	55	65.7	66.7	56.3	71.5		S	A.M. Fine and clear. A.M. Cloudy—showery.
♄ 21	29.912	61.9	29.943	68.0	57	57.4	66.7	53.6	67.6	0.086	N	A.M. Overcast. P.M. Fine and clear—light clouds.
☉ 22	30.001	62.3	29.991	64.5	53	57.5	61.7	53.0	66.6		NNW	Lightly overcast.
☽ 23	29.892	62.3	29.842	67.7	58	58.7	70.7	52.3	73.2		NNW	A.M. Overcast. P.M. Nearly cloudless.
♂ 24	29.840	64.3	29.817	69.7	60	60.5	69.7	55.7	72.6		N	A.M. Overcast. P.M. Nearly cloudless.
☿ 25	29.860	65.9	29.846	70.8	60	64.7	71.7	54.6	72.5		N	Fine and clear—light clouds and wind.
○ ♃ 26	29.849	57.7	29.804	66.8	55	57.6	67.3	51.5	69.2		E	A.M. Overcast. P.M. Clear and cloudless.
♀ 27	29.792	60.7	29.790	65.3	52	52.7	63.7	50.6	64.8		NNE	Lightly cloudy—light wind.
♄ 28	29.929	58.7	29.971	64.7	47	57.7	63.6	50.7	65.3		N	Lightly cloudy.
☉ 29	29.975	59.5	29.932	61.7	54	54.7	55.5	52.8	55.6		E	Rain.
☽ 30	29.910	59.7	29.934	63.7	50	55.3	62.3	49.7	63.7	0.333	NNE	A.M. Cloudy. P.M. Fine—light clouds.
♂ 31	29.997	63.7	29.942	63.4	50	61.3	62.3	51.3	63.6		E	Fine and clear—light clouds.
	Mean 29.930	Mean 59.7	Mean 29.907	Mean 62.0	Mean 49.0	Mean 56.2	Mean 62.2	Mean 47.1	Mean 63.0	Sum 1.055		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. .... 9 A.M. 29.851 | 3 P.M. 29.821

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge .... =83 feet 2¼ in.

.... above the mean level of the Sea (presumed about) .... =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR JUNE, 1831.

1831. June.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	External Thermometer. Fahrenheit. 3 P.M.	External Thermometer. Self-registering. Lowest.	External Thermometer. Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
☿ 1	29.937	66.8	29.981	66.4	52	61.7	64.7	49.6	67.3		NNE	Fine and clear—light clouds and wind.
♃ 2	30.153	67.7	30.113	66.4	44	62.7	70.0	48.6	70.6		N	Fine—light clouds and wind.
♀ 3	30.188	66.7	30.157	67.7	52	58.7	70.3	49.9	70.3		E	Cloudless—hazy.
♄ 4	30.196	64.4	30.188	68.5	47	60.7	68.7	48.7	69.5		NNE	Fine—light clouds and wind.
☉ 5	30.136	69.7	30.081	71.7	52	67.7	73.4	55.3	74.8		W	Fine—clouds and haze—light wind. Rain P.M.
☽ 6	30.081	65.3	30.078	65.6	43	57.4	59.4	51.3	60.3	0.025	N	Fine and clear—cloudy—light brisk wind.
♂ 7	30.074	60.4	30.007	63.3	42	54.3	58.4	49.9	61.3		NNW	Cloudy—light wind.
☿ 8	29.847	65.7	29.826	66.9	52	61.0	67.3	49.5	67.6		NNW	Fine—cloudy—light wind.
♃ 9	29.878	68.8	29.843	67.7	51	65.7	70.4	51.4	72.7		WSW	Fine—light clouds. Light rain P.M.
● ♀ 10	29.752	73.4	29.771	69.1	52	65.6	67.4	57.5	70.7	0.047	W	Fine—cloudy.
♄ 11	29.634	69.4	29.734	68.7	53	65.3	70.4	57.7	71.7	0.011	WSW	Clear—cloudy—light showers and unsteady wind.
☉ 12	29.798	68.7	29.877	70.3	58	67.6	70.7	60.3	72.3		WSW	Fair—lightly cloudy.
☽ 13	29.902	68.3	29.985	68.7	58	64.7	68.5	59.3	71.7	0.028	W	Cloudy—light wind. Showers through the day.
♂ 14	30.173	69.4	30.137	69.6	58	66.7	72.6	57.7	73.7	0.289	W	Clear and cloudless.
☿ 15	29.914	76.2	29.871	70.3	56	69.8	66.5	56.7	74.7		SSW	A.M. Very clear. P.M. Light rain.
♃ 16	29.856	70.2	29.830	69.4	52	67.3	69.7	57.3	73.5	0.036	WSW	Fine and clear—cloudy.
♀ 17	29.948	67.7	29.987	66.7	55	65.6	67.0	55.7	69.3		WSW	Clear—cloudy—showery.
♄ 18	30.019	64.7	30.026	67.9	63	63.6	67.0	56.7	69.6	0.055	SSE var.	A.M. Lowering—light wind. P.M. Clear—Cloudy.
☉ 19	29.970	69.7	30.023	70.0	56	67.6	71.0	60.7	73.3		SSW	A.M. Cloudy—light wind. P.M. Cloudless.
☽ 20	30.205	68.7	30.205	69.7	50	67.8	71.5	53.4	73.4		W	A.M. Very clear. P.M. Fine—cloudy.
♂ 21	30.226	73.4	30.175	71.5	53	69.7	75.3	55.6	75.7		S	Fine and clear—nearly cloudless.
☿ 22	30.230	70.5	30.220	71.6	53	69.5	73.4	56.7	73.8		WSW	Fine—light haze—cloudy.
♃ 23	30.232	70.4	30.180	71.3	57	67.7	71.0	58.5	72.7		NNE	Fine—hazy.
♀ 24	29.941	65.8	29.854	67.7	55	61.3	66.7	57.8	68.7	0.103	WSW	A.M. Rain. P.M. Fine and clear—light wind.
○ ♄ 25	29.791	65.7	29.763	64.9	51	63.6	61.3	55.7	66.0	0.011	W	Cloudy. Light showers P.M.
☉ 26	29.636	64.3	29.658	66.3	55	58.6	62.5	53.4	67.6	0.150	NNW	Fine and clear—light unsteady wind. Light rain, morning and evening.
☽ 27	29.928	69.6	29.850	66.3	55	66.7	64.6	51.7	69.4	0.128	WSW	A.M. Fine—light clouds. P.M. Rain.
♂ 28	29.927	62.8	29.984	65.4	57	58.7	60.7	55.5	62.3	0.225	NE	Cloudy—light brisk wind.
☿ 29	30.034	62.6	30.004	66.2	53	57.3	62.4	54.4	65.6		N	Overcast. Light rain P.M.
♃ 30	30.006	61.7	30.023	66.3	56	57.7	65.4	54.6	66.3	0.047	NW	Overcast—light wind.
	Mean 29.987	Mean 67.6	Mean 29.981	Mean 68.1	Mean 53.0	Mean 63.7	Mean 67.6	Mean 54.7	Mean 69.9	Sum 1.052		

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